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ON THE DIFFERENTIAL EQUATION FOR THE GENERALIZED WEIERSTRASS $\wp$ FUNCTION

Знайдено диференціальне рівняння для узагальненої квазі-еліптичної $\wp$ функції Вейєрштрасса $\wp_{\alpha\beta}$.

Ключові слова: еліптична функція, квазі-еліптична функція, $\wp$-функція Вейєрштрасса, узагальнена квазі-еліптична $\wp$-функція Вейєрштрасса.

The differential equation for the generalized quasi-elliptic Weierstrass function $\wp_{\alpha\beta}$ is found.

Keywords: elliptic function, quasi-elliptic function, the Weierstrass $\wp$-function, generalized quasi-elliptic Weierstrass $\wp$-function.

Introduction.

Let ω_1, ω_2 be complex numbers, such that $Im\frac{\omega_2}{\omega_1} > 0$. A meromorphic in $\mathbb{C}$ function g is called **elliptic** [2] if for every $u \in \mathbb{C}$

$$g(u+\omega_1) = g(u), \;\; g(u+\omega_2) = g(u).$$

We deal with quasi-elliptic functions, which are a direct natural generalization of elliptic functions which are widely used, and not only in mathematics. Thus the investigation of such functions and their properties is quite interesting. One of these functions is the Weierstrass $\wp$ function,

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right),$$

$$\omega = m\omega_1 + n\omega_2, \;\; m,n \in \mathbb{Z}.$$

It is basic in the theory of Weierstrass. The Weierstrass $\wp$ function and its derivative play an important role for a representation of elliptic functions. Each elliptic function can be expressed in terms of $\wp$ and $\wp'$.

The Weierstrass $\wp$ function satisfies the nonlinear ordinary differential equation [3],

$$\wp'^2(u) = 4\wp^3(u) - g_2\wp(u) - g_3,$$

where

$$g_2 = 60\sum_{\omega\neq 0}\frac{1}{\omega^4}, \quad g_3 = 140\sum_{\omega\neq 0}\frac{1}{\omega^6}.$$

The expressions g_2 and g_3 are called the elliptic invariants of the function $\wp$ (see [3] for more details).

This differential equation is of great importance in the applications of Weierstrass elliptic function in physics. For a physicist it is sometimes useful to even conceive this differential equation as the definition of Weierstrass elliptic function [5].

The last equation play a great role in the Weierstrass elliptic function expansion method [1], which allows to seek new types of doubly periodic solutions of nonlinear wave equations in mathematical physics. This equation is useful to solve the following nonlinear wave equations: new integrable Davey–Stewartson-type equation, the (2 + 1)- dimensional modified KdV equation, the generalized Hirota equation in 2 + 1 dimensions, the Generalized KdV equation, the (2 + 1)-dimensional modified Novikov–Veselov equations, (2 + 1)-dimensional generalized system of modified KdV equation, the coupled Klein–Gordon equation, and the (2 + 1)-dimensional generalization of coupled nonlinear Schrodinger equation [1]. The Weierstrass elliptic function expansion method and algorithm is also applied to other many nonlinear wave equations, which arise in mathematical physics. It is important to note that this method is based on differential equation, which classic Weierstrass $\wp$ function satisfies.

Also the Weierstrass $\wp$ function and it's differential equation have significant applications in classical mechanics, including a point particle in a cubic, sinusoidal or hyperbolic potential, quantum mechanics, namely the n = 1 Lamé potential (see [5]).

Now let's return to quasi-elliptic functions.

Definition. [4] *Let* $p = e^{i\alpha}$, $q = e^{i\beta}$, *where* $\alpha, \beta \in \mathbb{R}$. *A meromorphic in* $\mathbb{C}$ *function* g *is called* **quasi-elliptic**, *if there exist* $\omega_1, \omega_2 \in \mathbb{C}^*$, $Im\frac{\omega_2}{\omega_1} > 0$, *such that for every* $u \in \mathbb{C}$

$$g(u+\omega_1) = pg(u), \ \ g(u+\omega_2) = qg(u).$$

The class of quasi-elliptic functions is denoted by $\mathcal{QE}$. The concept of quasi-elliptic functions was first introduced in [4]. A quasi-elliptic analogue $\wp_{\alpha\beta}$ of classic Weierstrass $\wp$ function is constructed in [4] in such a way that the classic Weierstrass $\wp$ function turns out to be a partial case of $\wp_{\alpha\beta}$ if $\alpha = \beta = 0$. That is why we call $\wp_{\alpha\beta}$ generalized Weierstrass $\wp$ function. Therefore the problem of finding a differential equation for the generalized Weierstrass function $\wp_{\alpha\beta}$ arises naturally and this is the main purpose of the present paper. Also, it is a logical continuation of [4].

