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EXISTENCE CONDITIONS AND ASYMPTOTICS FOR SOLUTIONS OF ONE CLASS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

For a differential equation of the second order of the form $y'' = \alpha_0 p(t)\varphi_0(y)|y'|^{\sigma_1}$, where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\longrightarrow]0,+\infty[$ is continuous function, $\varphi_0 : \Delta_{Y_i} \longrightarrow]0,+\infty[$ is continuous regularly varying as $y \to Y_0$ the function of σ_0 order, and $\sigma_0+\sigma_1 = 1$, Δ_{Y_i} $(i \in \{0,1\})$ is a one-side neighborhood of Y_i and $Y_i \in \{0;\pm\infty\}$ $(i \in \{0,1\})$, the question of the existence of solutions for which $\lim\limits_{t\uparrow\omega} y^{(i)}(t) = Y_i$ $(i \in \{0,1\})$ is considered. Involvement in the 1980s in V.Marič, M. Tomič's works in the study of two-term second-order differential equations $y'' = p(t)\varphi(y)$ with regularly varying nonlinearities in zero made it possible to find two-sides estimates of solutions tending to zero as $t \to +\infty$. Further study of two-term second-order differential equations with regularly varying nonlinearities, the right side of which preserves the sign in the neighborhood of singular point (both finite or equals $\pm\infty$) is carried out by Evtukhov V.M. on $P_\omega(\lambda_0)-$solutions, which arises in the study of generalized $n-$th order Emden - Fowler equations. Among the set of such solutions of equation under study we distinguish a fairly wide class of so-called $P_\omega(Y_0,Y_1,\lambda_0)$-solutions (generalization of $P_\omega(\lambda_0)-$solutions). The set of all $P_\omega(Y_0,Y_1,\lambda_0)-$solutions by its asymptotic properties separate into 4 disjoint classes of solutions corresponding to the values of λ_0: $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$ is nonsingular case, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty$ are particular cases. This type of solution was previously introduced in the study of the two-term equation $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, where, $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\longrightarrow]0,+\infty[$ is continuous function, $\varphi_i : \Delta_{Y_i} \longrightarrow]0,+\infty[$ $(i = 0,1)$ are regularly varying as$z \to Y_i$ $(i = 0,1)$ functions of σ_i $(i = 0,1)$ orders, and $\sigma_0+\sigma_1 \neq 1$. The case $\sigma_0+\sigma_1 = 1$ corresponds to the so-called semilinear differential equations, which have a number of properties of both linear and nonlinear differential equations. Thus, for an equation $y'' = p(t)|y|^{1-\lambda}|y'|^{\lambda}\mathrm{sgn}\ y$ with some constraints on a function p (in particular, if the function preserves the sign, it is locally absolutely continuous and $\int\limits_a^\omega p^{\frac{1}{2-\lambda}}(t)\,dt = +\infty$, $\lim\limits_{t\to\omega} p'(t)p^{\frac{\lambda-3}{2-\lambda}}(t) = l_0$ $(|l_0| \le +\infty)$, asymptotic representations are found as $t \to \omega$ for all types of proper solutions of this equation by Evtukhov V.M.. Here, for the equation we are studying, the necessary as well as sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$- solutions are found, asymptotic representations of such solutions and their first-order derivatives are established, and the number of parametric families of such solutions is indicated.

Key words and phrases: two-term equation, $P_\omega(Y_0,Y_1,\lambda_0)$-solutions, regularly varying function, asymptotic representations of solutions, one-, two-parameter family of solutions.

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Introduction

Consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)|y'|^{\sigma_1}, \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$, $\varphi_0 : \Delta_{Y_0} \longrightarrow]0, +\infty[$ is continuous and regular varying as $y \to Y_0$ function of orders σ_0 , Δ_{Y_i} $(i \in \{0, 1\})$ is a one-side neighborhood of Y_i and $Y_i \in \{0; \pm\infty\}$ $(i \in \{0, 1\})$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} 1 & \text{if eigher } Y_i = +\infty \quad \text{or} \\ & Y_i = 0 \quad \text{and} \quad \Delta_{Y_i} \quad \text{is right neighborhood of the point } 0, \\ -1 & \text{if eigher } Y_i = -\infty \quad \text{or} \\ & Y_i = 0 \quad \text{and} \quad \Delta_{Y_i} \quad \text{is left neighborhood of the point } 0, \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \quad \text{for} \quad Y_0 = \pm\infty \quad \text{and} \quad \mu_0\mu_1 < 0 \quad \text{for} \quad Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of equation (1) defined in a left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \qquad \text{for } t \in [t_0, \omega[\ , \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \quad (i = 0, 1). \tag{3}$$

We study equation (1) on class $P_\omega(Y_0, Y_1, \lambda_0)$- solutions, that defined as follows.

