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INVERSE PROBLEM FOR A 2D STRONGLY DEGENERATE HEAT EQUATION WITH INTEGRAL OVERDETERMINATION CONDITIONS

The existence of a classical solution to the problem of identification of two time-dependent coefficients in the 2D heat equation is considered. We suppose that unknown coefficients vanish at the initial moment of time as a power with exponent greater than 1. Existence is established via Schauder fixed-point theorem.

Key words and phrases: Inverse problem, 2D strongly degenerate heat equation, integral overdetermination conditions.

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Introduction

An inverse problem for a 2D strongly degenerate anisotropic heat equation in a rectangular domain is considered. Such problems are of growing importance to various branches of physics, economics, etc. [1, 2, 3]. The first results on identification of unknown coefficients in degenerate equations were obtained in [4, 5]. More systematic investigation of inverse problems for one-dimensional degenerate parabolic equations was made in [6, 7, 8]. The transition to 2D degenerate equations was realized only in the case when unknown coefficients depend on the time variable. For weakly degenerate parabolic equations, inverse problems with one or two unknown leading coefficients [9, 10] were studied. An inverse problem for 2D strongly degenerate heat equation with one unknown coefficient was investigated in [11].

In this paper we establish conditions for existence of solution to an inverse problem for a 2D strongly degenerate heat equation with two unknown coefficients that have different behavior of power type at the initial moment of time.

1 Problem statement

In the domain $Q_T := \{(x, y, t) : 0 < x < h, 0 < y < l, 0 < t < T\}$ we consider a two-dimensional heat equation with two unknown leading coefficients depending on the time

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variable. We suppose that each of unknown coefficients vanishes at the initial moment of time as a power t^{β_i} where $\beta_i \geq 1, i \in \{1, 2\}$. We will use mixed Dirichlet-Neumann boundary conditions, and the additional conditions (also called *overdetermination conditions*) will be of integral type. The problem is to find a triplet of functions $(a_1(t), a_2(t), u(x, y, t)), a_i(t) > 0, t \in (0, T], i \in \{1, 2\}$ satisfying the degenerate heat equation

$$u_t = a_1(t)u_{xx} + a_2(t)u_{yy} + f(x, y, t), \quad (x, y, t) \in Q_T, \tag{1}$$

initial condition

$$u(x, y, 0) = \varphi(x, y, 0), \quad (x, y) \in \overline{D} := [0, h] \times [0, l], \tag{2}$$

boundary conditions

$$u(0, y, t) = \mu_{11}(y, t),\ u(h, y, t) = \mu_{12}(y, t), \quad (y, t) \in [0, l] \times [0, T], \tag{3}$$

$$u_y(x, 0, t) = \mu_{21}(x, t),\ u_y(x, l, t) = \mu_{22}(x, t), \quad (x, t) \in [0, h] \times [0, T] \tag{4}$$

and overdetermination conditions

$$\iint\limits_D u(x, y, t)dx\,dy = \mu_{31}(t), \quad \iint\limits_D xu(x, y, t)dx\,dy = \mu_{32}(t), \quad t \in (0, T]. \tag{5}$$

Our aim is to determine the conditions of existence of classical solution to the problem (1)-(5).

2 Reduction of the problem (1)-(5) to an equation system for $a_1(t), a_2(t)$

Assume temporarily that coefficients $a_i = a_i(t) > 0, t \in (0, T]$ from the space $C[0, T]$ are known. Reduce the problem (1)-(5) to a system of equations with respect to $a_1(t), a_2(t)$ with the aid of overdetermination conditions (5). Differentiating them with respect to t and using equation (1) we obtain

$$a_1(t)\int_0^l (hu_x(h, y, t) + \mu_{11}(y, t) - \mu_{12}(y, t))dy + a_2(t)\int_0^h x(\mu_{22}(x, t) - \mu_{21}(x, t))dx$$
$$= \mu_{32}'(t) - \iint\limits_D xf(x, y, t)dx\,dy, \quad t \in [0, T], \tag{6}$$

$$a_1(t)\int_0^l (u_x(h, y, t) - u_x(0, y, t))dy + a_2(t)\int_0^h (\mu_{22}(x, t) - \mu_{21}(x, t))dx$$
$$= \mu_{31}'(t) - \iint\limits_D f(x, y, t)dx\,dy, \quad t \in [0, T]. \tag{7}$$

Transform the system of equations (6), (7), substituting the values related to u. Using the Green function $G_{12}(x,y,t,\xi,\eta,\tau)$ we find the solution to the direct problem (1)-(4)

$$u(x,y,t)=\int_0^l\int_0^h G_{12}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta+\int_0^t\int_0^l G_{12_\xi}(x,y,t,0,\eta,\tau)a_1(\tau)$$
$$\times\mu_1(\eta,\tau)d\eta d\tau-\int_0^t\int_0^l G_{12_\xi}(x,y,t,h,\eta,\tau)a_1(\tau)\mu_2(\eta,\tau)d\eta d\tau-\int_0^t\int_0^h G_{12}(x,y,t,\xi,0,\tau)$$
$$\times a_2(\tau)\nu_1(\xi,\tau)d\xi d\tau+\int_0^t\int_0^h G_{12}(x,y,t,\xi,l,\tau)a_2(\tau)\nu_2(\xi,\tau)d\xi d\tau$$
$$+\int_0^t\iint_D G_{12}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau,\quad (x,y,t)\in\overline{Q}_T. \tag{8}$$

