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CENTER CONDITIONS FOR A CUBIC DIFFERENTIAL SYSTEM HAVING AN INTEGRATING FACTOR

We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one irreducible invariant cubic. The presence of a center at $O(0,0)$ is proved by constructing integrating factors.

Key words and phrases: cubic differential system, the problem of the center, invariant cubic curve, integrating factor.

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Introduction

We consider the cubic system of differential equations

$$\dot{x} = y + p_2(x,y) + p_3(x,y) \equiv P(x,y), \quad \dot{y} = -x + q_2(x,y) + q_3(x,y) \equiv Q(x,y), \tag{1}$$

where $p_j(x,y)$ and $q_j(x,y)$ are homogeneous polynomials of degree j and $P(x,y)$, $Q(x,y) \in \mathbb{R}[x,y]$ are coprime polynomials. The origin $O(0,0)$ is a singular point for (1) with purely imaginary eigenvalues, i.e. a focus or a center. The purpose of this paper is to find verifiable conditions under which $O(0,0)$ is a center.

Although the problem of the center dates from the end of the 19th century, it is completely solved only for: quadratic systems $\dot{x} = y + p_2(x,y)$, $\dot{y} = -x + q_2(x,y)$; cubic symmetric systems $\dot{x} = y + p_3(x,y)$, $\dot{y} = -x + q_3(x,y)$; Kukles system $\dot{x} = y$, $\dot{y} = -x + q_2(x,y) + q_3(x,y)$ and a few particular cases in families of polynomial systems of higher degree.

If the cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of finding a finite number of necessary and sufficient conditions for the center is still open. It was possible to find a finite number of conditions for the center only in some particular cases (see, for example, [2], [3], [7], [9], [11], [10], [12]).

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1 Invariant algebraic curves and integrating factors

It is known from Poincaré and Lyapunov that a singular point $O(0,0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral [8]

$$x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x,y) = C$$

in the neighborhood of $O(0,0)$ or an analytic integrating factor of the form [1]

$$\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y), \tag{2}$$

where F_k and μ_k are homogeneous polynomials of degree k.

We study the problem of the center for a cubic system (1) assuming that the system has an irreducible invariant algebraic curve.

Definition 1. *An algebraic curve $\Phi(x,y) = 0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x,y]$ is said to be an invariant algebraic curve of system (1) if*

$$\frac{\partial \Phi}{\partial x} P(x,y) + \frac{\partial \Phi}{\partial y} Q(x,y) = \Phi(x,y) K(x,y), \tag{3}$$

for some polynomial $K(x,y) \in \mathbb{C}[x,y]$ called the cofactor of the invariant algebraic curve $\Phi(x,y) = 0$.

The conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system (1) with two distinct invariant straight lines were obtained in [4].

In [3] the problem of the center was solved for system (1) with: at least three invariant straight lines; two invariant straight lines and one irreducible invariant conic. The center conditions for system (1) with two invariant straight lines and one irreducible invariant cubic

$$x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0 \tag{4}$$

where found in [5]. The presence of a center in these papers was proved by using the method of Darboux integrability and the rational reversibility.

The goal of this paper is to obtain the center conditions for a cubic differential system (1) with an irreducible invariant cubic curve of the form (4) by constructing integrating factors.

2 Cubic systems with one invariant cubic

Let us write the cubic system (1) in the form

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{5}$$

where $P(x,y), Q(x,y)$ are coprime polynomials in $\mathbb{R}[x,y]$. The origin $O(0,0)$ is a singular point which is a center or a focus (a fine focus) for (5).

Assume that the cubic system (5) has a real invariant cubic curve of the form (4). By rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x$, $y \to \alpha y$), we can make the curve to pass through a point $(0, 1)$. In this case the invariant cubic curve looks as

$$\Phi \equiv a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 - y^3 + x^2 + y^2 = 0. \qquad (6)$$

In this Section we determinate the condition under which the cubic system (5) has an irreducible invariant cubic curve of the form (6).

