Baksa V.P., Bandura A.I., Skaskiv O.B.

ON EXISTENCE OF MAIN POLYNOMIAL FOR ANALYTIC VECTOR-VALUED FUNCTIONS OF BOUNDED L-INDEX IN THE UNIT BALL

In this paper, we present necessary and sufficient conditions of boundedness of $\mathbb{L}$-index in joint variables for vector-functions analytic in the unit ball, where $\mathbf{L} = (l_1, l_2) : \mathbb{B}^2 \to \mathbb{R}^2_+$ is a positive continuous vector-function, $\mathbb{B}^2 = \{z \in \mathbb{C}^2 : |z| = \sqrt{|z_1|^2 + |z_2|^2} \le 1\}$. These conditions describe local behavior of homogeneous polynomials (so-called a main polynomial) with power series expansion for analytic vector-valued functions in the unit ball. These results use a bidisc exhaustion of a unit ball.

Key words and phrases: bounded index, bounded **L**-index in joint variables, analytic function, unit ball, main polynomial, homogeneous polynomial.

Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, Lviv, Ukraine (Baksa V.P., Skaskiv O.B.)
Department of Advanced Mathematics,
Ivano-Frankivsk National Technical University of Oil and Gas, Ivano-Frankivsk, Ukraine (Bandura A. I.)
e-mail: *vitalinabaksa@gmail.com (Baksa V.P.), andriykopanytsia@gmail.com (Bandura A.I.), olskask@gmail.com (Skaskiv O.B.)*

1 Introduction

We need some standard notations (for example see [5, 4, 6]). Let $\mathbb{R}_+ = [0; +\infty)$, $\mathbf{0} = (0, 0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1, 1) \in \mathbb{R}^2_+$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $|(z, \omega)| = \sqrt{|z|^2 + |\omega|^2}$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$, we will use formal notations without violation of the existence of these expressions: $AB = (a_1b_1, a_2b_2)$, $A/B = (a_1/b_1, a_2/b_2)$, $A^B = (a_1^{b_1}, a_2^{b_2})$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, 2\}$; the relation $A \le B$ is defined in the similar way. For $K = (k_1, k_2) \in \mathbb{Z}^2_+$ let us denote $K! = k_1! \cdot k_2!$. Addition, multiplication by scalar and conjugation in $\mathbb{C}^2$ is defined componentwise. For $z \in \mathbb{C}^2$, $w \in \mathbb{C}^2$ we define $\langle z, w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2$, where $\overline{w}_1, \overline{w}_2$ is the complex conjugate of w_1, w_2.

The bidisc $\{(z, \omega) \in \mathbb{C}^2 : |z - z_0| < r_1, |\omega - \omega_0| < r_2\}$ is denoted by $\mathbb{D}^2((z_0, \omega_0), R)$, its skeleton $\{(z, \omega) \in \mathbb{C}^2 : |z - z_0| = r_1, |\omega - \omega_0| = r_2\}$ is denoted by $\mathbb{T}^2((z_0, \omega_0), R)$, the closed polydisc $\{(z, \omega) \in \mathbb{C}^2 : |z - z_0| \le r_1, |\omega - \omega_0| \le r_2\}$ is denoted by $\mathbb{D}^2[(z_0, \omega_0), R]$, $\mathbb{D}^2 =$

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$\mathbb{D}^2(\mathbf{0};\mathbf{1})$, $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. The open ball $\{(z,\omega)\in\mathbb{C}^2:\sqrt{|z-z_0|^2+|\omega-\omega_0|^2}<r\}$ is denoted by $\mathbb{B}^2((z_0,\omega_0),r)$, the sphere $\{(z,\omega)\in\mathbb{C}^2:\sqrt{|z-z_0|^2+|\omega-\omega_0|^2}=r\}$ is denoted by $\mathbb{S}^2((z_0,\omega_0),r)$, and the closed ball $\{z\in\mathbb{C}^2:\sqrt{|z-z_0|^2+|\omega_0-\omega_0|^2}\le r\}$ is denoted by $\mathbb{B}^2[(z_0,\omega_0),r]$, $\mathbb{B}^2=\mathbb{B}^2(\mathbf{0},\mathbf{1})$, $\mathbb{D}=\mathbb{B}^1=\{z\in\mathbb{C}:|z|<1\}$.