The Green function is defined by the equality

$$G_{ij}(x,y,t,\xi,\eta,\tau)=\frac{1}{4\pi\sqrt{(\theta_1(t)-\theta_1(\tau))(\theta_2(t)-\theta_2(\tau))}}\sum_{m,n=-\infty}^{\infty}\left(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\right)\right.$$
$$\left.+(-1)^i\exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\right)\right)\left(\exp\left(-\frac{(y-\eta+2ml)^2}{4(\theta_2(t)-\theta21(\tau))}\right)\right)$$
$$\left.+(-1)^j\exp\left(-\frac{(y+\eta+2ml)^2}{4(\theta_2(t)-\theta_2(\tau))}\right)\right),\quad i,j\in\{1,2\},\theta_k(t):=\int_0^t a_k(\sigma)d\sigma,\ k\in\{1,2\},$$

where the values $i,j=1$ correspond to Dirichlet boundary conditions and $i,j=2$ to Neumann conditions by x and y respectively. One can see that 2D Green function G_{ij} may be presented as a product of two 1D Green functions for corresponding heat equations: $G_{ij}(x,y,t,\xi,\eta,\tau)=G_i(x,t,\xi,\tau)G_j(y,t,\eta,\tau)$. Moreover, it is easy to verify that $G_{1_x}(x,t,\xi,\tau)=-G_{2_\xi}(x,t,\xi,\tau)$.

Suppose that the following assumptions hold:

(A1) $\varphi\in C^{1,0}(\overline{D}),\mu_{1i}\in C^{2,1}([0,l]\times(0,T]),\mu_{2i}\in C^{1,0}([0,h]\times(0,T]),f\in C^{1,0,0}(\overline{Q}_T),\mu_{3i}\in C^1([0,T]),i\in\{1,2\}$.

Here we used the notations from [12]. Now, find the derivative $u_x(x,y,t)$ and transform it by integrating by parts:

$$u_x(x,y,t)=\int_0^l\int_0^h G_{22}(x,y,t,\xi,\eta,0)\varphi_\xi(\xi,\eta)d\xi d\eta$$

$$-\int_0^t\int_0^l G_{22}(x,y,t,0,\eta,\tau)(\mu_{11_\tau}(\eta,\tau)-a_2(\tau)\mu_{11_{\eta\eta}}(\eta,\tau)-f(0,\eta,\tau))d\eta d\tau$$

$$+\int_0^t\int_0^l G_{22}(x,y,t,h,\eta,\tau)(\mu_{12_\tau}(\eta,\tau)-a_2(\tau)\mu_{12_{\eta\eta}}(\eta,\tau)-f(h,\eta,\tau))d\eta d\tau$$

$$-\int_0^t\int_0^h G_{22}(x,y,t,\xi,0,\tau)a_2(\tau)\mu_{21_\xi}(\xi,\tau)d\xi d\tau+\int_0^t\int_0^h G_{22}(x,y,t,\xi,l,\tau)$$

$$\times a_2(\tau)\mu_{22_\xi}(\xi,\tau)d\xi d\tau+\int_0^t\iint_D G_{22}(x,y,t,\xi,\eta,\tau)f_\xi(\xi,\eta,\tau)d\xi d\eta d\tau. \qquad (9)$$

To find $\int_0^l u_x(x,y,t)dy$, note that by direct calculation we get $\int_0^h G_2(x,t,\xi,\tau)dx=1$. Hence, we obtain

$$\int_0^l u_x(x,y,t)dy=\int_0^h G_2(x,t,\xi,0)d\xi\int_0^l\varphi_\xi(\xi,\eta)d\eta-\int_0^t G_2(x,t,0,\tau)d\tau\int_0^l(\mu_{11_\tau}(\eta,\tau)$$

$$-f(0,\eta,\tau))d\eta+\int_0^t G_2(x,t,h,\tau)d\tau\int_0^l(\mu_{12_\tau}(\eta,\tau)-f(h,\eta,\tau))d\eta$$

$$+\int_0^t G_2(x,t,0,\tau)a_2(\tau)(\mu_{22}(0,\tau)-\mu_{21}(0,\tau))d\tau-\int_0^t G_2(x,t,h,\tau)a_2(\tau)(\mu_{22}(h,\tau)$$

$$-\mu_{21}(h,\tau))d\tau-\int_0^t\int_0^h G_2(x,t,\xi,\tau)a_2(\tau)\mu_{21_\xi}(\xi,\tau)d\xi d\tau+\int_0^t\int_0^h G_2(x,t,\xi,\tau)$$

$$\times a_2(\tau)\mu_{22_\xi}(\xi,\tau)d\xi d\tau+\int_0^t\int_0^h G_2(x,t,\xi,\tau)d\xi d\tau\int_0^l f_\xi(\xi,\eta,\tau)d\eta. \qquad (10)$$

Return to the system (6), (7). To find its positive solution on $(0,T]$, consider an auxiliary algebraic system of equations

$$a_1x+b_1y=c_1,\quad a_2x+b_2y=c_2$$

where $a_i>0, b_i>0, c_i>0, i\in\{1,2\}$. The solution of this system is defined by a formula

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1},\quad y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}.$$