The rest of this work is organized as follows: in Section 2, we give some auxiliary results concerning quasi-elliptic functions. Section 3 is devoted to the construction of differential equation for generalized Weierstrass function $\wp_{\alpha\beta}$. In Section 4 it is shown that this differential equation indeed is a generalization of the classic one.

1. Auxiliary results

Let ω_1, $\omega_2 \in \mathbb{C}^*$ and $Im\frac{\omega_2}{\omega_1} > 0$. We say that two points $u \in \mathbb{C}$ and $v \in \mathbb{C}$, are **congruent modulo the periods** ω_1 and ω_2 or **equivalent** if $u - v = m\omega_1 + n\omega_2$, where $m, n \in \mathbb{Z}$ [3].

In the finite complex plane we take a point u_0, and construct the parallelogram with vertices u_0, $u_0+\omega_1$, $u_0+\omega_1+\omega_2$, $u_0+\omega_2$. The vertex u_0 and the adjacent sides of the boundary, exclusive of their other end points, are considered as belonging to parallelogram, the rest of the boundary being excluded. The resulting point set is called a **period parallelogram**. In other words, points belonging to the parallelogram have a form

$$u_0 + r_1\omega_1 + r_2\omega_2 \quad (0 \le r_1 < 1,\, 0 \le r_2 < 1).$$

By $\prod(u_0)$ we denote this point set.

Theorem A. [3] *For any point* $u \in \mathbb{C}$ *there is a unique point in a period parallelogram which is equivalent to* u.

The following lemmas describe the properties of quasi-elliptic functions. These lemmas are analogues of classic results for elliptic functions. The Lemma 1 being a consequence of Theorem A.

Lemma 1. *Every holomorphic quasi-elliptic function is constant.* **Доведення.** This lemma can be easily proved using the Liouville's theorem.

Lemma 2. *Let* $f \in \mathcal{QE}$. *Then* f *has equal numbers of zeros and poles (counted according to their multiplicities) in every period parallelogram* $\prod(u_0)$. **Доведення.** It follows immediately from the argument principle applied to a period parallelogram.

Consider the function

$$G_{\alpha\beta}(u) = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right)e^{i(m\alpha+n\beta)}, \tag{1}$$

where $\omega_1, \omega_2 \in \mathbb{C}$, $Im\frac{\omega_2}{\omega_1} > 0$, $\omega = m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{R}$. Note that $G_{\alpha\beta}$ is meromorphic in $\mathbb{C}$ (see [4] for more details).

Obviously, G_{00} coincides with the classic Weierstrass $\wp$ function.

Definition. [4] *Let* $\alpha \neq 0 \mod 2\pi$, $\beta \neq 0 \mod 2\pi$. *The function of the form*

$$\wp_{\alpha\beta}(u) = G_{\alpha\beta}(u) + C_{\alpha\beta} =$$

$$= \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right)e^{i(m\alpha+n\beta)} + C_{\alpha\beta},$$

where

$$C_{\alpha\beta} = \frac{G_{\alpha\beta}\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_{\alpha\beta}\left(-\frac{\omega_1}{2}\right)}{e^{i\alpha}-1} =$$

$$= \frac{G_{\alpha\beta}\left(\frac{\omega_2}{2}\right) - e^{i\beta}G_{\alpha\beta}\left(-\frac{\omega_2}{2}\right)}{e^{i\beta}-1}$$

is called **the generalized Weierstrass** $\wp$ **function.**

Remark 1. *For the sake of completeness, in the case* $\alpha = \beta = 0 \mod 2\pi$, *we define* $C_{00} = 0$. *Then* $\wp_{00} = \wp$.

Theorem B. [4] *The fuction* $\wp_{\alpha\beta}$ *is quasi-elliptic with* $p = e^{i\alpha}$, $q = e^{i\beta}$.

Thus, constructed in such a way function $\wp_{\alpha\beta}$ is a kind of quasi-elliptic analogue of the classic Weierstrass $\wp$ function. We will show, that the function $\wp_{\alpha\beta}$ satisfies certain differential equation as the original function $\wp$ does.