Let $F(z,\omega)=(f_1(z,\omega),f_2(z,\omega))$ be an analytic vector-function in $\mathbb{B}^2$. Then at a point $(a,b)\in\mathbb{B}^2$ the function $F(z,\omega)$ has a bivariate Taylor expansion:

$$F(z,\omega)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}C_{kl}(z-a)^k(\omega-b)^m,$$

where $C_{km}=\frac{1}{k!m!}\left(\frac{\partial^{k+m}f_1(z,\omega)}{\partial z^k\partial\omega^m},\frac{\partial^{k+m}f_2(z,\omega)}{\partial z^k\partial\omega^m}\right)\Big|_{z=a,\omega=b}=\frac{1}{k!m!}F^{(k,m)}(a,b)$.

Let $\mathbf{L}(z,\omega)=(l_1(z,\omega),l_2(z,\omega))$, where $l_j(z,\omega):\mathbb{B}^2\to\mathbb{R}^2_+$ is a positive continuous function such that

$$\forall(z,\omega)\in\mathbb{B}^2:\quad l_j(z,\omega)>\frac{\beta}{1-\sqrt{|z|^2+|\omega|^2}},\tag{1}$$

$j\in\{1,2\}$, where $\beta>\sqrt{2}$ is a some constant.

The norm for the vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ is defined as the sup-norm:

$$\|F(z,\omega)\|=\max_{1\le j\le 2}\{|f_j(z,\omega)|\}.$$

We write

$$F^{(i,j)}(z,\omega)=\frac{\partial^{i+j}F(z,\omega)}{\partial z^i\partial\omega^j}=\left(\frac{\partial^{i+j}f_1(z,\omega)}{\partial z^i\partial\omega^j},\frac{\partial^{i+j}f_2(z,\omega)}{\partial z^i\partial\omega^j}\right).$$

An analytic vector-function $F:\mathbb{B}^2\to\mathbb{C}^2$ is said to be of bounded $\mathbf{L}$-index (in joint variables) [1, 2, 3], if there exists $n_0\in\mathbb{Z}_+$ such that

$$\forall(z,\omega)\in\mathbb{B}^2\ \ \forall(i,j)\in\mathbb{Z}^2_+:$$

$$\frac{\|F^{(i,j)}(z,\omega)\|}{i!j!l_1^i(z,\omega)l_2^j(z,\omega)}\le\max\left\{\frac{\|F^{(k,m)}(z,\omega)\|}{k!m!l_1^k(z,\omega)l_2^m(z,\omega)}:k,m\in\mathbb{Z}_+,k+m\le n_0\right\}.\tag{2}$$

The least such integer n_0 is called the $\mathbf{L}$-index in joint variables of the vector-function F and is denoted by $N(F,\mathbf{L},\mathbb{B}^2)$. The concept of boundedness of $\mathbf{L}$-index in joint variables were considered for other classes of analytic functions. They are differed domains of analyticity: the unit ball [5, 4, 12, 13], the polydisc [7, 11], the Cartesian product of the unit disc and complex plane [8], n-dimensional complex space [12, 14]. Vector-valued functions of one and several complex variables having bounded index were considered in [16, 18, 15, 20, 19, 17].

The function class $Q(\mathbb{B}^2)$ is defined as following: $\forall R\in\mathbb{R}^2_+,|R|\le\beta,j\in\{1,2\}:$

$$0<\lambda_{1,j}(R)\le\lambda_{2,j}(R)<\infty$$

where

$$\lambda_{1,j}(R)=\inf_{(z_0,\omega_0)\in\mathbb{B}^2}\inf\left\{\frac{l_j(z,\omega)}{l_j(z_0,\omega_0)}:(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\},\tag{3}$$

$$\lambda_{2,j}(R)=\sup_{(z_0,\omega_0)\in\mathbb{B}^2}\sup\left\{\frac{l_j(z,\omega)}{l_j(z_0,\omega_0)}:(z,\omega)\in\mathbb{D}^2[(z_0,\omega_0),R/\mathbf{L}(z_0,\omega_0)]\right\}.\tag{4}$$

We need the following theorem.