It has a positive solution if

$$b_2c_1 - b_1c_2 > 0, \quad a_1c_2 - a_2c_1 > 0, \quad a_1b_2 - a_2b_1 > 0.$$

In turn, these conditions are fulfilled if

$$\frac{a_1}{a_2} > \frac{c_1}{c_2} > \frac{b_1}{b_2}. \tag{11}$$

Apply this result to the system (6), (7). It follows that the positive solution of the system (6), (7) exists if the following conditions hold:

$$\mu_{22}(x,t) - \mu_{21}(x,t) \geq 0, \mu_{22}(x,t) \not\equiv \mu_{21}(x,t), (x,t) \in [0,h] \times (0,T], \quad \int\limits_0^l (\mu_{11}(\eta,t)$$

$$- \mu_{12}(\eta,t))d\eta > 0, \quad \mu'_{31}(t) - \iint\limits_D f(x,y,t)dx\,dy > 0, \qquad \mu'_{32}(t) - \iint\limits_D xf(x,y,t)dx\,dy > 0,$$

$$h > \frac{\mu'_{32}(t) - \iint\limits_D xf(x,y,t)dx\,dy,}{\mu'_{31}(t) - \iint\limits_D f(x,y,t)dx\,dy} > \frac{\int\limits_0^h x(\mu_{22}(x,t) - \mu_{21}(x,t))dx}{\int\limits_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx}$$

$$\int\limits_0^l u_x(0,y,t)dy \geq 0, \quad \int\limits_0^l (u_x(h,y,t) - u_x(0,y,t))dy > 0, \quad t \in (0,T]. \tag{12}$$

Note that two last conditions in (12) are expressed via $u(x,y,t)$. In order to study the first of them, put $x = 0$ in (10) and consider one of the terms within:

$$\int\limits_0^t G_2(0,t,0,\tau)a_2(\tau)(\mu_{22}(0,\tau) - \mu_{21}(0,\tau))d\tau - \int\limits_0^t G_2(0,t,h,\tau)a_2(\tau)(\mu_{22}(h,\tau)$$

$$- \mu_{21}(h,\tau))d\tau = \int\limits_0^t (G_2(0,t,0,\tau) - G_2(0,t,h,\tau))a_2(\tau)(\mu_{22}(0,\tau) - \mu_{21}(0,\tau))d\tau$$

$$+ \int\limits_0^t G_2(0,t,h,\tau)a_2(\tau)(\mu_{22}(0,\tau) - \mu_{21}(0,\tau) - \mu_{22}(h,\tau) + \mu_{21}(h,\tau))d\tau.$$

As $G_2(0,t,0,\tau) - G_2(0,t,h,\tau) \geq 0$, we have $\int\limits_0^l u_x(0,y,t)dy > 0$ if the following assumptions are fulfilled:

$$\int\limits_0^l (\mu_{11_t}(y,t) - f(0,y,t))dy < 0, \int\limits_0^l (\mu_{12_t}(y,t) - f(h,y,t))dy > 0, t \in [0,T], \int\limits_0^l \varphi_x(x,y)dy > 0,$$

$$\mu_{21_x}(x,t) \leq 0, \mu_{22_x}(x,t) \geq 0, \mu_{22}(x,t) - \mu_{21}(x,t) \geq 0, \mu_{22}(x,t) \not\equiv \mu_{21}(x,t), \int\limits_0^l f_x(x,y,t)dy$$

$$\geq 0, (x,t) \in [0,h] \times (0,T], \mu_{22}(0,t) - \mu_{21}(0,t) - \mu_{22}(h,t) + \mu_{21}(h,t) \geq 0. \tag{13}$$

Find from (10)

$$\int_0^l (u_x(h,y,t)-u_x(0,y,t))dy = \int_0^h (G_2(h,t,\xi,0)-G_2(0,t,\xi,0)d\xi \int_0^l \varphi_\xi(\xi,\eta)d\eta$$
$$-\int_0^t (G_2(h,t,0,\tau)-G_2(0,t,0,\tau))d\tau \int_0^l (\mu_{11_\tau}(\eta,\tau)-f(0,\eta,\tau)+\mu_{12_\tau}(\eta,\tau)-f(h,\eta,\tau))d\eta$$
$$+\int_0^t (G_2(h,t,0,\tau)-G_2(0,t,0,\tau))a_2(\tau)(\mu_{22}(0,\tau)-\mu_{21}(0,\tau)+\mu_{22}(h,\tau)-\mu_{21}(h,\tau))d\tau$$
$$+\int_0^t\int_0^h (G_2(h,t,\xi,\tau)-G_2(0,t,\xi,\tau))a_2(\tau)(\mu_{22_\xi}(\xi,\tau)-\mu_{21_\xi}(\xi,\tau))d\xi d\tau$$
$$+\int_0^t\int_0^h (G_2(h,t,\xi,\tau)-G_2(0,t,\xi,\tau))d\xi d\tau \int_0^l f_\xi(\xi,\eta,\tau)d\eta. \tag{14}$$

Taking into account the representation of the Green function, consider

$$G_2(0,t,\xi,\tau)-G_2(h,t,\xi,\tau)=\frac{1}{\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}}\sum_{n=-\infty}^{\infty}\Bigg(\exp\bigg(-\frac{(\xi+2nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\bigg)$$
$$-\exp\bigg(-\frac{(\xi+(2n+1)h)^2}{4(\theta_1(t)-\theta_1(\tau))}\bigg)\Bigg)=\frac{1}{\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}}\sum_{n=-\infty}^{\infty}(-1)^n\exp\bigg(-\frac{(\xi+nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\bigg)$$
$$:=\tilde{G}(0,t,\xi,\tau).$$

It is easy to check that $\tilde{G}(x,t,\xi,\tau)$ is the Green function for the heat equation

$$u_t=a_1(t)u_{xx},\quad (x,t)\in(0,h/2)\times(0,T)$$

with boundary conditions

$$\tilde{G}_x(0,t,\xi,\tau)=0,\quad \tilde{G}(h/2,t,\xi,\tau)=0.$$

In virtue of properties of Green functions, $\tilde{G}(x,t,\xi,\tau)\geq 0, (x,t),(\xi,\tau)\in[0,h/2]\times[0,T]$.