Theorem 1. *The cubic differential system* (5) *has an invariant cubic curve of the form* (6) *if and only if one of the following three sets of conditions holds*

(c_1) $d = 2(a - 4f - 3r - 6)$, $l = -[(f + r + 1)a_{12} + b]$, $k = [(54 - 6a + 36f + 24r)a_{12} + 6ac - 12bf - 8br - 18b - 30cf - 20cr - 45c + 12fg + 8gr + 18g]/6$, $m = 4af + 2ar + 6a - a_{12}^2 + a_{12}c - 12f^2 - 14fr - 36f - 4r^2 - 21r - 27$, $n = [-8af - 8ar - 12a - 2a_{12}^2 + (2b + c)a_{12} + 24f^2 + 40fr + 72f + 16r^2 + 60r + 54]/2$, $p = [(2f + 6)a_{12} - 3c]/2$, $q = 2ab + 2ac + (6f + 3r + 9 - 3a)a_{12} - 4bf - 2br - 6b - 4cf - 2cr - 6c - 2fg - 2gr - 3g$, $s = [-12a_{12}^2 + (14b + 17c - 8g)a_{12} - 4b^2 - 10bc + 4bg - 6c^2 + 6cg]/6$;

(c_2) $d = (3 - 2a + 2f - a_{12}^2)/2$, $g = (a_{12}^3 - 27a_{12} + 18b + 18c)/18$, $k = [a_{12}^5 + (6a + 54)a_{12}^3 - 27ca_{12}^2 + (162a - 972f - 324r - 2187)a_{12} + 729c + 486p]/162$, $l = -(b + fa_{12} + ra_{12} + a_{12})$, $m = (a_{12}^4 + (6a - 162f - 48r - 486)a_{12}^2 - 162a + 216ca_{12} + 72pa_{12} + 486f + 324r + 729)/108$, $n = [(10f + 4r + 19)a_{12}^2 + 6(b - c - p)a_{12} - 18f - 12r + 6a - 27]/6$, $q = [-a_{12}^5 + (6f + 54 - 6a)a_{12}^3 - 36(b + c)a_{12}^2 + (891 - 54a + 378f + 108r)a_{12} - 324c - 216p]/108$, $s = [a_{12}((2b + 3c)a_{12}^2 - 5a_{12}^3 + (36f + 12r - 6a + 81)a_{12} - 27c - 18p)]/54$;

(c_3) $d = (2f - 2a + 3 - a_{12}^2 + 81t^2)/2$, $g = (a_{12}^3 - 243a_{12}t^2 - 27a_{12} + 18b + 18c + 1458t^3)/18$, $k = [2a_{12}^5 + 3a_{12}^3(4a - 216t^2 + 27) + 27a_{12}^2(9t - 2c + 108t^3) + 81a_{12}(2a - 36at^2 - 6f - 4r + 486t^4 - 81t^2 - 9) + 729t(24at^2 + 2a + 6ct - 6f - 4r - 324t^4 - 27t^2 - 9)]/324$, $l = -[(f + r + 1)a_{12} + b]$, $m = [-a_{12}^4 + 18ta_{12}^3 - 6a_{12}^2(a + 9f + 4r + 36) + 54a_{12}(2at + 2c - 6ft - 4rt - 27t^3 - 9t) + 81(6f - 6at^2 - 2a + 90ft^2 + 48rt^2 + 4r + 81t^4 + 162t^2 + 9)]/108$, $n = [a_{12}^4 - 9ta_{12}^3 + 3a_{12}^2(2a + 2f + 4r - 27t^2 - 7) + 9a_{12}(4b + 2c - 6at + 18ft + 12rt + 81t^3 + 27t) + 18(2a - 108ft^2 - 6f - 108rt^2 - 4r - 189t^2 - 9)]/36$, $p = [-a_{12}^3 + 9a_{12}^2t + 3a_{12}(45 - 2a + 18f + 4r + 27t^2) + 27(2at - 2c - 6ft - 4rt - 27t^3 - 9t)]/36$, $q = [-a_{12}^5 + 6a_{12}^3(10 - a + f + 54t^2) - 18a_{12}^2(2b + 2c + 81t^3 + 3t) + 9a_{12}(162at^2 - 2a - 162ft^2 + 6f + 4r - 2187t^4 - 702t^2 + 9) + 162t(9 - 54at^2 - 2a + 18bt + 18ct + 54ft^2 + 6f + 4r + 729t^4 + 108t^2)]/108$, $s = [-9a_{12}^4 + a_{12}^3(4b + 6c - 9t) + 3a_{12}^2(9 - 2a + 6f + 4r + 729t^2) + 27ta_{12}(9 - 2a - 36bt - 54ct + 6f + 4r - 405t^2) + 486t^2(2a + 12bt + 18ct - 6f - 4r - 27t^2 - 9)]/108$.