Theorem 1 ([3]). *Let* $\mathbf{L} \in Q(\mathbb{B}^2)$. *An analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has bounded* $\mathbf{L}$*-index in joint variables if and only if there exist* $p \in \mathbb{Z}_+$ *and* $c \in \mathbb{R}_+$ *such that for each* $(z, \omega) \in \mathbb{B}^2$ *inequality holds*

$$\max\left\{ \frac{\|F^{(i,j)}(z,\omega)\|}{l_1^i(z,\omega) l_2^j(z,\omega)} : i+j=p+1 \right\} \le c \max\left\{ \frac{\|F^{(k,m)}(z,\omega)\|}{l_1^k(z,\omega) l_2^m(z,\omega)} : k+m \le p \right\}. \tag{5}$$

2 Properties of a power series expansion of analytic vector-functions in the unit ball.

Let $(z_0, w_0) \in \mathbb{B}^2$. We expand a vector-function $F : \mathbb{B}^2 \longrightarrow \mathbb{C}^2$ in vector-valued power series

$$F(z,w) = \sum_{k=0}^{\infty} P_k(z-z_0, w-w_0) = \sum_{k=0}^{\infty} \sum_{i+j=k} B_{ij}(z-z_0)^i (w-w_0)^j, \tag{6}$$

where P_k is a homogeneous polynomial of degree k,

$$B_{ij} = \frac{F^{(i,j)}(z_0,w_0)}{i!j!} = \left(\frac{f_1^{(i,j)}(z_0,w_0)}{i!j!}, \frac{f_2^{(i,j)}(z_0,w_0)}{i!j!} \right).$$

The polynomial P_{k_0}, $k_0 \in \mathbb{Z}_+$, is called *a main polynomial* in series (6) on $\mathbb{T}^2((z_0, w_0), R)$, if for every $(z, w) \in \mathbb{T}^2((z_0, w_0), R)$ inequality holds

$$\|\sum_{k \ne k_0} P_k(z-z_0, w-w_0)\| \le \frac{1}{2} \max\left\{ \|B_{i,j}\| r_1^i r_2^j : i+j = k_0 \right\}.$$

The following Theorem 2 and 3 have proofs which are similar to proofs of corresponding theorems in [9, 4, 10].

Theorem 2. *Let* $\mathbf{L} \in Q(\mathbb{B}^2)$. *If an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has bounded L-index in joint variables then there exists* $p \in \mathbb{Z}_+$ *such that for each* $d \in \left(0; \frac{\beta}{\sqrt{2}}\right]$ *there exists* $\eta(d) \in (0; d)$ *such that for each* $(z_0, w_0) \in \mathbb{B}^2$ *and for some* $r = r(d, (z_0, w_0)) \in (\eta(d), d)$ *and some* $\nu_0 = \nu_0(d, (z_0, w_0)) \le p$ *the polynomial* p_{ν_0} *is main in series* (6) *on* $\mathbb{T}^2\left((z_0, w_0), \frac{r\mathbf{1}}{\mathbf{L}(z_0,w_0)}\right)$.

Proof. Let F be an analytic vector-function of bounded $\mathbf{L}$–index in joint variables with $N = N(F, \mathbf{L}, \mathbb{B}^2) < +\infty$ and n_0 be the $\mathbf{L}$-index in joint variables at the point $(z_0, w_0) \in \mathbb{B}^2$, that is n_0 the least such that inequality (2) holds in (z_0, w_0). Then for every $(z_0, w_0) \in \mathbb{B}^2$ one has $n_0 \le N$.