For some continuous on $[0,h]$ function ψ, transform the integral

$$\int_0^h (G_2(h,t,\xi,\tau)-G_2(0,t,\xi,\tau))\psi(\xi)d\xi=\frac{1}{\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}}\int_0^h\sum_{n=-\infty}^{\infty}(-1)^{n+1}$$
$$\times\exp\bigg(-\frac{(\xi+nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\bigg)\psi(\xi)d\xi=\frac{1}{\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}}\int_0^{h/2}\sum_{n=-\infty}^{\infty}(-1)^{n+1}$$
$$\times\exp\bigg(-\frac{(\xi+nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\bigg)\psi(\xi)d\xi+\frac{1}{\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}}\int_{h/2}^h\sum_{n=-\infty}^{\infty}(-1)^{n+1}$$
$$\times\exp\bigg(-\frac{(\xi+nh)^2}{4(\theta_1(t)-\theta_1(\tau))}\bigg)\psi(\xi)d\xi.$$

After substitution $\zeta = h - \xi$ in the second integral, we obtain

$$\int_0^h (G_2(h,t,\xi,\tau) - G_2(0,t,\xi,\tau))\psi(\xi)d\xi = \int_0^{h/2} \tilde{G}(0,t,\xi,\tau)(\psi(h-\xi) - \psi(\xi))d\xi.$$

The same transformation is applied also to the last integral in (14). Hence, the expression (14) is reduced to the form

$$\int_0^l (u_x(h,y,t) - u_x(0,y,t))dy = \int_0^{h/2} \tilde{G}(0,t,\xi,\tau)d\xi \int_0^l (\varphi_\xi(h-\xi,\eta) - \varphi_\xi(\xi,\eta))d\eta$$
$$+ \int_0^t \tilde{G}(0,t,0,\tau)d\tau \int_0^l (\mu_{11_\tau}(\eta,\tau) - f(0,\eta,\tau) + \mu_{12_\tau}(\eta,\tau) - f(h,\eta,\tau))d\eta$$
$$- \int_0^t \tilde{G}(0,t,\xi,\tau)a_2(\tau)(\mu_{22}(0,\tau) - \mu_{21}(0,\tau) + \mu_{22}(h,\tau) - \mu_{21}(h,\tau))d\tau$$
$$+ \int_0^t \int_0^{h/2} \tilde{G}(0,t,\xi,\tau)a_2(\tau)(\mu_{22_\xi}(h-\xi,\tau) - \mu_{21_\xi}(h-\xi,\tau) - \mu_{22_\xi}(\xi,\tau) + \mu_{21_\xi}(\xi,\tau))d\xi d\tau$$
$$+ \int_0^t \int_0^{h/2} \tilde{G}(0,t,\xi,\tau)d\xi d\tau \int_0^l (f_\xi(h-\xi,\eta,\tau) - f_\xi(\xi,\eta,\tau))d\eta. \quad (15)$$

Taking into account (12)-(15), we conclude that the system (6), (7) has a positive solution if the following assumptions hold:

$$\textbf{(A2)} \int\limits_0^l \varphi_x(x,y)dy > 0, \mu_{21_x}(x,t) \le 0, \mu_{22_x}(x,t) \ge 0, \int\limits_0^l f_x(x,y,t)dy \ge 0, \mu_{22}(x,t) \not\equiv \mu_{21}(x,t),$$

$$\mu_{22}(x,t) - \mu_{21}(x,t) \ge 0, \int\limits_0^l (\varphi_x(h-x,\eta) - \varphi_x(x,\eta))d\eta \ge 0, \int\limits_0^l (\mu_{11_t}(\eta,t) - f(0,\eta,t)$$

$$+ \mu_{12_t}(\eta,t) - f(h,\eta,t))d\eta \ge 0, \mu_{22_x}(h-x,t) - \mu_{21_x}(h-x,t) - \mu_{22_x}(x,t) + \mu_{21_x}(x,t) \ge 0,$$

$$\int\limits_0^l (f_x(h-x,\eta,t) - f_x(x,\eta,t))d\eta \ge 0, (x,t) \in [0,h] \times (0,T], \int\limits_0^l (\mu_{11}(y,t) - \mu_{12}(y,t))dy > 0,$$

$$\int\limits_0^l (\mu_{11_t}(y,t) - f(0,y,t))dy < 0, \int\limits_0^l (\mu_{12_t}(y,t) - f(h,y,t))dy > 0, \mu_{22}(0,t) - \mu_{21}(0,t)$$

$$- \mu_{22}(h,t) + \mu_{21}(h,t) \ge 0, \mu'_{31}(t) - \iint\limits_D f(x,y,t)dx\,dy > 0, \mu'_{32}(t) - \iint\limits_D xf(x,y,t)dx\,dy > 0,$$

$$h > \frac{\mu'_{32}(t) - \iint\limits_D xf(x,y,t)dx\,dy,}{\mu'_{31}(t) - \iint\limits_D f(x,y,t)dx\,dy} > \frac{\int\limits_0^h x(\mu_{22}(x,t) - \mu_{21}(x,t))dx}{\int\limits_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx}, \quad t \in [0,T]. \tag{16}$$