Proof. By Definition 1, the cubic curve (6) is an invariant cubic curve for system (5) if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$ such that

$$P(x, y)\frac{\partial\Phi}{\partial x} + Q(x, y)\frac{\partial\Phi}{\partial y} \equiv \Phi(x, y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y). \qquad (7)$$

Identifying the coefficients of the monomials x^iy^j in (7), we reduce this identity to a system of fifteen equations

$$\{U_{ij} = 0, \ \ i + j = 3,\, 4,\, 5\} \tag{8}$$

for the unknowns $a_{30}, a_{21}, a_{12}, c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$.

When $i + j = 3$, we find that $c_{10} = 2a - a_{21}$, $c_{01} = 2c - 2g - 2a_{12} + 3a_{30}$, $d = (2f - 2a + 3a_{21} + 3)/2$, $g = (3a_{30} - 3a_{12} + 2b + 2c)/2$. Then we express c_{02}, c_{11}, c_{20} and s from the equations $\{U_{05}, U_{14}, U_{23}, U_{32}\}$ of (8)

$c_{02} = -a_{12}r - 3l$, $c_{11} = -a_{12}^2 r - a_{12}l - a_{12}p - 2a_{21}r - 3n$, $c_{20} = -a_{12}^3 r - a_{12}^2 l - pa_{12}^2 - 3ra_{12}a_{21} - ma_{12} - na_{12} - 2la_{21} - 2pa_{21} - 3ra_{30} - 3q$, $s = [-ra_{12}^4 - (l + p)a_{12}^3 - (4ra_{21} + m + n)a_{12}^2 - (3la_{21} + 3pa_{21} + 4ra_{30} + k + q)a_{12} - 2ra_{21}^2 - 2(m + n)a_{21} - 3(l + p)a_{30}]/3$

and calculate the resultant of the equation U_{50} and U_{41} with respect to q. We obtain

$$Res(U_{50}, U_{41}, q) = f_1 f_2,$$

where $f_1 = ra_{12}^2 + (l+p)a_{12} + ra_{21} + m + n$, $f_2 = 4a_{12}^3 a_{30} - a_{12}^2 a_{21}^2 + 18a_{12}a_{21}a_{30} - 4a_{21}^3 + 27a_{30}^2$.

Let $f_1 = 0$, then $n = -(ra_{12}^2 + (l + p)a_{12} + ra_{21} + m)$. In this case we have

$$U_{41} \equiv g_1 g_2 = 0,\ U_{50} \equiv g_1 h_1 = 0,$$

where $g_1 = (ra_{12} + l + p)a_{21} + ra_{30} + k + q$, $g_2 = a_{12}^2 + 3a_{21}$, $h_1 = a_{12}a_{21} + 9a_{30}$.

Assume that $g_1 = 0$, then $q = -(ra_{30} + (l + p)a_{21} + ra_{12}a_{21} + k)$. In this case we express l, m, k and p from the equations of (8), and we obtain the set of conditions (c_1) for the existence of the invariant cubic

$$(3a_{12} - 2b - 2c + 2g)x^3 + 3(2a - 6f - 4r - 9)x^2 y + 3a_{12}xy^2 - 3y^3 + 3(x^2 + y^2) = 0.$$

Assume that $g_1 \neq 0$, then the equations $U_{50} = 0$ and $U_{41} = 0$ of (8) yield

$$a_{30} = (-a_{12}a_{21})/9,\ a_{21} = (-a_{12}^2)/3.$$

In this case $f_2 \equiv 0$ and we obtain the set of conditions which is contained in (c_2) $(p = (-6aa_{12}^2 + 54a - a_{12}^4 + 90fa_{12}^2 + 12ra_{12}^2 + 252a_{12}^2 - 108ca_{12} - 162f - 108r - 243)/(72a_{12}))$.