Define

$$a_{i,j}^* = \frac{\|B_{i,j}\|}{\mathbf{L}^{i,j}(z_0,w_0)} = \frac{\|F^{i,j}(z_0,w_0)\|}{i!j!\mathbf{L}^{i,j}(z_0,w_0)},$$
$$a_\nu = \max\{a_{i,j}^* : i+j=\nu\}, c = 2\{(N+3)!3! + (N+1)C_{N+1}^N\} = 2\{(N+3)!3! + (N+1)^2\}.$$

Let $d \in \left(0; \frac{\beta}{\sqrt{2}}\right]$ be an arbitrary number. Put $r_t = \frac{d}{(d+1)c^t}$, $\mu_t = \max\{a_\nu r_t^\nu : \nu \in \mathbb{Z}_+\}$, $s_t = \min\{\nu : a_\nu r_t^\nu = \mu_t\}$ for $t \in \mathbb{Z}_+$.

Since $(z_0, w_0) \in \mathbb{B}^2$ is a fixed point, for every $(k,m) \in \mathbb{Z}_+^2$ inequality $a^*_{k,m} \leq \max\{a^*_{i,j} : i+j \leq n_0\}$ holds. Then $a_\nu \leq a_{n_0}$ for every $\nu \in \mathbb{Z}_+$. Then for every $\nu > n_0$ with $r_0 < 1$ we have $a_\nu r_0^\nu < a_{n_0} r_0^{n_0}$. Thus, $s_0 \leq n_0$. Since $cr_t = r_{t-1}$, we obtain $\nu > s_{t-1}$ $(r_{t-1} < 1)$.

$$a_{s_{t-1}} r_t^{s_{t-1}} = a_{s_{t-1}} r_{t-1}^{s_{t-1}} c^{-s_{t-1}} = a_\nu r_t^\nu c^{\nu - s_{t-1}} \geq c a_\nu r_t^\nu. \tag{7}$$

Hence, $s_t \leq s_{t-1}$ for every $t \in \mathbb{N}$. Thus,

$$\mu_0 = \max\{a_\nu r_0^\nu : \nu \leq n_0\},$$
$$\mu_t = \max\{a_\nu r_t^\nu : \nu \leq s_{t-1}\}, t \in \mathbb{N}.$$

Let us introduce additional notations for $t \in \mathbb{N}$

$$\mu_0^* = \max\{a_\nu r_0^\nu : s_0 \neq \nu \leq n_0\}, s_0^* = \min\{k : k \neq s_0, a_\nu r_0^\nu = \mu_0^*\},$$

$$\mu_t^* = \max\{a_\nu r_t^\nu : s_t \neq \nu \leq s_{t-1}\}, s_t^* = \min\{k : k \neq s_t, a_\nu r_t^\nu = \mu_t^*\}.$$

Now we will prove that there exists $t_0 \in \mathbb{Z}_+$, for which

$$\frac{\mu_{t_0}^*}{\mu_{t_0}} \leq \frac{1}{c}. \tag{8}$$

On the contrary, suppose that for each $t \in \mathbb{Z}_+$ the next inequality holds

$$\frac{\mu_{t_0}^*}{\mu_{t_0}} > \frac{1}{c}. \tag{9}$$

For $s_t^* < s_t$ we have

$$a_{s_t^*} r_{t+1}^{s_t^*} = \frac{a_{s_t^*} r_t^{s_t^*}}{c^{s_t^*}} = \frac{\mu_t^*}{c^{s_t^*}} > \frac{\mu_t}{c^{s_t^*+1}} = \frac{a_{s_t} r_t^{s_t}}{c^{s_t^*+1}} = \frac{a_{s_t} r_{t+1}^{s_t}}{c^{s_t^*+1} - s_t} \geq a_{s_t} r_{t+1}^{s_t}.$$

For each $\nu > s_t^*$, $\nu \neq s_t$ (that is $\nu - 1 \geq s_t^*$) we educe

$$a_{s_t^*} r_{t+1}^{s_t^*} = \frac{a_{s_t^*} r_t^{s_t^*}}{c^{s_t^*}} \geq \frac{a_\nu r_t^\nu}{c^{s_t^*}} \geq \frac{a_\nu r_t^\nu}{c^{\nu-1}} = c a_\nu r_{t+1}^\nu.$$

Thus, $a_{s_t^*} r_{t+1}^{s_t^*} > a_\nu r_{t+1}^\nu$ for all $\nu > s_t^*$. Then

$$s_{t+1} \leq s_t^* \leq s_t - 1. \tag{10}$$

If $s_t < s_t^* \leq s_{t-1}$ then the equality $s_{t+1} = s_t$ can be valid. Indeed, $s_{t+1} \leq s_t$. And with $s_{t+1} < s_t$ we have $s_{t+1} < s_{t-1}$. It implies (10) .