Now find the solution of the system (6), (7):

$$a_1(t) = \Bigg(\Bigg(\mu'_{32}(t) - \iint\limits_D xf(x,y,t)dx\,dy\Bigg) \int\limits_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx - \Bigg(\mu'_{31}(t)$$

$$- \iint\limits_D f(x,y,t)dx\,dy\Bigg) \int\limits_0^h x(\mu_{22}(x,t) - \mu_{21}(x,t))dx\Bigg)\Delta^{-1}(t), t \in [0,T], \tag{17}$$

$$a_2(t) = \Bigg(\Bigg(\mu'_{31}(t) - \iint\limits_D f(x,y,t)dx\,dy\Bigg) \int\limits_0^l (hu_x(h,y,t) + \mu_{11}(y,t) - \mu_{12}(y,t))dy$$

$$- \Bigg(\mu'_{32}(t) - \iint\limits_D xf(x,y,t)dx\,dy\Bigg) \int\limits_0^l (u_x(h,y,t) - u_x(0,y,t))dy\Bigg)\Delta^{-1}(t), t \in [0,T], \tag{18}$$

where

$$\Delta(t) := \int\limits_0^l (hu_x(h,y,t) + \mu_{11}(y,t) - \mu_{12}(y,t))dy \int\limits_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx$$

$$- \int\limits_0^l (u_x(h,y,t) - u_x(0,y,t))dy \int\limits_0^h x(\mu_{22}(x,t) - \mu_{21}(x,t))dx. \tag{19}$$

3 Investigation of the system (17)-(18)

To establish the behavior of $\Delta(t)$ as $t \to 0$, we present it as

$$\Delta(t) = \int_0^l u_x(h,y,t)dy \int_0^h (h-x)(\mu_{22}(x,t) - \mu_{21}(x,t))dx + \int_0^l u_x(0,y,t)dy \int_0^h x(\mu_{22}(x,t)$$

$$- \mu_{21}(x,t))dx + \int_0^l (\mu_{11}(y,t) - \mu_{12}(y,t))dy \int_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx. \quad (20)$$

Find $\int_0^l u_x(0,y,t)dy$ from (10) and estimate the expressions

$$\int_0^h G_2(0,t,\xi,0)d\xi \int_0^l \varphi_\xi(\xi,\eta)d\eta \le C_1 \int_0^h G_2(0,t,\xi,0)d\xi,$$

$$-\int_0^t G_2(0,t,0,\tau)d\tau \int_0^l (\mu_{11_\tau}(\eta,\tau) - f(0,\eta,\tau))d\eta \le C_2 \int_0^t G_2(0,t,0,\tau)d\tau,$$

$$\int_0^t G_2(0,t,h,\tau)d\tau \int_0^l (\mu_{12_\tau}(\eta,\tau) - f(h,\eta,\tau))d\eta \le C_3 \int_0^t G_2(0,t,h,\tau)d\tau,$$

$$\int_0^t G_2(0,t,0,\tau)a_2(\tau)(\mu_{22}(0,\tau) - \mu_{21}(0,\tau))d\tau \le C_4 \int_0^t G_2(0,t,0,\tau)a_2(\tau)d\tau,$$

$$\int_0^t G_2(0,t,h,\tau)a_2(\tau)(\mu_{22}(h,\tau) - \mu_{21}(h,\tau))d\tau \le C_5 \int_0^t G_2(0,t,h,\tau)a_2(\tau)d\tau,$$

$$-\int_0^t \int_0^h G_2(0,t,\xi,\tau)a_2(\tau)\mu_{21_\xi}(\xi,\tau)d\xi d\tau + \int_0^t \int_0^h G_2(0,t,\xi,\tau)a_2(\tau)\mu_{22_\xi}(\xi,\tau)d\xi d\tau$$

$$\le C_6 \int_0^t \int_0^h G_2(0,t,\xi,\tau)a_2(\tau)d\xi d\tau,$$

$$\int_0^t \int_0^h G_2(0,t,\xi,\tau)d\xi d\tau \int_0^l f_\xi(\xi,\eta,\tau)d\eta \le C_7 \int_0^t \int_0^h G_2(0,t,\xi,\tau)d\xi d\tau.$$

Here and below $C_i, i = 1,2,3,...$, mean constants depending on given data. Denote $a_i(t) := t^{\beta_i} b_i(t)$ and suppose that $\beta_i \ge 1, b_i(t) > 0, t \in [0,T], i \in \{1,2\}$. Taking into account Green functions estimates [13], we get

$$\int_0^t G_2(0,t,0,\tau)d\tau \le \int_0^t \left(\frac{C_{11}}{\sqrt{\theta_1(t) - \theta_1(\tau)}} + C_{12} \right) d\tau \le \frac{C_{13} t^{\frac{1-\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + C_{12}T, \; G_2(0,t,h,\tau) \le C_{14}.$$

On the other hand,

$$\int_0^t G_2(0,t,0,\tau)d\tau \geq \int_0^t \frac{d\tau}{\sqrt{\pi(\theta_1(t)-\theta_1(\tau))}} \geq \frac{C_{15}t^{\frac{1-\beta_1}{2}}}{\sqrt{\max\limits_{[0,T]} b_1(t)}}.$$