Assume that $f_2 = 0$ and let $f_1 \neq 0$. The equation $f_2 = 0$ admits the following parametrization

$$a_{30} = (a_{12}^3 - 243a_{12}t^2 + 1458t^3)/27,\ a_{21} = (81t^2 - a_{12}^2)/3.$$

In this case we have $U_{41} \equiv e_1 e_2 = 0$, $U_{50} \equiv e_1 e_2 (a_{12} - 9t) = 0$, where

$e_1 = t^2$, $e_2 = 4ra_{12}^3 + 9(l + p + 6rt)a_{12}^2 + 9(9lt + 2m + 2n + 9pt + 108rt^2)a_{12} + 27(k + 27lt^2 + 3mt + 3nt + 27pt^2 + q + 135rt^3)$.

Suppose that $e_1 = 0$, then $t = 0$. In this case we express l, n, q, k and m from the equations $\{U_{ij} = 0, \ \ i + j = 4\}$ of (8). We get the set of conditions (c_2) for the existence of the invariant cubic

$$(a_{12}x - 3y)^3 + 27(x^2 + y^2) = 0.$$

Suppose that $e_2 = 0$ and let $e_1 \neq 0$. In this case we express $q = 0$ from the equation $e_2 = 0$ and l, n, k from the equations $\{U_{04} = 0, U_{13} = 0, U_{22} = 0\}$ of (8).

We calculate the resultant of the equation U_{40} and U_{31} with respect to m. We obtain

$$Res(U_{40}, U_{31}, m) = i_1 i_2 i_3,$$

where $i_1 = (a_{12} - 9t)^2 + 9 \neq 0$, $i_2 = (a_{12} + 18t)^2 + 9 \neq 0$, $i_3 = a_{12}^3 - 9ta_{12}^2 + 3a_{12}(2a - 18f - 4r - 27t^2 - 45) + 9(4p - 6at + 6c + 18ft + 12rt + 81t^3 + 27t)$.

Let $i_3 = 0$. We express p from the equation $i_3 = 0$. Then the equations $F_{40} = 0$ and $F_{31} = 0$ yield $m = [-a_{12}^4 + 18ta_{12}^3 - 6a_{12}^2(a + 9f + 4r + 36) + 54a_{12}(2at + 2c - 6ft - 4rt - 27t^3 - 9t) + 81(9 - 6at^2 - 2a + 90ft^2 + 6f + 48rt^2 + 4r + 81t^4 + 162t^2)]/108$.

In this case we obtain the set of conditions (c_3) for the existence of the invariant cubic

$$(a_{12}x + 18tx - 3y)(a_{12}x - 9tx - 3y)^2 + 27(x^2 + y^2) = 0.$$

$\square$

3 CUBIC SYSTEMS WITH AN INTEGRATING FACTOR

Let the cubic system (5) have an irreducible invariant cubic curve, i.e. at least one of the conditions of Theorem 1 holds. In this section we find the center conditions for cubic system (5) with one invariant cubic curve by constructing an integrating factor of the form

$$\mu = \frac{1}{(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 - y^3 + x^2 + y^2)^h}, \tag{9}$$

where h is a real parameter.

According to [3] the function (9) is an integrating factor for system (1) if and only if the following identity holds

$$P(x,y)\frac{\partial \mu}{\partial x} + Q(x,y)\frac{\partial \mu}{\partial y} + \mu\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = 0. \tag{10}$$

The identity (10) can be used to find integrating factors of the cubic system (5) with invariant algebraic curves (5).