Therefore, from inequalities $s_{t+1}^* \leq s_t$ and $s_t^* \neq s_{t+1}$ we have $s_{t+1}^* < s_{t+1}$. Hence, instead (10) we have

$$s_{t+2} \leq s_{t+1}^* \leq s_{t+1} - 1 = s_t - 1.$$

Then, if for all $t \in \mathbb{Z}_+$ is true (9) , then for each $t \in \mathbb{Z}_+$ one of two is executed: or $s_{t+2} \leq s_{t+1} \leq s_t - 1$, or $s_{t+2} \leq s_t - 1$, $s_{t+2} \leq s_t - 1$, since $s_{t+2} \leq s_{t+1}$. Then we have

$$s_t \leq s_{t-2} - 1 \leq ... \leq s_{t-2[/2]} - [t/2] \leq s_0 - [t/2] \leq N - [t/2].$$

In other words, $s_t < 0$ with $t > 2N + 1$. It is a contradiction. Therefore, there exists $t_0 \leq 2N + 1$, for which (8) is true.

Put $r = r_{t_0}$, $\eta(d) = \frac{d}{(d+1)c^{2(N+1)}}$, $p = N$ and $\nu_0 = s_{t_0}$. Then for all $(i + j) \neq \nu_0 = s_{t_0}$ on $\mathbb{T}^2\left((z_0, w_0), \frac{r\mathbf{e}}{\mathbf{L}(z_0,w_0)}\right)$ in view of (7) and (8) we obtain that

$$\|B_{ij}\||(z - z_0)^i (w - w_0)^j| = a^*_{i,j} r^{i+j} \leq a_{i+j} r^{i+j} \leq \frac{1}{c} a_{s_{t_0}} r_{t_0}^{s_{t_0}} = \frac{1}{c} a_{\nu_0} r^{\nu_0}.$$

Thus, for $(z, w) \in \mathbb{T}^2\left((z_0, w_0), \frac{r\not\in}{\mathbf{L}(z_0,w_0)}\right)$

$$\left\|\sum_{i+j\neq\nu_0} B_{i,j}(z - z_0)^i (w - w_0)^j\right\| \leq \sum_{i+j\neq\nu_0} a^*_{i,j} r^{i+j} \leq \sum_{\nu=0}^{\infty} a_\nu C^\nu_{\nu+1} r_\nu =$$

$$= \sum_{\nu=0}^{s_{t_0}-1} a_\nu C^\nu_{\nu+1} r_\nu + \sum_{\nu=s_{t_0-1}+11}^{\infty} a_\nu C^\nu_{\nu+1} r_\nu. \tag{11}$$

We estimate two sums in (11). Then in view of inequality (8) we obtain that $\mu^*_{t_0} \leq \frac{1}{c}\mu_{t_0}$ or $\max\{a_\nu r_{t_0} : \nu \neq s_{t_0}, \nu \leq s_{t_0-1}\} \leq \frac{1}{c} \max\{a_\nu r_{t_0} : \nu \neq s_{t_0}, \nu \leq s_{t_0-1}\}$, we have $a_\nu r^\nu \leq \frac{1}{c} a_{\nu_0} r^{\nu_0}$. From (10) we have

$$\sum_{\substack{\nu=0,\\ \nu\neq s_{t_0}}}^{s_{t_0-1}} a_\nu C^\nu_{\nu+1} r_\nu \leq \frac{a_{\nu_0} r^{\nu_0}}{c} \sum_{\nu=0}^{N} C^\nu_{\nu+1} \leq \frac{a_{\nu_0} r^{\nu_0}}{c}(N + 1)^2. \tag{12}$$