Hence, we obtain

$$\frac{C_{16}t^{\frac{1-\beta_1}{2}}}{\sqrt{\max\limits_{[0,T]} b_1(t)}} \leq \int_0^l u_x(0,y,t)dy \leq \frac{C_{17}t^{\frac{1-\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{18}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} \int_0^t \frac{\tau^{\beta_2} b_2(\tau)d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}}$$
$$+ C_{19}\int_0^t \tau^{\beta_2} b_2(\tau)d\tau.$$

Transform the integral

$$\int_0^t \frac{\tau^{\beta_2} b_2(\tau)d\tau}{\sqrt{t^{\beta+1}-\tau^{\beta+1}}} \leq t^{\beta_2-\beta_1/2}\int_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau\left(\dfrac{\tau}{t}\right)^{\beta}}} \leq t^{\beta_2-\beta_1/2}\int_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}}.$$

Finally, we get

$$\frac{C_{16}t^{\frac{1-\beta_1}{2}}}{\sqrt{\max\limits_{[0,T]} b_1(t)}} \leq \int_0^l u_x(0,y,t)dy \leq \frac{C_{17}t^{\frac{1-\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{20}t^{\beta_2-\beta_1/2}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} \int_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}}$$
$$+ C_{21}t^{\beta_2}\int_0^t b_2(\tau)d\tau \quad t\in(0,T]. \tag{21}$$

Similarly we find

$$\frac{C_{22}t^{\frac{1-\beta_1}{2}}}{\sqrt{\max\limits_{[0,T]} b_1(t)}} \leq \int_0^l u_x(h,y,t)dy \leq \frac{C_{23}t^{\frac{1-\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{24}t^{\beta_2-\beta_1/2}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} \int_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}}$$
$$+ C_{25}t^{\beta_2}\int_0^t b_2(\tau)d\tau \quad t\in(0,T]. \tag{22}$$

It means that the following inequality is true:

$$\frac{C_{26}t^{\frac{1-\beta_1}{2}}}{\sqrt{\max\limits_{[0,T]} b_1(t)}} \leq \Delta(t) \leq \frac{C_{27}t^{\frac{1-\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{28}t^{\beta_2-\beta_1/2}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} \int_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{29}t^{\beta_2}$$
$$\times \int_0^t b_2(\tau)d\tau,\ t\in(0,T]. \tag{23}$$

Present (17) as

$$a_1(t) = \Delta^{-1}(t) \int_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx \Big(\mu'_{31}(t) - \iint_D f(x,y,t)dx\,dy \Big)$$
$$\times \left(\frac{\mu'_{32}(t) - \iint_D xf(x,y,t)dx\,dy,}{\mu'_{31}(t) - \iint_D f(x,y,t)dx\,dy} - \frac{\int_0^h x(\mu_{22}(x,t) - \mu_{21}(x,t))dx}{\int_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx} \right), \quad t \in (0,T]. \tag{24}$$

Suppose that the following representations hold:

$$\mu'_{31}(t) - \iint_D f(x,y,t)dx\,dy \equiv \varkappa_1(t)t^{\frac{\beta_1+1}{2}},\ \mu'_{32}(t) - \iint_D xf(x,y,t)dx\,dy \equiv \varkappa_2(t)t^{\frac{\beta_1+1}{2}}, \tag{25}$$

where $\varkappa_i(t) > 0, t \in [0,T], i \in \{1,2\}$. In virtue of assumptions (16) and (25), we obtain from (22):

$$b_1(t) \le C_{30}\sqrt{\max_{[0,T]} b_1(t)},$$

or

$$b_1(t) \le B_1 < \infty, \quad t \in [0,T], \tag{26}$$

where constant B_1 is determined by given data.

Investigate the equation (18). Transform it as follows:

$$a_2(t) = \Bigg(\int_0^l u_x(h,y,t)dy \Big(h\mu'_{31}(t) - h\iint_D f(x,y,t)dx\,dy - \mu'_{32}(t) + \iint_D xf(x,y,t)dx\,dy \Big)$$
$$+ \int_0^l u_x(0,y,t)dy \Big(\mu'_{32}(t) - \iint_D xf(x,y,t)dx\,dy \Big) + \Big(\mu'_{31}(t) - \iint_D f(x,y,t)dx\,dy \Big)$$
$$\times \int_0^l (\mu_{11}(y,t) - \mu_{12}(y,t))dy \Bigg) \Delta^{-1}(t), \quad t \in (0,T]. \tag{27}$$

In addition to the assumptions from above, suppose that

$$h\mu'_{31}(t) - \mu'_{32}(t) + \iint_D (x-h)f(x,y,t)dx\,dy > 0, \quad t \in (0,T]. \tag{28}$$

To estimate $a_2(t)$, note that from (15), (16) we have the inequality

$$\int_0^l u_x(h,y,t)dy \ge \int_0^l u_x(0,y,t)dy.$$

Taking it and (20) into account, we obtain from (27)

$$a_2(t) \le \left(\mu_{31}'(t) - \iint\limits_D f(x,y,t)dx\,dy\right)\left(h\int\limits_0^l u_x(h,y,t)dy + \int\limits_0^l (\mu_{11}(y,t)-\mu_{12}(y,t))dy\right)$$
$$\times\left(\int\limits_0^l u_x(0,y,t)dy\int\limits_0^h x(\mu_{22}(x,t)-\mu_{21}(x,t))dx\right)^{-1}.$$