Theorem 2. *The cubic system (5) has an integrating factor of the form (9) if and only if one of the following three conditions holds*

(i) $d = 2a$, $f = (-3)/2$, $k = a(c-2l-2b)$, $m = 2(c-2b-2l)(b+l)$, $n = (c-2b-4l)(b+l)$, $p = [3(2b+2l-c)]/2$, $q = 2a(c-3l-2b)$, $r = 0$, $s = [(3c-6b-8l)(2b-c+g+3l)]/3$, $h = (c-2b)/(2l)$;

(ii) $d = 2(a-4f-3r-6)$, $k = [6(4r+9+6f-a)(c-2b) - h((5c-2g-10b)(4r+9+6f) + 6a(2b-c))]/(6h)$, $l = [(2b-c)(r+f+1) - bh(2f+2r+3)]/h$, $m = [h^2(2b(c-2b) - (r+2f+3)(4r+6f-2a+9)) + (2b-c)(4bh-ch+c-2b)]/h^2$, $n = [2h^2((2r+2f+3)(4r+6f-2a+9)+b(c-2b)) + (2b-c)(6bh-ch+2c-4b)]/(2h^2)$, $p = [h(4bf+12b-3c) - 2(2b-c)(f+3)]/(2h)$, $q = -[(2r+3+2f)gh + (2b-c)(2h-3)a - (r+3+2f)(2b-c)(2h-3)]/h$, $s = [(4bh-6b-2ch+3c+2gh)(c-2b)(3h-4)]/(6h^2)$, $h = [2(4f+3r+6)]/(6f+4r+9)$;

(iii) $d = -[60a+(4b+3c)^2]/45$, $f = (d-2a-15)/10$, $g = [(4b+3c)^3 - 450b + 225c]/2250$, $k = [((4b+3c)^4 + (150a+225)(4b+3c)^2 + 4500b(4b+3c) + 101250a - 101250r)(4b+3c)]/506250$, $q = [(15a+b(4b+3c)-15r)(4b+3c) - 90k]/45$, $l = [(4b+3c)^3 + (150a-450r+225)(4b+3c) - 2250b]/2250$, $p = (-6l - r(4b+3c))/3$, $m = [(4b+3c)^4 + 75(2a-4r+3)(4b+3c)^2 - 9000b(4b+3c) + 101250(r-a)]/33750$, $n = [45m + r(4b+3c)^2 + 15b(4b+3c) + 225(a-r)]/45$, $s = [(4b+3c)^2(15(r-a) - b(4b+3c))]/3375$, $h = 5/3$.

Proof. Let the cubic system (5) have and an invariant cubic $\Phi = 0$ of the form (6). In this case at least one set of the conditions (c_1), (c_2), (c_3) from Theorem 1 holds. The system (5) will have an integrating factor of the form (9) if and only if the identity (10) holds.

1. Let the set of conditions (c_1) hold. Then identifying the coefficients of the monomials $x^i y^j$ in (10), we obtain a system of five equations

$$\{F_{ij} = 0, \quad i + j = 1, 2\} \tag{11}$$

for the unknowns a_{12}, h and the coefficients of system (5).

The equation $F_{01} = 0$ of (11) yields $c = a_{12}h - 2bh + 2b$ and $F_{10} = 0$ becomes $F_{10} \equiv (6f + 4r + 9)h - 2(4f + 3r + 6) = 0$.

If $f = -(4r + 9)/6$, then $r = 0$ and $h = (c - 2b)/(2l)$. In this case we obtain the set of conditions (i) for the existence of the integrating factor (9) with $h = (c - 2b)/(2l)$ and

$$\Phi \equiv 2(2b - c + g + 3l)x^3 + 6ax^2y + 6(b + l)xy^2 - 3y^3 + 3(x^2 + y^2) = 0.$$

If $f \neq -(4r + 9)/6$, then we obtain the set of conditions (ii) for the existence of the integrating factor (9) with $h = [2(4f + 3r + 6)]/(6f + 4r + 9)$ and

$$\Phi \equiv (4bh - 6b - 2ch + 3c + 2gh)x^3 + 3h(2a - 6f - 4r - 9)x^2y+$$
$$+3xy^2(2bh - 2b + c) - 3hy^3 + 3h(x^2 + y^2) = 0.$$

2. Let the set of conditions (c_2) hold. Then identifying the coefficients of the monomials $x^i y^j$ in (10), we obtain a system of five equations

$$\{G_{ij} = 0, \quad i + j = 1, 2\} \tag{12}$$

for the unknowns a_{12}, h and the coefficients of system (5).