For all $\nu \geq s_{t_0-1} + 1$ we have $a_\nu r^\nu_{t_0-1} \leq \mu_{t_0-1}$. Thus, $a_\nu r^\nu_{t_0} = \frac{a_\nu r^\nu_{t_0-1}}{c^\nu} \leq \frac{\mu_{t_0-1}}{c^\nu}$. From (8) we have

$$\sum_{\nu=s_{t_0-1}+1}^{\infty} a_\nu C^\nu_{\nu+1} r^\nu \mu_{t_0-1} C^\nu_{\nu+1} \frac{1}{c^\nu} \leq a_{s_{t_0-1}} r_{t_0}^{s_{t_0-1}} c^{s_{t_0-1}} \left(\sum_{\nu=s_{t_0-1}+1}^{\infty} x^{\nu+2}\right)'' |_{x=\frac{1}{c}} =$$

$$= \frac{a_{\nu_0} r^{\nu_0}}{c} c^{s_{t_0-1}} \left\{\frac{x^{s_{t_0-1}+3}}{1 - x}\right\}'' |_{x=\frac{1}{c}} = \frac{a_{\nu_0} r^{\nu_0}}{c} c^{s_{t_0-1}} \sum_{j=0}^{2} C^j_2 2!(s_{t_0-1} + 3) \times ... \times$$

$$\times (s_{t_0-1} + 4 - j) \cdot \frac{x^{s_{t_0-1}+3-j}}{(1 - x)^{3-j}} |_{x=\frac{1}{c}} \leq \frac{a_{\nu_0} r^{\nu_0}}{c} c^{s_{t_0-1}} 2!(N + 3)! \frac{(1/c)^{s_{t_0-1}+3-j}}{(1 - 1/c)^{3-j}} =$$

$$= 2!(N + 3)! \frac{a_{\nu_0} r^{\nu_0}}{c} \sum_{j=0}^{2} \frac{1}{(c - 1)^{3-j}} \leq 3!(N + 3)! \frac{a_{\nu_0} r^{\nu_0}}{c} \tag{13}$$

for $c \geq 2$. Then from (11)–(13) we obtain

$$\left\|\sum_{i+j\neq\nu_0} B_{i,j}(z - z_0)^i (w - w_0)^j\right\| \leq \frac{((N + 1)C^N_{N+1} + 3!(N + 3)!)a_{\nu_0} r^{\nu_0}}{c} \leq \frac{1}{2} a_{\nu_0} r^{\nu_0}.$$

Then the polynomial p_{ν_0} is main in (6) on $\mathbb{T}^2\left((z_0, w_0), \frac{r\mathbf{e}}{\mathbf{L}(z_0,w_0)}\right)$. □

Theorem 3. *Let* $\mathbf{L} \in Q(\mathbb{B}^2)$. *If there exist* $p \in \mathbb{Z}_+$, $d \in (0;1]$, $\eta \in (o;d)$ *such that for each* $(z_0, w_0) \in \mathbb{B}^2$ *and for some* $R = (r_1, r_2)$ *with* $r_j = r_j(d, (z_0, w_0)) \in (\eta, d)$, $j \in \{1, 2\}$ *and for some* $\nu_0 = \nu_0(d, (z_0, w_0)) \le p$ *a polynomial* p_{ν_0} *is main in* (6) *on* $\mathbb{T}^2\left((z_0, w_0), \frac{R}{\mathbf{L}(z_0,w_0)}\right)$, *then an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has bounded* L*-index in joint variables.*

Proof. Suppose that exist $p \in \mathbb{Z}_+$, $d \le 1$ and $\eta \in (o;d)$ such that for each $(z_0, w_0) \in \mathbb{B}^2$ and for some $R = (r_1, r_2)$ with $r_j = r_j(d, (z_0, w_0)) \in (\eta, d)$, $j \in \{1, 2\}$ and for some $\nu_0 = \nu_0(d, (z_0, w_0)) \le p$ polynomial p_{ν_0} is main in (6) in $\mathbb{T}^2\left((z_0, w_0), \frac{R}{\mathbf{L}(z_0,w_0)}\right)$. We put $r_0 = \max\limits_{1\le j\le 2} r_j$. Then

$$\left\|\sum_{i+j\ne\nu_0} B_{i,j}(z-z_0)^i(w-w_0)^j\right\| = \left\|F(z,w) - \sum_{i+j=\nu_0} B_{i,j}(z-z_0)^i(w-w_0)^j\right\| \le$$
$$\le \frac{a_{\nu_0} r_0^{\nu_0}}{2}.$$