2. The differential equation for $\wp_{\alpha\beta}$

It is known [2], that in the neighborhood of the point $u = 0$

$$\frac{1}{(u-\omega)^2} = -\left(\frac{1}{u-\omega}\right)' =$$

$$= -\left(-\frac{1}{\omega}\sum_{k=0}^{+\infty}\left(\frac{u}{\omega}\right)^k\right)' = \frac{1}{\omega}\sum_{k=1}^{+\infty}\frac{ku^{k-1}}{\omega^k},$$

where $\omega = m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}\backslash\{0, 0\}$. Thus,

$$\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} = \frac{2u}{\omega^3} + \frac{3u^2}{\omega^4} + \frac{4u^3}{\omega^5} + \cdots =$$

$$= \sum_{k=2}^{+\infty}\frac{ku^{k-1}}{\omega^{k+1}},\ \omega = m\omega_1 + n\omega_2,\ m, n \in \mathbb{Z}\backslash\{0, 0\}.$$

From this we can conclude that in the neighborhood of the origin the function $\wp_{\alpha\beta}(u)$ has the form

$$\wp_{\alpha\beta}(u) = \frac{1}{u^2} + C_{\alpha\beta} + \sum_{\omega\neq 0}\sum_{k=2}^{+\infty}\frac{ku^{k-1}}{\omega^{k+1}}e^{i(m\alpha+n\beta)}.$$

The above expression also may be written as

$$\wp_{\alpha\beta}(u) = \frac{1}{u^2} + C_{\alpha\beta} + \sum_{k=2}^{+\infty}ku^{k-1}\sum_{\omega\neq 0}\frac{e^{i(m\alpha+n\beta)}}{\omega^{k+1}}.$$

To shorten notations, set

$$A_k = A_k(\alpha, \beta) = \sum_{\omega\neq 0}\frac{e^{i(m\alpha+n\beta)}}{\omega^{k+1}}, \quad k = 2, 3, \ldots$$

Hence, we can rewrite $\wp_{\alpha\beta}$ as follows

$$\wp_{\alpha\beta}(u) = \frac{1}{u^2} + C_{\alpha\beta} + \sum_{k=2}^{+\infty}A_k k u^{k-1}. \qquad (2)$$

Therefore,

$$\wp'_{\alpha\beta}(u) = -\frac{2}{u^3} + \sum_{k=2}^{+\infty}A_k k(k-1)u^{k-2},$$

and

$$\wp'^2_{\alpha\beta}(u) = \frac{4}{u^6} - \frac{4}{u^6}\sum_{k=2}^{+\infty}A_k k(k-1)u^{k-2} +$$

$$+\left(\sum_{k=2}^{+\infty}A_k k(k-1)u^{k-2}\right)^2 =$$

$$= \frac{4}{u^6} - \frac{8A_2}{u^3} - \frac{24A_3}{u^2} - \frac{48A_4}{u} -$$

$$-\frac{4}{u^3}\sum_{k=5}^{+\infty}A_k k(k-1)u^{k-2} +$$

$$+\left(\sum_{k=2}^{+\infty}A_k k(k-1)u^{k-2}\right)^2.$$

Also we need to calculate $\wp^2_{\alpha\beta}(u)$ and $\wp^3_{\alpha\beta}(u)$. We have

$$\wp^2_{\alpha\beta}(u) = \frac{1}{u^4} + C^2_{\alpha\beta} + \left(\sum_{k=2}^{+\infty}A_k k u^{k-1}\right)^2 +$$

$$+\frac{2C_{\alpha\beta}}{u^2} + \frac{2}{u^2}\sum_{k=2}^{+\infty}A_k k u^{k-1} + 2C_{\alpha\beta}\sum_{k=2}^{+\infty}A_k k u^{k-1} =$$

$$= \frac{1}{u^4} + \frac{2C_{\alpha\beta}}{u^2} + \frac{4A_2}{u} + C^2_{\alpha\beta} + \left(\sum_{k=2}^{+\infty}A_k k u^{k-1}\right)^2 +$$

$$+\frac{2}{u^2}\sum_{k=3}^{+\infty}A_k k u^{k-1} + 2C_{\alpha\beta}\sum_{k=2}^{+\infty}A_k k u^{k-1}.$$

Similarly,

$$\wp^3_{\alpha\beta}(u) = \frac{1}{u^6} + \frac{3C_{\alpha\beta}}{u^4} + \frac{6A_2}{u^3} + (9A_3 + 3C^2_{\alpha\beta})\frac{1}{u^2} +$$

$$+(12A_4 + 12A_2C_{\alpha\beta})\frac{1}{u} + \frac{3}{u^4}\sum_{k=5}^{+\infty}A_k k u^{k-1} +$$

$$+(3C_{\alpha\beta}+\frac{1}{u^2})\left(\sum_{k=2}^{+\infty}A_k k u^{k-1}\right)^2+C_{\alpha\beta}^3+$$

$$+\left(\sum_{k=2}^{+\infty}A_k k u^{k-1}\right)^3+(3C_{\alpha\beta}^2+\frac{4A_2}{u})\sum_{k=2}^{+\infty}A_k k u^{k-1}+$$

$$+\frac{6C_{\alpha\beta}}{u^2}\sum_{k=3}^{+\infty}A_k k u^{k-1}+\frac{2}{u^2}\sum_{k=2}^{+\infty}A_k k u^{k-1}\sum_{k=3}^{+\infty}A_k k u^{k-1}.$$