The equation $G_{01} = 0$ yields $c = a_{12}h - 2bh + 2b$. We express f and p from the equations $G_{10} = 0$ and $G_{02} = 0$ of (12). We obtain that $G_{11} \equiv u_1 u_2 u_3 = 0$, where $u_1 = 6(3h - 4)a + (3h - 4)a_{12}^2 - 6r$, $u_2 = (6h - 11)a_{12}^2 + 9$, $u_3 = 3h - 5$.

The case $u_1 = 0$ is contained in (i). Assume that $u_1 \neq 0$ and let $u_2 = 0$. Then $h = (11a_{12}^2 - 9)/(6a_{12}^2)$ and $F_{20} = a_{12}^2 + 9 \neq 0$.

Assume that $u_1 u_2 \neq 0$ and let $u_3 = 0$, then $h = 5/3$. In this case we determine the set of conditions (iii) for the existence of the integrating factor (9) with $h = 5/3$ and

$$\Phi \equiv ((4b + 3c)x - 15y)^3 + 3375(x^2 + y^2) = 0.$$

3. Let the set of conditions (c_3) hold. Then identifying the coefficients of the monomials $x^i y^j$ in (10), we obtain a system of five equations

$$\{H_{ij} = 0, \quad i + j = 1, 2\} \tag{13}$$

for the unknowns a_{12}, h and the coefficients of system (5).

The equations $H_{01} = 0, H_{10} = 0$ of (13) yield $c = a_{12}h - 2bh + 2b$, $f = [a_{12}^2(3 - 2h) + 3(6a - 4ah + 54ht^2 - 81t^2 - 3)]/6$, respectively. We obtain that $H_{20} \equiv 2a_{12}(5 - 3h) + 9t(11 - 6h) = 0$ and $H_{02} \equiv 2a_{12}(3h - 5) + 9t = 0$. From the equations $H_{20} = 0$ and $H_{02} = 0$ we get $a_{12} = (-9t)/[2(3h - 5)]$ and $h = 2$. Then $H_{11} \equiv 9(81t^2 + 4) \neq 0$.

In the case (c_3) we cannot construct an integrating factor of the form (9) using the invariant cubic curve of the form (6). Theorem 2 is proved. □

Remark 1. *It is easy to verify that the center conditions obtained in Theorem 2 generalize the center conditions obtained in Lemma 4.3 of [6].*

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Розглянуто двовимірну кубічну диференціальну систему

$$\dot{x} = y + p_2(x, y) + p_3(x, y), \quad \dot{y} = -x + q_2(x, y) + q_3(x, y)$$

із особливою точкою $O(0; 0)$ і з чисто уявними коренями характеристичного рівняння $\lambda_{1,2} = \pm i$, де $p_j(x, y)$ і $q_j(x, y)$ однорідні многочлени степеня j. Для даної системи вивчено проблему розрізнення центра і фокуса за наявності однієї алгебраїчної інваріантної кривої третього порядку. У роботі отримані необхідні і достатні умови існування незвідної інваріантної кривої третього порядку $a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2 = 0$, де $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$.

Доведено, що якщо інваріантна крива має один із таких трьох виглядів $\Phi_1 \equiv 2(2b - c + g + 3l)x^3 + 6ax^2y + 6(b + l)xy^2 - 3y^3 + 3(x^2 + y^2) = 0$, $\Phi_2 \equiv (24g - 4bf - 6b + 2cf + 3c +

$16fg + 12gr)x^3 + 6(2a - 6f - 4r - 9)(4f + 3r + 6)x^2y + 3(4bf + 4br + 6b + 6cf + 4cr + 9c)xy^2 + 6(4f + 3r + 6)(x^2 + y^2 - y^3) = 0$, $\Phi_3 \equiv ((4b + 3c)x - 15y)^3 + 3375(x^2 + y^2) = 0$, то кубічна диференціальна система має інтегруючі множники

$$\mu = \Phi_1^{(2b-c)/(2l)},\ \mu = \Phi_2^{-2(4f+3r+6)/(6f+4r+9)},\ \mu = \Phi_3^{-5/3},$$

які визначені в деякому околі початку координат.

Для кубічної диференціальної системи з інтегруючим множником одержано три нові умови існування центра в особливій точці $O(0;0)$.