In virtue of (21), (22), (25), (26), we conclude

$$a_2(t) \le t^{\beta_1}\left(\frac{C_{31}t^{\frac{1-\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{32}t^{\beta_2-\beta_1/2}}{\sqrt{\min\limits_{[0,T]} b_1(t)}}\int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{33}t^{\beta_2}\int\limits_0^t b_2(\tau)d\tau + C_{34}\right)\sqrt{\max\limits_{[0,T]} b_1(t)}$$
$$\le t^{\frac{\beta_1+1}{2}}\left(\frac{C_{35}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{36}t^{\beta_2-1/2}}{\sqrt{\min\limits_{[0,T]} b_1(t)}}\int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{37}t^{\frac{\beta_1-1}{2}+\beta_2}\int\limits_0^t b_2(\tau)d\tau + C_{38}t^{\frac{\beta_1-1}{2}}\right)$$

or

$$b_2(t) \le \frac{C_{35}t^{\frac{\beta_1+1}{2}-\beta_2}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{36}t^{\frac{\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}}\int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{37}t^{\beta_1}\int\limits_0^t b_2(\tau)d\tau + C_{38}t^{\beta_1-\beta_2}.$$

It is easy to see that the above inequality has sense if $\beta_2 = \frac{\beta_1+1}{2}$. In this case we obtain

$$b_2(t) \le \frac{C_{35}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \frac{C_{36}t^{\frac{\beta_1}{2}}}{\sqrt{\min\limits_{[0,T]} b_1(t)}}\int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{37}t^{\beta_1}\int\limits_0^t b_2(\tau)d\tau + C_{38}t^{\frac{\beta_1-1}{2}},\ t\in[0,T]. \quad (29)$$

Transform the integral

$$\int\limits_0^t b_2(\tau)d\tau = \int\limits_0^t \frac{\sqrt{t-\tau}\,b_2(\tau)d\tau}{\sqrt{t-\tau}} \le C_{39}\int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}}$$

and reduce the inequality (29) to the form

$$b_2(t) \le \frac{C_{35}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + \left(\frac{C_{40}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + C_{41}\right)\int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{42}, \quad t\in[0,T]. \quad (30)$$

The inequality (30) is similar to the following one:

$$v(t) \le C_{43} + C_{44}\int\limits_0^t \frac{v(\tau)d\tau}{\sqrt{t-\tau}}, \quad t\in[0,T]. \quad (31)$$

To solve it, we find from (31)

$$\int\limits_0^t \frac{v(\sigma)d\sigma}{\sqrt{t-\sigma}} \le C_{45} + C_{44}\int\limits_0^t \frac{d\sigma}{\sqrt{t-\sigma}}\int\limits_0^\sigma \frac{v(\tau)d\tau}{\sqrt{\sigma-\tau}} \le C_{45} + C_{46}\int\limits_0^t v(\tau)d\tau.$$

Applying this estimate to (31) we obtain the Bellman-Gronwall inequality

$$v(t) \le C_{47} + C_{48} \int_0^t v(\tau)d\tau, \quad t \in [0,T], \tag{32}$$

where C_{47}, C_{48} are determined by T, C_{43}, C_{44}. The solution of (32) is given by

$$v(t) \le C_{47} e^{C_{48} t}, \quad t \in [0,T]. \tag{33}$$

Hence, we have the following inequality for the solution of (30):

$$b_2(t) \le \left(\frac{C_{35}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + C_{42} \right) \exp\left(\left(\frac{C_{40}}{\sqrt{\min\limits_{[0,T]} b_1(t)}} + C_{41} \right) t \right), \quad t \in [0,T]. \tag{34}$$

Return to the equation (24) and estimate $a_1(t)$ from below using (16), (23), (25):

$$b_1(t) \ge \frac{C_{49} t^{\frac{1-\beta_1}{2}} \sqrt{\min\limits_{[0,T]} b_1(t)}}{C_{27} t^{\frac{1-\beta_1}{2}} + C_{28} t^{\beta_2 - \beta_1/2} \int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{29} t^{\beta_2} \sqrt{\min\limits_{[0,T]} b_1(t)} \int\limits_0^t b_2(\tau)d\tau}.$$

In virtue of (26) and assumption $\beta_2 = \frac{\beta_1+1}{2}$, we get

$$b_1(t) \ge \frac{C_{49} \sqrt{\min\limits_{[0,T]} b_1(t)}}{C_{27} + C_{28}\sqrt{t} \int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{50} t^{\beta_1} \int\limits_0^t b_2(\tau)d\tau}. \tag{35}$$

Note that $C_{28}\sqrt{t} \int\limits_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{50} t^{\beta_1} \int\limits_0^t b_2(\tau)d\tau \to 0$ as $t \to 0$. Then $\exists T_0 \in (0,T]$:

$$C_{28}\sqrt{t} \int_0^t \frac{b_2(\tau)d\tau}{\sqrt{t-\tau}} + C_{50} t^{\beta_1} \int_0^t b_2(\tau)d\tau \le C_{27}, \quad t \in [0,T_0]. \tag{36}$$

One easily derives from here the estimate

$$b_1(t) \ge B_2 > 0, \quad t \in [0,T_0]. \tag{37}$$

Applying (37) to (34), we get

$$b_2(t) \le B_3 < \infty, \quad t \in [0,T_0]. \tag{38}$$

Here B_2, B_3 are known constants.

