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REGULAR GROWTH OF FOURIER COEFFICIENTS OF THE LOGARITHMIC DERIVATIVE OF ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH

We establish a criterion for the improved regular growth of entire functions of positive order with zeros on a finite system of rays in terms of Fourier coefficients of their logarithmic derivative.

Key words and phrases: entire function of completely regular growth, entire function of improved regular growth, logarithmic derivative, Fourier coefficients, finite system of rays.

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1 Introduction

Let f be an entire function, let $f(0)=1$, let $F(z):=zf'(z)/f(z)$, $z=re^{i\varphi}$, let (λ_n) be the sequence of its zeros, let $\Omega=\{|\lambda_n| : n\in\mathbb{N}\}$, let p be the least nonnegative integer number for which $\sum_{n\in\mathbb{N}}|\lambda_n|^{-p-1}<+\infty$, let $n_k(r,f):=\sum\limits_{|\lambda_n|\le r} e^{-ik\arg\lambda_n}$, $k\in\mathbb{Z}$, let $n(r,\psi;f):=\sum\limits_{|\lambda_n|\le r,\,\arg\lambda_n=\psi} 1$, let Q_ρ be the coefficient of z^ρ in the exponential factor in the Hadamard–Borel representation ([12, p. 24]) of an entire function f of order $\rho\in(0,+\infty)$, and let

$$c_k(r,\log|f|):=\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\log|f(re^{i\varphi})|\,d\varphi,\quad k\in\mathbb{Z},\quad r>0,$$

$$c_k(r,F):=\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}F(re^{i\varphi})\,d\varphi,\quad k\in\mathbb{Z},\quad r>0,\quad r\notin\Omega,$$

be a *Fourier coefficients* of the functions $\log|f(re^{i\varphi})|$ and $F(re^{i\varphi})$, respectively. A set $C\subset\mathbb{C}$ is called a C^0-*set* ([12, p. 90]) if it can be covered by a system of disks $\{z : |z-a_k|<s_k\}$, $k\in\mathbb{N}$, satisfying $\sum\limits_{|a_k|\le r} s_k=o(r)$ as $r\to+\infty$. A set $E\subset[0,+\infty)$ is called a E_0-*set* ([12, p. 96]) if $\operatorname{mes}(E\cap[0,r])=o(r)$ as $r\to+\infty$.

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An entire function f of order $\rho \in (0,+\infty)$ with the indicator $h(\varphi)$ is called an *entire function of completely regular growth* in the sense of Levin and Pfluger ([12, p. 139]) if there exists a C^0-set such that

$$\log|f(re^{i\varphi})| = r^\rho h(\varphi) + o(r^\rho), \quad C^0 \not\ni re^{i\varphi} \to \infty,$$

uniformly in $\varphi \in [0,2\pi)$. Numerous investigations have been devoted to the development of the Levin-Pfluger theory of entire functions and generalization of its results to other classes of functions (see [1, 3, 11, 12]). At present, many different conditions are known that are necessary and sufficient for the completely regular growth of entire functions. In particular, from [2, 4, 5] it follows a criterion for the completely regular growth of entire functions of positive order in terms of Fourier coefficients of their logarithmic derivative.

Theorem A ([2, 4, 5])**.** *For an entire function f of order $\rho \in (0,+\infty)$ to be a function of completely regular growth, it is necessary and sufficient that for all $k \in \mathbb{Z}$*

$$c_k(r,F) = d_k r^\rho + o(r^\rho), \quad r \to +\infty, \quad r \notin E_0, \quad d_k \in \mathbb{C}.$$

In [7, 15] (see also [8, 9, 10, 14]), the notion of entire function of improved regular growth was introduced, and a criterion for this regularity was obtained in terms of the distribution of zeros under the condition that they are located on a finite system of rays. In [6], this notion was generalized to subharmonic functions. Criterion for the improved regular growth of entire functions of positive order with zeros on a finite system of rays in terms of their Fourier coefficients was established in [8]. Asymptotic behavior of entire functions of improved regular growth with zeros on a finite system of rays in the metric of $L^p[0,2\pi]$ was described in [10].

An entire function f is called a function of *improved regular growth* ([7, 8, 9, 10, 14, 15]) if for certain $\rho \in (0,+\infty)$ and $\rho_1 \in (0,\rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that

$$\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

If an entire function f is of improved regular growth, then it has the order ρ and indicator $h(\varphi)$ ([15]).

The aim of the present paper is to establish an analog of Theorem A for the class of entire functions of improved regular growth with zeros on a finite system of rays. Our main result is the following theorem.

Theorem 1. *An entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for certain $\rho_2 \in (0,\rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0+1,\dots,k_0+m-1\}$, one has*

$$c_k(r,F) = d_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty, \quad r \notin \Omega, \quad d_k \in \mathbb{C}. \qquad (1)$$

2 Preliminaries

In the proof of Theorem 1, we use the following auxiliary statements.