Definition 1. *A solution y of equation (1) on interval $[t_0, \omega[\subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, it, in addition to (3), it satisfies the condition*

$$\lim_{t\uparrow\omega} \frac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions have different asymptotic properties. For $\lambda_0 \in \mathbb{R} \setminus \{1\}$ in [2] such ratios

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1}, \tag{4}$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if} \quad \omega = +\infty, \\ t - \omega & \text{if} \quad \omega < +\infty, \end{cases}$$

are established.Let us emphasize that for $\lambda_0 = 0$ the existence of $\lim\limits_{t\uparrow\omega} \frac{y''(t)\pi_\omega(t)}{y'(t)}$ (finite or equal to $\pm\infty$) is assumed.

Note that the numbers μ_0, μ_1 determine the signs of any $P_\omega(Y_0, Y_1, \lambda_0)$- solution of equation (1) and its derivative in a left neighborhood of ω. In addition, the sign of the second derivative of any $P_\omega(Y_0, Y_1, \lambda_0)$-solution of equation (1) in a left neighborhood ω coincides with α_0. Then taking into account (2), we have

$$\alpha_0\mu_1 > 0 \quad \text{as} \quad Y_1 = \pm\infty \quad \text{and} \quad \alpha_0\mu_1 < 0 \quad \text{as} \quad Y_1 = 0. \tag{5}$$

By the definition of a regularly varying function ([1], Chap. 1, Sec. 1.1 9-10 of the Russian translation), each of the functions φ_0 admits a representation of the form

$$\varphi_0(z) = |z|^{\sigma_0} L_0(z),$$

where $L_0 : \Delta_{Y_i} \longrightarrow]0, +\infty[$ is a continuous function slowly varying as $y \to Y_0$ and satisfying the condition

$$\lim_{y \to Y_0} \frac{L(\lambda y)}{L(y)} = 1 \quad \text{for any} \quad \lambda > 0, \tag{6}$$

and the condition is satisfied uniformly for λ on any interval $[c, d] \subset]0, +\infty[$. Moreover, there exist continuously differentiable functions (see [1], Chap. 1, Sec. 1.1 10-15 of the Russian translation]) $L_{00} : \Delta_{Y_i} \longrightarrow]0, +\infty[$ slowly varying as $y \to Y_0$ and satisfying the conditions

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{L_0(y)}{L_{00}(y)} = 1, \qquad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y L'_{00}(y)}{L_{00}(y)} = 0. \tag{7}$$

Asymptotic representations and conditions of the existence of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions in case $\sigma_0 + \sigma_1 \neq 1$ are obtained in [6] for differential equation in general view. In each of the cases $\lambda_0 \in \mathbf{R} \setminus \{0, 1\}$, $\lambda_0 = 0, \lambda_0 = 1, \lambda_0 = \pm\infty$ a condition $(RN)_{\lambda_0}$ is imposed on the right-hand side of the equation under which the equation becomes close in a sense to the two-term as $t \uparrow \omega$.

Here we study the behavior of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions in case $\sigma_0 + \sigma_1 = 1$ and $\lambda_0 \in \mathbb{R} \setminus \{1\}$, when it becomes close in some sense to the linear, which is studied in detail in the monograph [5]. The purpose of this article is to generalize the results from work [3] on equation (1).