To establish the estimate of $a_2(t)$ from below, we find from (27)

$$a_2(t) \ge \int_0^l u_x(0,y,t)dy \left(\mu'_{32}(t) - \iint_D x f(x,y,t)dx\,dy \right) \Delta^{-1}(t).$$

After (38), (39) the estimate of $\Delta(t)$ from (23) becomes following:

$$\Delta(t) \leq C_{51} t^{\frac{1-\beta_1}{2}} + C_{52}.$$

Similarly we get from (21):

$$\int_0^l u_x(0,y,t)dy \geq C_{53} t^{\frac{1-\beta_1}{2}}$$

.

Finally, we find

$$b_2(t) \geq B_4 > 0, \quad t \in [0, T_0], \tag{39}$$

where the constant B_4 is defined by given data.

4 Existence of solution of the problem (1)-(5)

Rewrite the equation (1) as

$$u_t = b_1(t) t^{\beta_1} u_{xx} + b_2(t) t^{\beta_2} u_{yy} + f(x,y,t), \quad (x,y,t) \in Q_T, \tag{40}$$

and consider the inverse problem (40), (2)-(5) with unknowns $(b_1(t), b_2(t), u(x,y,t))$.

Theorem. *Suppose that in addition to* **(A1)**, **(A2)**, *the following assumptions hold:*

(A3) $h\mu_{31}'(t) - \mu_{32}'(t) + \iint\limits_D (x-h) f(x,y,t)dx\,dy > 0$, $\mu_{31}'(t) - \iint\limits_D f(x,y,t)dx\,dy \equiv \varkappa_1(t) t^{\frac{\beta_1+1}{2}}$, $\mu_{32}'(t) - \iint\limits_D x f(x,y,t)dx\,dy \equiv \varkappa_2(t) t^{\frac{\beta_1+1}{2}}$, $\varkappa_i(t) > 0, t \in [0,T], i \in \{1,2\}$;

(A4) *consistency conditions of zero order and conditions* $\iint\limits_D \varphi(x,y)dx\,dy = \mu_{31}(0)$, $\iint\limits_D x\varphi(x,y)dx\,dy = \mu_{32}(0), \quad t \in (0,T]$.

If $\beta_2 = \frac{\beta_1+1}{2}$, *then there exists a solution* $(b_1(t), b_2(t), u(x,y,t))$ *of the problem (40), (2)-(5) defined for* $(x,y) \in \overline{D}, t \in [0,T_0]$ *such that* $(b_1, b_2, u) \in \left(C([0,T_0])\right)^2 \times C^{2,2,1}(D \times (0,T_0)) \cap C^{1,0,0}(\overline{D} \times (0,T_0]), b_i(t) > 0, t \in [0,T_0], i \in \{1,2\}$. *The number* $T_0 \in (0,T]$ *is determined by given data.*

Proof. Reduce the system (24), (27) to the form

$$b_1(t) = t^{\frac{1-\beta_1}{2}} \varkappa_1(t) \int_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx \left(\frac{\mu'_{32}(t) - \iint_D x f(x,y,t)dx\,dy,}{\mu'_{31}(t) - \iint_D f(x,y,t)dx\,dy} - \frac{\int_0^h x(\mu_{22}(x,t) - \mu_{21}(x,t))dx}{\int_0^h (\mu_{22}(x,t) - \mu_{21}(x,t))dx} \right) \Delta^{-1}(t), \quad t \in (0,T], \tag{41}$$

$$b_2(t) = \Big((h\varkappa_1(t) - \varkappa_2(t)) \int_0^l u_x(h,y,t)dy + \varkappa_2(t) \int_0^l u_x(0,y,t)dy + \varkappa_1(t) \int_0^l (\mu_{11}(y,t) - \mu_{12}(y,t))dy \Big) \Delta^{-1}(t), \quad t \in (0,T]. \tag{42}$$

Evidently, the system (41), (42) is equivalent to the problem (40), (2)-(5). On the other hand, the estimates (26), (38)-(40) remain valid for its solution on $[0, T_0]$. Consider the system (41), (42) as an operator equation

$$\omega = P\omega, \tag{43}$$

where $(b_1, b_2) := \omega, (P_1, P_2) := P$ and operators P_1, P_2 are defined by the right hand sides of the equations (41), (42), respectively. Consider the equation (43) in the domain $\mathcal{N} := \{(b_1, b_2) \in (C([0, T_0]))^2 : B_2 \le b_1 \le B_1, B_4 \le b_2 \le B_3\}$ of the Banach space $(C([0, T_0]))^2$. In virtue of estimates (26), (38)-(40), the operator P maps $\mathcal{N}$ onto itself. As it is shown in [7], [11], the operator is compact on $\mathcal{N}$. Then it follows from Schauder fixed-point theorem that the operator has at least one fixed point in $\mathcal{N}$. It means that the system (41), (42) has a solution $(b_1, b_2) \in (C([0, T_0]))^2$. Putting it in (8), we get a solution of the direct problem (40), (2)-(4) with appropriate smoothness [12]. □

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Микола Іванчов, Віталій Власов *Обернена задача для двовимірного рівняння теплопровідності зі сильним виродженням. Випадок інтегральних умов перевизначення.* // Буковинський матем. журнал — 2019. — Т.7, №1. — С. 32–47.

Розглядається існування класичного розв'язку оберненої задачі знаходження двох залежних від часу коефіцієнтів у двовимірному рівнянні теплопровідності. Припускаємо, що невідомі коефіцієнти зникають у початковий момент часу як степенева функція із показником, більшим 1. Існування доводиться з допомогою теореми Шаудера про нерухому точку.