Hence, in view of Cauchy's integral formula we obtain that

$$\left\|B_{i,j}(z-z_0)^i(w-w_0)^j\right\| = a^*_{i,j} r_1^i r_2^j \le \frac{a_{\nu_0} r_0^{\nu_0}}{2}, \qquad \forall i,j \in \mathbb{Z}^2_+, \qquad i+j \ne \nu_0,$$

and for all $i + j = \nu \ne \nu_0$ one has

$$a_\nu r_1^i r_2^j \le \frac{a_{\nu_0} r_0^{\nu_0}}{2}. \tag{14}$$

Suppose that F is of unbounded $\mathbf{L}$-index in joint variables. By Theorem 1 for each $p_1 \in \mathbb{Z}_+$ and $c > 1$ $\exists (z_0, w_0) \in \mathbb{B}^2$ such that

$$\max\left\{\frac{\|F^{(i,j)}(z_0,w_0)\|}{l_1^i(z_0,w_0) l_2^j(z_0,w_0)} : i+j = p_1+1\right\} > c \cdot \max\left\{\frac{\|F^{(k,m)}(z_0,w_0)\|}{l_1^k(z_0,w_0) l_2^m(z_0,w_0)} : k+m \le p_1\right\}.$$

Put $p_1 = p$ and $c = \left(\frac{(p+1)!}{\eta^{p+1}}\right)^2$. Then for $z_0(p_1, c)$, $w_0(p_1, c)$ one has

$$\max\left\{\frac{\|F^{(i,j)}(z_0,w_0)\|}{i!j! l_1^i(z_0,w_0) l_2^j(z_0,w_0)} : i+j = p_1+1\right\} >$$
$$> \frac{1}{\eta^{p+1}} \max\left\{\frac{\|F^{(k,m)}(z_0,w_0)\|}{k!m! l_1^k(z_0,w_0) l_2^m(z_0,w_0)} : k+m \le p\right\},$$

that is, $a_{p+1} > \frac{a_{\nu_0}}{\eta^{p+1}}$. We obtain $a_{p+1} r_0^{p+1} > \frac{a_{\nu_0} r_0^{p+1}}{\eta^{p+1}} \ge a_{\nu_0} r_{\nu_0}$.

The last inequality contradicts (14). Thus, the vector-function F has bounded $\mathbf{L}$-index in joint variables. □

References

[1] Baksa V.P., *Analytic vector-functions in the unit ball having bounded* **L***-index in joint variables*, Carpathian Mathematical Publications (in print).

[2] Baksa V.P., Bandura A.I., Skaskiv O.B., *Analogs of Fricke's theorems for analytic vector-valued functions in the unit ball having bounded* **L***-index in joint variables*, submitted to Proceedings of IAMM of NASU.

[3] Baksa V.P., Bandura A.I., Skaskiv O.B., *Analogs of Hayman's theorem and of logarithmic criterion for analytic vector-valued functions in the unit ball having bounded* **L***-index in joint variables.* submitted to Matematica Slovaca.