We choose a number $b \in \Delta_{Y_0}$ such that the inequality

$$|b| < 1 \quad \text{for} \quad Y_0 = 0, \qquad b > 1 \ (b < -1) \quad \text{for} \quad Y_0 = +\infty \ (Y_0 = -\infty)$$

is respected and put

$$\begin{array}{l} \Delta_{Y_0}(b) = [b, Y_0[\quad \text{if} \quad \Delta_{Y_0} \quad \text{is a left neighborhood of} \quad Y_0, \\ \Delta_{Y_0}(b) =]Y_0, b] \quad \text{if} \quad \Delta_{Y_0} \quad \text{is a right neighborhood of} \quad Y_0. \end{array}$$

Now we introduce auxiliary functions and notation as follows:

$$\Phi : \Delta_{Y_0}(b) \longrightarrow \mathbb{R}, \quad \Phi(y) = \int_B^y \frac{ds}{sL_0(s)}, \quad B = \begin{cases} b & \text{if} \ \int_b^{Y_0} \frac{ds}{sL_0(s)} = \pm\infty, \\ Y_0 & \text{if} \ \int_b^{Y_0} \frac{ds}{sL_0(s)} = const, \end{cases}$$

$$Z = \lim_{y \to Y_0} \Phi(y) = \begin{cases} 0 & \text{if} \quad B = Y_0, \\ +\infty & \text{if} \quad B = b \quad \text{and} \quad \mu_0\mu_1 > 0, \\ -\infty & \text{if} \quad B = b \quad \text{and} \quad \mu_0\mu_1 < 0, \end{cases} \qquad \mu_2 = \begin{cases} 1 & \text{if} \quad B = b, \\ -1 & \text{if} \quad B = Y_0, \end{cases} \tag{8}$$

$$I_0(t) = \int_{A_0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\, d\tau, \quad I_1(t) = \int_{A_1}^{t} p(\tau)|\pi_\omega(\tau)|^{-\sigma_1} d\tau,$$

where the integration limits $A_i \in \{a; \omega\}$ $(i = 0, 1)$ are chosen so as to ensure that the integrals I_i $(i = 0, 1)$ tend either to zero or to $\pm\infty$ as $t \uparrow \omega$.

Note that due to the choice μ_0, μ_1, μ_2

$$\operatorname{sign}\Phi(y) = \mu_0\mu_1\mu_2 \quad \text{as} \quad y \in \Delta_{Y_0}(b) \setminus \{b\}. \tag{9}$$

Since the function Φ is strictly monotonic on the interval $\Delta_{Y_0}(b)$ and the range of its value is the interval

$$\Delta_Z(c) = \begin{cases} [c, Z[& \text{if} \quad \mu_0 > 0, \\ ]Z, c] & \text{if} \quad \mu_0 < 0, \end{cases}$$

where $c = \Phi(b)$, then for it there is a continuously differentiable inverse function Φ^{-1} : $\Delta_Z(c) \to \Delta_{Y_0}(b)$, for which $\lim\limits_{z\to Z} \Phi^{-1} = Y_0$.

It is easy to check that the function $\Phi(y)$ is slowly varying at $y \to Y_0$. Consequently, the inverse to it $\Phi^{-1}(z)$ at $z \to Z$ is a rapidly varying function. The question remains what the function $L\left(\Phi^{-1}(z)\right)$ will be like at $z \to Z$. In some cases (for example, for functions with a finite limit at $y \to Y_0$, or for functions of the form $|ln|y||^{k_1}$, $ln^{k_2}|ln|y||$, $k_1 \in \mathbb{R} \setminus \{1\}$, $k_2 \in \mathbb{R}$, $exp\left(|ln|y||^{k_3}\right)$, $0 < k_3 < 1$) it is regularly varying at $y \to Y_0$.

In addition, by virtue of the choice μ_0, μ_1, μ_2 we have $sign\Phi(y) = \mu_0\mu_1\mu_2$ at $y \in \Delta_{Y_0}(b) \setminus \{b\}$.