In view of these expansions,

$$\wp_{\alpha\beta}'^2(u)-4\wp_{\alpha\beta}^3(u)+12C_{\alpha\beta}\wp_{\alpha\beta}^2(u)-$$
$$-16A_2\wp_{\alpha\beta}'(u)+(60A_3+12C_{\alpha\beta}^2-$$
$$-24C_{\alpha\beta})\wp_{\alpha\beta}(u)=$$
$$=(48A_2-96A_4-48A_2C_{\alpha\beta})\frac{1}{u}+H(u),$$

where H is an entire function of the form

$$H(u)=2A_2(1-16A_2)-140A_5+$$
$$+8C_{\alpha\beta}^3-24C_{\alpha\beta}^2+60A_3C_{\alpha\beta}+$$
$$+\sum_{k=1}^{+\infty}\{(1-16A_2)A_{k+2}(k+2)(k+1)-$$
$$-4A_{k+5}(k+7)(k+5)\}u^k-$$
$$-4\left(\sum_{k=1}^{+\infty}A_{k+1}(k+1)u^k\right)^3-$$
$$-\frac{4}{u^2}\left(\sum_{k=1}^{+\infty}A_{k+1}(k+1)u^k\right)^2+$$
$$+(60A_3-\frac{16A_2}{u}-8\sum_{k=0}^{+\infty}A_{k+3}(k+3)u^k-$$
$$-4C_{\alpha\beta}^3+36C_{\alpha\beta}^2-24C_{\alpha\beta}^2)\sum_{k=1}^{+\infty}A_{k+1}(k+1)u^k.$$

In other words, the function $\wp_{\alpha\beta}$ satisfies the differential equation

$$\wp_{\alpha\beta}'^2(u)=4\wp_{\alpha\beta}^3(u)-12C_{\alpha\beta}\wp_{\alpha\beta}^2(u)+$$
$$+16A_2\wp_{\alpha\beta}'(u)-(60A_3+12C_{\alpha\beta}^2-$$
$$-24C_{\alpha\beta})\wp_{\alpha\beta}(u)+$$
$$+(48A_2-96A_4-48A_2C_{\alpha\beta})\frac{1}{u}+H(u), \quad (3)$$

where H is an entire function given above.

On the connection with the classic differential equation for $\wp$

The important point to note here that the differential equation for $\wp_{\alpha\beta}$ is a generalization of the classic one for $\wp$.

Consider the case $\alpha=\beta=0 \mod 2\pi$. Since $C_{00}=0$, $A_{2k}(0,0)=\sum_{\omega\neq 0}\frac{1}{\omega^{2k+1}}=0$ for $k\in\mathbb{N}$, and $A_3(0,0)=\frac{g_2}{60}$, then equation (3) takes the form

$$\wp'^2(u)=4\wp^3(u)-g_2\wp(u)+H(u), \quad (4)$$

or

$$\wp'^2(u)-4\wp^3(u)+g_2\wp(u)=H(u), \quad (5)$$

The function on the left hand side of (5) is elliptic. Since H is holomorphic, then by Lemma 1, H is constant. Thus, $H(u)=H(0)$ for all u. Since $H(0)=-140A_5=-g_3$, then (4) can be rewritten as follows

$$\wp'^2(u)=4\wp^3(u)-g_2\wp(u)-g_3,$$

which is the classic differential equation for the classic Weierstrass $\wp$ function.

REFERENCES

1. *Chen, Y., Yan, Zh.* (2006). The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations: Solitons and Fractals, 29, 948-964.

2. *Hellegouarch, Y.* (2002). Invitation to the Mathematics of Fermat-Wiles. Academic Press.

3. *Hurwitz, A.* (1968). Function theory. Moscow. Nauka (in Russian).

4. *Khrystiyanyn, A., Lukivska, Dz.* (2017). Quasi-elliptic functions: Ufa Math. J., 9(4), 129-136.

5. *Pastras, G.* (2017). Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics: arXiv:1706.07371v2 [math-ph] 6 Sep 2017.