Lemma 1 ([7, 15])**.** *An entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for a certain $\rho_3 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$*

$$n(t, \psi_j; f) = \Delta_j t^\rho + o(t^{\rho_3}), \quad t \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{2}$$

and, in addition, for $\rho \in \mathbb{N}$ and certain $\rho_4 \in (0, \rho)$ and $\delta_f \in \mathbb{C}$, one has

$$\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_4 - \rho}), \quad r \to +\infty. \tag{3}$$

In this case,

$$h(\varphi) = \sum_{j=1}^{m} h_j(\varphi), \quad \rho \in (0, +\infty) \setminus \mathbb{N},$$

where $h_j(\varphi)$ is the 2π-periodic function defined on the interval $[\psi_j, \psi_j + 2\pi)$ by the equality $h_j(\varphi) = \frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi - \psi_j - \pi)$. In the case $\rho \in \mathbb{N}$, we have

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^{m} h_j(\varphi), & \rho = p, \\ Q_\rho \cos\rho\varphi, & \rho = p + 1, \end{cases}$$

where $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_j(\varphi)$ is the 2π-periodic function defined on the interval $[\psi_j, \psi_j + 2\pi)$ by the equality $h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j)\sin\rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho}\cos\rho(\varphi - \psi_j)$.

Lemma 2 ([8])**.** *Let f be an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$. If f is of improved regular growth, then for a certain $\rho_5 \in (0, \rho)$ and each $k \in \mathbb{Z}$, one has*

$$c_k(r, \log|f|) = c_k r^\rho + o(r^{\rho_5}), \quad r \to +\infty, \tag{4}$$

where

$$c_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi = \frac{\rho}{\rho^2 - k^2}\sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, \quad \Delta_j \in [0, +\infty), \tag{5}$$

if ρ is a noninteger number, and

$$c_k = \begin{cases} \dfrac{\rho}{\rho^2 - k^2}\displaystyle\sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, & |k| \ne \rho = p, \\ \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho}\displaystyle\sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j}, & k = \rho = p, \\ 0, \quad |k| \ne \rho = p + 1, \\ \dfrac{Q_\rho}{2}, \quad k = \rho = p + 1, \end{cases} \tag{6}$$

if $\rho \in \mathbb{N}$. Conversely, if for certain $\rho_5 \in (0, \rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$, relation (4) with c_k defined by (5) and (6) be true, then f is an entire function of improved regular growth.

3 Proof of Theorem 1

Necessity. Let f be an entire function of improved regular growth of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$. Then, by Lemma 1, for a certain $\rho_3 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$ holds (2) and, according to Lemma 2, for a certain $\rho_5 \in (0, \rho)$ and each $k \in \mathbb{Z}$, one has (4) with c_k defined by (5) and (6). In view of this, since

$$n_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} n(r, \psi_j; f), \quad k \in \mathbb{Z},$$

and ([13, p. 43])

$$c_k(r, F) = n_k(r, f) + k^2 \int_0^r \frac{c_k(t, \log|f|)}{t}\, dt + k c_k(r, \log|f|), \quad k \in \mathbb{Z}, \quad r \notin \Omega,$$

then using (2), (4)–(6), for a certain $\rho_2 \in (0, \rho)$ and each $k \in \mathbb{Z}$, we obtain

$$c_k(r, F) = d_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty, \quad r \notin \Omega,$$

where

$$d_k = \frac{\rho}{\rho - k} \sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, \tag{7}$$

if ρ is a noninteger number, and (for $\rho = p + 1$ equality (2) holds with $\Delta_j = 0$, because $\sum_{n \in \mathbb{N}} |\lambda_n|^{-p-1} < +\infty$)

$$d_k = \begin{cases} \dfrac{\rho}{\rho - k} \displaystyle\sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, & |k| \ne \rho = p, \\ \rho \tau_f e^{i\theta_f} + \dfrac{1}{2} \displaystyle\sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j}, & k = \rho = p, \\ 0, \quad |k| \ne \rho = p + 1, \\ \rho Q_\rho, \quad k = \rho = p + 1, \end{cases} \tag{8}$$

if $\rho \in \mathbb{N}$. Thus, the relation (1) holds.