1 Section with results

Theorem. *Let $\lambda_0 \in \mathbb{R} \setminus \{1\}$ and let the function $L_0\left(\Phi^{-1}(z)\right)$ is regular varying of γ-th order as $z \to Z$, moreover, let the order σ_0 of the function φ_0 regularly varying as $y \to Y_0$ satisfy the condition $\sigma_0 + \sigma_1 = 1$. Besides for $\lambda_0 = 0$ exists (finite or equal to $\pm\infty$) $\lim\limits_{t\uparrow\omega} \frac{|\pi_\omega(t)|^{\sigma_0} p(t)}{I_1(t)}$. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$- solutions of the differential equation (1), it is necessary and, if the condition*

$$(\sigma_0 + \lambda_0)\left((\sigma_0 + \lambda_0)(1 + \gamma) - \gamma\right) \neq 0 \tag{10}$$

is satisfied, sufficient that, along with inequality (2), (5) the conditions

$$\lim_{t\uparrow\omega} \frac{|\pi_\omega(t)|^{\sigma_0} p(t)}{I_1(t)} = -\beta, \qquad \lim_{t\uparrow\omega} \mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0 - 1|^{\sigma_0} I_0(t) = Z, \tag{11}$$

$$\lim_{t\uparrow\omega} I_1(t)\pi_\omega(t) L_0\left(\Phi^{-1}(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0 - 1|^{\sigma_0} I_0(t))\right) = -\frac{|\lambda_0|^{\sigma_0}}{|\lambda_0 - 1|^{1+\sigma_0}}, \tag{12}$$

and the sign conditions

$$\mu_2 I_0(t) > 0, \quad \alpha_0\mu_1(\lambda_0 - 1)\pi_\omega(t) > 0 \quad \textit{for} \quad t \in]a, \omega[\tag{13}$$

hold. Moreover, each solution of this kind admits the asymptotic representations

$$\Phi(y(t)) = \mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_0(t)[1+o(1)], \tag{14}$$

$$\frac{y'(t)}{y(t)} = -\mu_0\mu_1\beta|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)L_0\left(\Phi^{-1}(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_0(t))\right) \quad as \quad t\uparrow\omega, \tag{15}$$

and if $\beta(\sigma_0+\lambda_0)(\lambda_0-1)<0$ *such solutions form a one-parameter family if* $(\sigma_0+\lambda_0)\left((\sigma_0+\lambda_0)(1+\gamma)-\gamma\right)h_2(t)>0$ *for* $t\in]a,\omega[$ *and two-parameter family if* $(\sigma_0+\lambda_0)\left((\sigma_0+\lambda_0)(1+\gamma)-\gamma\right)h_2(t)<0$ *for* $t\in]a,\omega[$.

Proof. *Necessity.* Let $\lambda_0\in\mathbf{R}\setminus\{0,1\}$ and и $y:[t_0,\omega[\to\Delta_{Y_0}$ be an arbitrary $P_\omega(Y_0,Y_1,\lambda_0)-$ solution of equation(1). Then there is a number $t_1\in[t_0,\omega[$ such that $y^{(k)}(t)\neq 0$ $(k=0,1,2)$, $\operatorname{sign} y^{(k)}(t)=\mu_k$ $(k=0,1)$ at $t\in[t_1,\omega[$. In addition, the definition of the $P_\omega(Y_0,Y_1,\lambda_0)-$ solution for $\lambda_0\in\mathbf{R}\setminus\{0,1\}$ (for $\lambda_0=0$ в in the case of existence $\lim\limits_{t\uparrow\omega}\frac{y''(t)\pi_\omega(t)}{y'(t)}$) immediately implies the fulfillment of limit equalities (4), using which, taking into account $\sigma_0+\sigma_1=1$, from equation (1) we have

$$y''(t)=\alpha_0 p(t)|y(t)|\left|\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\right|^{\sigma_1}L_0(y(t))[1+o(1)] \quad \text{as} \quad t\uparrow\omega.$$

From the last equality we have

$$\frac{y''(t)}{y(t)L_0(y(t))}=\alpha_0\mu_0\left|\frac{\lambda_0}{(\lambda_0-1)}\right|^{\sigma_1}p(t)|\pi_\omega(t)|^{-\sigma_1}[1+o(1)] \quad \text{as} \quad t\uparrow\omega, \tag{16}$$

whence, taking into account the second of relations (4), we obtain the equality

$$\frac{y'(t)}{y(t)L_0(y(t))}=\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}p(t)|\pi_\omega(t)|^{\sigma_0}[1+o(1)] \quad \text{as} \quad t\uparrow\omega. \tag{17}$$

Integrating the last relation on a segment $[A_0,t]$, we obtain (14). In addition, by virtue of (8), (9) from (14) implies the first of the sign conditions (13) and the second of the limit equalities (11).