[4] Bandura A. I., Skaskiv O. B. *Analytic functions in the unit ball of bounded* **L***-index: asymptotic and local properties.* Mat. Stud. 2017, **48** (1), 37–73. doi: 10.15330/ms.48.1.37-73

[5] Bandura A., Skaskiv O. *Sufficient conditions of boundedness of* **L***-index and analog of Hayman's Theorem for analytic functions in a ball.* Stud. Univ. Babeş-Bolyai Math. 2018, **63** (4), 483–501. doi:10.24193/subbmath.2018.4.06

[6] Bandura A., Skaskiv O. *Functions analytic in the unit ball having bounded L-index in a direction.* Rocky Mountain J. Math. 2019, **49** (4), 1063–1092. doi: 10.1216/RMJ-2019-49-4-1063

[7] Bandura A., Petrechko N., Skaskiv O. *Maximum modulus in a bidisc of analytic functions of bounded* **L***-index and an analogue of Hayman's theorem.* Mat. Bohemica 2018, **143** (4), 339–354. doi: 10.21136/MB.2017.0110-16

[8] Bandura A.I., Skaskiv O.B., Tsvigun V.L. *Some characteristic properties of analytic functions in* $\mathbb{D}\times\mathbb{C}$ *of bounded* **L***-index in joint variables.* Bukovyn. Mat. Zh. 2018, **6** (1-2), 21–31.

[9] Bandura A., Petrechko N. *Properties of power series expansion of entire function of bounded* **L***-index in joint variables.* Visn. Lviv. Un-ty. Ser. Mech.-Math. 2016, Iss. **82**, 27–33.

[10] Bandura A.I., Petrechko N.V. *Properties of power series of analytic in a bidisc functions of bounded* **L***-index in joint variables.* Carpathian Math. Publ. 2017, **9** (1), 6–12. doi: 10.15330/cmp.9.1.6-12

[11] Bandura A.I., Petrechko N.V., Skaskiv O.B. *Analytic in a polydisc functions of bounded* **L***-index in joint variables.* Mat. Stud. 2016, **46** (1), 72–80. doi: 10.15330/ms.46.1.72-80

[12] Bandura A., Skaskiv O. *Analytic functions in the unit ball of bounded* **L***-index in joint variables and of bounded L-index in direction: a connection between these classes.* Demonstr. Math. 2019, **52** (1), 82-87. doi: 10.1515/dema-2019-0008

[13] Bandura A.I., Skaskiv O.B. *Partial logarithmic derivatives and distribution of zeros of analytic functions in the unit ball of bounded* **L***-index in joint variables.* J. Math. Sci. 2019, **239** (1), 17–29. doi: 10.1007/s10958-019-04284-z

[14] Bandura A.I., Skaskiv O.B. *Exhaustion by balls and entire functions of bounded* **L***-index in joint variables*, Ufa Math. J. 2019, **11** (1), 100-113. doi: 10.13108/2019-11-1-100

[15] Bordulyak M.T., Sheremeta M.M. *Boundedness of l-index of analytic curves.* Mat. Stud. 2011, **36** (2), 152–161.

[16] Heath L. F. *Vector-valued entire functions of bounded index satisfying a differential equation.* Journal of Research of NBS 1978, **83B** (1), 75–79.

[17] Nuray F., Patterson R.F. *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations.* Mat. Stud. 2018, **49** (1), 67–74. doi: 10.15330/ms.49.1.67-74

[18] Roy R., Shah S.M. *Growth properties of vector entire functions satisfying differential equations.* Indian J. Math. 1986, **28** (1), 25–35.

[19] Roy R., Shah S.M. *Vector-valued entire functions satisfying a differential equation.* J. Math. Anal. Appl. 1986, **116** (2), 349–362.

[20] Sheremeta M. *Boundedness of $l-M$-index of analytic curves.* Visnyk Lviv Un-ty. Ser. Mech.-Math. Iss. **75**, 226–231 (2011)

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У цій статті отримано необхідні і достатні умови обмеженості $\mathbb{L}$-індексу за сукупністю змінних для векторнозначних функкій, аналітичних в одиничній кулі, де $\mathbf{L} = (l_1, l_2) : \mathbb{B}^2 \to \mathbb{R}^2_+$ — додатна неперервна вектор-функкція, $\mathbb{B}^2 = \{z \in \mathbb{C}^2 : |z| = \sqrt{|z_1|^2 + |z_2|^2} \leq 1\}$. Ці умови описують локальне поводження однорідних багаточленів (так званих головних багаточленів) з розвинення у степеневий ряд аналітичних в одиничній кулі векторнозначних функцій. Отримані результати використовують бікругове вичерпання одиничної кулі.