Sufficiency. Let equality (1) is true. Then, using (1) and the relation ([13, p. 43])

$$n_k(r, f) = c_k(r, F) - k \int_0^r \frac{c_k(t, F)}{t}\, dt, \quad k \in \mathbb{Z},$$

for certain $\rho_2 \in (0, \rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$, we obtain

$$n_k(r, f) = d_k r^\rho - k \int_0^r (d_k t^{\rho - 1} + o(t^{\rho_2 - 1}))\, dt + o(r^{\rho_2}) = d_k (1 - k/\rho) r^\rho + o(r^{\rho_2}), \tag{9}$$

as $\Omega \not\ni r \to +\infty$, where d_k are defined by (7) and (8). Further, without loss of generality, we can assume that $k_0 = 0$. Then, by analogy with [8, p. 1957] (see also [11, p. 127]), for $k \in \{0, 1, \dots, m - 1\}$ we get

$$n_0(r, f) = n(r, \psi_1; f) + n(r, \psi_2; f) + \dots + n(r, \psi_m; f),$$

$$n_1(r,f) = e^{-i\psi_1}n(r,\psi_1;f) + e^{-i\psi_2}n(r,\psi_2;f) + \ldots + e^{-i\psi_m}n(r,\psi_m;f),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$n_{m-1}(r,f) = e^{-i(m-1)\psi_1}n(r,\psi_1;f) + e^{-i(m-1)\psi_2}n(r,\psi_2;f) + \ldots + e^{-i(m-1)\psi_m}n(r,\psi_m;f).$$

This is a system of linear equations for the unknowns $n(r,\psi_j;f)$, $j \in \{1,\ldots,m\}$. Its determinant is the nonzero Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ e^{-i\psi_1} & e^{-i\psi_2} & \ldots & e^{-i\psi_m} \\ \ldots & \ldots & \ldots & \ldots \\ e^{-i(m-1)\psi_1} & e^{-i(m-1)\psi_2} & \ldots & e^{-i(m-1)\psi_m} \end{vmatrix} \neq 0.$$

Therefore, the functions $n(r,\psi_j;f)$, $j \in \{1,\ldots,m\}$, can be represented as linear combinations of the functions $n_k(r,f)$, $k \in \{0,1,\ldots,m-1\}$. Solving this system by the Cramer rule and using (9), we obtain

$$n(r,\psi_j;f) = \Delta_j r^\rho + o(r^{\rho_3}), \quad r \to +\infty, \quad r \notin \Omega,$$

for a certain $\rho_3 \in (0,\rho)$ and each $j \in \{1,\ldots,m\}$. Since the functions $n(r,\psi_j;f)$ are continuous on $[0,+\infty) \setminus \Omega$, we get relation (2). Let us now prove the equality (3). Since ([13, p. 43])

$$c_k(r,F) = 2kc_k(r,\log|f|) + \sum_{|\lambda_n|\le r} \left(\frac{\overline{\lambda}_n}{r}\right)^k, \quad k \in \mathbb{N},$$

and for $k = \rho = p$ we have ([7, p. 21])

$$c_\rho(r,\log|f|) = \frac{1}{2}Q_\rho r^\rho + \frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\left(\frac{r}{\lambda_n}\right)^\rho - \left(\frac{\overline{\lambda}_n}{r}\right)^\rho\right),$$

then, using formulas (1), (8) and the identity $\sum_{j=1}^m \Delta_j e^{-i\rho\psi_j} = 0$, $\rho \in \mathbb{N}$, for a certain $\rho_4 \in (0,\rho)$ we get

$$\begin{aligned}\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} =& r^{-\rho}c_\rho(r,F) - \rho Q_\rho = d_\rho - \rho Q_\rho + o(r^{\rho_2-\rho}) \\ =& \rho(\tau_f e^{i\theta_f} - Q_\rho) + o(r^{\rho_4-\rho}) = \delta_f + o(r^{\rho_4-\rho}), \quad r \to +\infty.\end{aligned}$$

Hence, equality (3) holds for $\rho = p$. In the case $\rho = p+1$, condition (3) follows from (2) (see [7, Remark 2, p. 23]). Thus, according to Lemma 1, the entire function f is a function of improved regular growth. This completes the proof of Theorem 1.

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Нехай f — ціла функція, $f(0) = 1$, (λ_n) — послідовність її нулів, $\Omega = \{|\lambda_n| : n \in \mathbb{N}\}$ і $F(z) = zf'(z)/f(z)$, $z = re^{i\varphi}$. Ціла функція f називається функцією покращеного регулярного зростання, якщо для деяких $\rho \in (0, +\infty)$, $\rho_1 \in (0, \rho)$ і 2π-періодичної ρ-тригонометрично опуклої функції $h(\varphi) \not\equiv -\infty$ існує множина $U \subset \mathbb{C}$, яка міститься в об'єднанні кругів із скінченною сумою радіусів така, що $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$. В роботі доведено, що для того щоб ціла функція f порядку $\rho \in (0, +\infty)$ з нулями на скінченній системі променів $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, була функцією покращеного регулярного зростання, необхідно і достатньо, щоб для деяких $\rho_2 \in (0, \rho)$, $k_0 \in \mathbb{Z}$ і кожного $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$, виконувалось

$$c_k(r, F) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} F(re^{i\varphi})\, d\varphi = d_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty, \quad r \notin \Omega, \quad d_k \in \mathbb{C}.$$

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