For $\lambda_0\in\mathbf{R}\setminus\{1\}$ it is also obvious in view of (4) that the second of the sign conditions (13) is satisfied.

Given the equality

$$\left(\frac{y'(t)}{y(t)L_{00}(y(t))}\right)'=\frac{y''(t)}{y(t)L_{00}(y(t))}\left(1-\frac{(y'(t))^2}{y''(t)y(t)}-\frac{(y'(t))^2}{y''(t)y(t)}\frac{L'_{00}(y(t))y(t)}{L_{00}(y(t))}\right),$$

by virtue of the definition of a slowly varying function and the definition of a , $P_\omega(Y_0,Y_1,\lambda_0)-$ solution we have

$$\left(\frac{y'(t)}{y(t)L_{00}(y(t))}\right)'=\frac{y''(t)}{y(t)L_{00}(y(t))}\left(1-\lambda_0\right)(1+o(1)) \quad \text{as} \quad t\uparrow\omega,$$

from which in view of (16), (7) it follows

$$\frac{y'(t)}{y(t)L_0(y(t))}=-\mu_0\mu_1\beta|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)[1+o(1)] \quad \text{as} \quad t\uparrow\omega. \tag{18}$$

Comparing relations (17) и (18), we obtain the first of conditions (11). Also from (18) by virtue of (1) we have the condition

$$\frac{y''(t)\pi_\omega(t)}{y'(t)} = -\frac{\beta p(t)|\pi_\omega(t)|^{\sigma_0}}{(\lambda_0-1)I_1(t)}[1+o(1)] \quad \text{as} \quad t\uparrow\omega,$$

which in the case $\lambda_0 = 0$, due to the existence of the limit $\frac{p(t)|\pi_\omega(t)|^{\sigma_0}}{(\lambda_0-1)I_1(t)}$, guarantees the fulfillment of the asymptotic representations (4) for all $\lambda_0 \in \mathbb{R}\setminus\{1\}$. Further, we note that condition (14) since the function $L_0(\Phi(z))$ as $z \to Z$ is of a regularly varying order γ, implies that

$$L_0(y(t)) = L_0\left(\Phi^{-1}(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_0(t))\right)[1+o(1)] \quad \text{as} \quad t\uparrow\omega.$$

Due to the last equality, taking into account (18) we obtain (15), and also, multiplying both sides of (18) by $\pi_\omega(t)$, taking into account(4), we get condition (12).

Sufficiency. Suppose, along with (2), (5), (11) - (13), condition (10) is satisfied. Let us show that in this case the differential equation (1) has $P_\omega(Y_0, Y_1, \lambda_0)$- solutions admitting representations (14), (15) and clarify the question of the number of such solutions.

Applying to the differential equation (1) the transformation

$$\frac{y'(t)}{y(t)} = -\beta C I_1(t)L_0(\Phi^{-1}(CI_0(t))))[1+v_1(\tau)], \quad \Phi(y(t)) = CI_0(t)[1+v_2(\tau)], \tag{19}$$

$$\tau = \beta\ln|\pi_\omega(t)|, \quad C = \mu_0\mu_1\beta|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0},$$

we obtain the system of differential equations

$$\begin{cases} v_1' = & \beta h_1(\tau)\left(\dfrac{-\alpha_0\mu_0}{C}|g_1(\tau)|^{\sigma_1}H(\tau,v_2)|1+v_1|^{\sigma_1} + \right. \\ & \left. +\dfrac{\beta g_1(\tau)}{h_1(\tau)}(1+v_1)^2 - (1+v_1)(1+g_1(\tau)g_2(\tau))\right), \\ & \\ v_2' = & h_2(\tau)\left(-\dfrac{(1+v_1)}{H(\tau,v_2)} - h_1(\tau)(1+v_2)\right), \end{cases} \tag{20}$$

where

$$h_1(\tau(t)) = \frac{I_1'(t)\pi_\omega(t)}{I_1(t)}, \quad H(\tau(t),v_2) = \frac{L_0\left(\Phi^{-1}(CI_0(t)(1+v_2))\right)}{L_{00}\left(\Phi^{-1}(CI_0(t)\right)},$$

$$g_1(\tau(t)) = C\pi_\omega(t)I_1(t)L_{00}\left(\Phi^{-1}(CI_0(t))\right),$$

$$h_2(\tau(t)) = \frac{I_1(t)\pi_\omega(t)}{I_0(t)}, \quad g_2(\tau(t)) = \frac{\Phi^{-1}(CI_0(t))L_{00}'\left(\Phi^{-1}(CI_0(t)\right)}{L_{00}\left(\Phi^{-1}(CI_0(t)\right)}$$

Since the function $\tau(t) = \beta\ln|\pi_\omega(t)|$ is such that

$$\tau : [a_0,\omega[\longrightarrow [\tau_0,+\infty[\quad (\tau_0 = \beta\ln|\pi_\omega(a)|), \quad \tau'(t) > 0 \quad \text{as} \quad t\in[a_0,\omega[, \quad \lim_{t\uparrow\omega}\tau(t) = +\infty,$$

then by virtue of the first of conditions (11)

$$\begin{gathered} \lim_{\tau\to+\infty} h_1(\tau) = \lim_{t\to\omega} h(\tau(t)) = -1, \quad \lim_{\tau\to+\infty} h_2(\tau) = \lim_{t\to\omega} h(\tau(t)) = 0, \\ \int_{\tau_1}^{+\infty} |h_2(\tau)|\,d\tau = +\infty, \quad \lim_{\tau\to+\infty}\frac{h_2'(\tau)}{h_2(\tau)} = 0, \end{gathered} \tag{21}$$

where τ is any number from the interval $]\tau_0, +\infty[$.

In view of the second of conditions (11), (13) and (8), (9) there exists a number $t_1 \in]a, \omega[$ such, that $\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0 - 1|^{\sigma_0} I_0(t)(1 + v_2) \in \Delta_{Z(c)}$ for $t \in [t_1, \omega[$ и $|v_2| \leq \frac{1}{2}$. Consider system (19) on the set $[\tau_1, +\infty[\times\mathbb{R}^2_{\frac{1}{2}}$, where $\tau_1 = \beta \ln |\pi_\omega(t_1)|$, $\mathbb{R}^2_{\frac{1}{2}} = \{(v_1, v_2) \in \mathbb{R}^2 : |v_i| \leq 1/2, i = 1, 2\}$, on which the right-hand sides of the system are defined and continuous.

Since the function $L_0\left(\Phi^{-1}(z)\right)$ is regularly varying as $z \to Z$ of the order γ, it admits the representation $L_0\left(\Phi^{-1}(z)\right) = |z|^\gamma L(z)$, where L is the slowly varying function as $z \to Z$. Therefore, according to (6)

$$L_0\left(\Phi^{-1}(CI_0(t)(1+v_2)\right) = |CI_0(t)(1+v_2)|^\gamma L(CI_0(t)(1+v_2)) =$$

$$= |CI_0(t)|^\gamma |(1+v_2)|^\gamma L(CI_0(t))[1 + R(t, v_2)] = L_0\left(\Phi^{-1}(CI_0(t)\right)|(1+v_2)|^\gamma[1 + R(t, v_2)],$$

where

$$\lim_{t\uparrow\omega} R(t, v_2) = 0 \quad \text{uniformly over} \quad |v_2| \leq \frac{1}{2}.$$

Therefore, taking into account (7) we have

$$H(\tau, v_2) = |(1+v_2)|^\gamma[1 + r_1(t, v_2)], \qquad \frac{1}{H(\tau, v_2)} = |(1+v_2)|^\gamma[1 + r_2(t, v_2)],$$

where functions $r_i(t, v_2)$ $(i = 1, 2)$ are continuous on the set $[\tau_1, +\infty[\times\mathbb{R}^2_{\frac{1}{2}}$ and such that

$$\lim_{\tau\to+\infty} r_i(\tau, v_2) = 0 \quad (i = 1, 2) \quad \text{uniformly over} \quad |v_2| \leq \frac{1}{2}.$$

Obviously, that $\lim\limits_{t\uparrow\omega} \Phi^{-1}(CI_0(t)) = Y_0$, therefore, by virtue of (7), (11), (12)

$$\lim_{\tau\to+\infty} g_1(\tau) = -\frac{\beta\lambda_0}{\lambda_0 - 1} \qquad \lim_{\tau\to+\infty} g_2(\tau) = 0.$$

Now we rewrite system (20) in the form

$$\begin{cases} v_1' = & \beta\left(f_1(\tau, v_1, v_2) + \frac{\sigma_0+\lambda_0}{\lambda_0-1}v_1 - \frac{\gamma}{\lambda_0-1}v_2 + V_1(v_1, v_2)\right), \\ \\ v_2' = & h_2(\tau)\left(f_2(\tau, v_1, v_2) - v_1 + (1+\gamma)v_2 + V_2(v_1, v_2)\right), \end{cases} \tag{22}$$

where

$$\lim_{\tau\to+\infty} f_i(\tau, v_1, v_2) = 0 \quad (i = 1, 2) \quad \text{uniformly over} \quad (v_1, v_2)\mathbb{R}^2_{\frac{1}{2}},$$

$$\lim_{|v_1|+|v_2|\to 0} \frac{\partial V_i(v_1, v_2)}{\partial v_j} = 0 \quad (i, j = 1, 2),$$

whence it follows that $\lim\limits_{|v_1|+|v_2|\to 0} \frac{V_i(v_1,v_2)}{|v_1|+|v_2|} = 0 \quad (i = 1, 2)$. In addition, conditions (21) are satisfied. Thus, for system (20) the conditions of Theorem 2.6 from [4] are satisfied. Therefore, this system has at least one solution $(v_1, v_2) : [\tau_1, +\infty[\longrightarrow \mathbb{R}^2_{\frac{1}{2}}$ $(\tau_2 \geq \tau_1)$, tending to zero as $\tau \to +\infty$. Due to transformation (20) each such solution corresponds to a solution y of differential equation (1), admitting asymptotic representations (14), (15). It is easy to check that the indicated solution is a $P_\omega(Y_0, Y_1, \lambda_0)$- solution of equation (1).

Also, based on Theorem 2.6 in [4], it is easy to find the number of families of solutions to system (20). By virtue of (10), for $\lambda_0 \in \mathbb{R} \setminus \{1\}$, the determinant

$$\begin{vmatrix} \frac{\sigma_0+\lambda_0}{\lambda_0-1} & \frac{-\gamma}{\lambda_0-1} \\ -1 & \gamma+1 \end{vmatrix}$$

is nonzero. Therefore, for $\beta\frac{\sigma_0+\lambda_0}{\lambda_0-1} < 0$ and $h_2(\tau)\frac{\sigma_0+\lambda_0}{\lambda_0-1}\begin{vmatrix} \frac{\sigma_0+\lambda_0}{\lambda_0-1} & \frac{-\gamma}{\lambda_0-1} \\ -1 & \gamma+1 \end{vmatrix} < 0$ system (20) has a two-parameter family of solutions tending to zero as $\tau \to +\infty$. System (20) has a one-parameter family of solutions vanishing at infinity either for $\beta\frac{\sigma_0+\lambda_0}{\lambda_0-1} < 0$ and $h_2(\tau)\frac{\sigma_0+\lambda_0}{\lambda_0-1}\begin{vmatrix} \frac{\sigma_0+\lambda_0}{\lambda_0-1} & \frac{-\gamma}{\lambda_0-1} \\ -1 & \gamma+1 \end{vmatrix} > 0$ or $\beta\frac{\sigma_0+\lambda_0}{\lambda_0-1} > 0$ and $h_2(\tau)\frac{\sigma_0+\lambda_0}{\lambda_0-1}\begin{vmatrix} \frac{\sigma_0+\lambda_0}{\lambda_0-1} & \frac{-\gamma}{\lambda_0-1} \\ -1 & \gamma+1 \end{vmatrix} < 0$.
The theorem is completely proved. □

In what follows, equation (1) should be studied at $\sigma_0 + \sigma_1 = 1$ for values $\lambda_0 = 1$, $\lambda_0 = \pm\infty$. It is also possible to extend the results of this work to an equation of the form $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$ where, $p : [a, \omega[\longrightarrow]0, +\infty[$ is a continuous function $\varphi_i : \Delta_{Y_i} \longrightarrow]0, +\infty[$ $(i = 0, 1)$ and are a continuous regularly varying as $z \to Y_i$ $(i = 0, 1)$ functions of σ_i $(i = 0, 1)$ orders.

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Для диференціального рівняння другого порядку виду $y'' = \alpha_0 p(t)\varphi_0(y)|y'|^{\sigma_1}$, де $\alpha_0 \in \{-1,1\}$,$p : [a,\omega[\longrightarrow]0,+\infty[$-неперервна функція, $\varphi_0 : \Delta_{Y_i} \longrightarrow]0,+\infty[$ -неперервна правильно змінна при $y \to Y_0$ функція порядку σ_0, причому $\sigma_0 + \sigma_1 = 1$, Δ_{Y_i}- односторонній окіл Y_i, $Y_i \in \{0, \pm\infty\}$ $(i \in \{0,1\})$ розглянуто питання існування розв'язків, для яких $\lim\limits_{t\uparrow\omega} y^{(i)}(t) = Y_i$ $(i \in \{0,1\})$.

Залучення у 80-х рр. XX ст. в працях V.Marič, M. Tomič при вивченні двочленних диференціальних рівнянь другого порядку з правильно змінними в нулі нелінійностями $y'' = p(t)\varphi(y)$ дало змогу вказати двубічні оцінки розв'язків, що прямують до нуля при $t \to +\infty$. Подальше вивчання двочленних диференціальних рівнянь другого порядку з правильно змінними нелінійностями, права частина яких зберігає в околі особливій точки (як скінченній, так и рівній $\pm\infty$) знак, проведено на виділеному В.М.Євтуховим класі $P_\omega(\lambda_0)-$ розв'язків, що виникає при дослідженні узагальнених рівняннях Емдена - Фаулера $n-$го порядку. Серед множини розв'язків вивчаемого рівняяня відокремлюємо достатньо широкий клас т. з. $P_\omega(Y_0, Y_1, \lambda_0)$- розв'язків (узагальнення $P_\omega(\lambda_0)-$ розв'язків). Множина усіх $P_\omega(Y_0, Y_1, \lambda_0)-$ розв'язків за своїми асимптотичними властивостями розпадається на 4 непертинаючихся класів розв'язків, що відповідають наступним значенням λ_0: $\lambda_0 \in \mathbb{R} \setminus \{0,1\}-$ неособливий випадок, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty-$ особливі випадки. Такого типу розв'язки раніше було уведено при вивченні двочленного рівняння $y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')$, де $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\longrightarrow]0,+\infty[$–неперервна функція, $\varphi_i : \Delta_{Y_i} \longrightarrow]0,+\infty[$ $(i = 0,1)$ –неперервні правильно змінні при $z \to Y_i$ $(i = 0,1)$ функції порядків σ_i $(i = 0,1)$, причому $\sigma_0 + \sigma_1 \neq 1$. Випадок $\sigma_0 + \sigma_1 = 1$ відповідає т.з. полулінійним диференціальним рівнянням, яким притаманні властивості як лінійних, так и нелінійних диференціальних рівнянь. Так, для рівняння $y'' = p(t)|y|^{1-\lambda}|y'|^{\lambda}\mathrm{sgn}\, y$ при деяких обмеженнях на функцію p (зокрема, якщо функція $p : [a,\omega[\longrightarrow]0,+\infty[$ зберігає знак, локально абсолютно неперервна і $\int\limits_a^\omega p^{\frac{1}{2-\lambda}}(t)\, dt = +\infty$, $\lim\limits_{t\to\omega} p'(t)p^{\frac{\lambda-3}{2-\lambda}}(t) = l_0$ $(|l_0| \leq +\infty)$, В.М.Євтуховим знайдено асимптотичні зображення при $t \to \omega$ усіх типів правильних розв'язків цього рівняння. Тут для рівняння, що вивчаємо, знайдено необхідні, а також достатні умови існування $P_\omega(Y_0, Y_1, \lambda_0)$- розв'язків, встановлено асимптотичні зображення таких розв'язків та їх похідних першого порядку, вказано кількість параметричних сімей таких розв'язків.