УДК 517.55

©2018 **A.I. Bandura[1], O.B. Skaskiv[2], V. L. Tsvigun[2]**

[1] Ivano-Frankivsk National Technical University of Oil and Gas
[2] Ivan Franko National University of Lviv

SOME CHARACTERISTIC PROPERTIES OF ANALYTIC FUNCTIONS IN $\mathbb{D}\times\mathbb{C}$ OF BOUNDED L-INDEX IN JOINT VARIABLES

Вивчаються характеристичні властивості функцій обмеженого **L**-індексу за сукупністю змінних, які аналітичні в $\mathbb{D}\times\mathbb{C}$. Отримані твердження є аналогами відомих критеріїв для функцій, які аналітичні в одиничній кулі, в полікрузі та для цілих функцій від декількох змінних. Вони описують оцінки максимуму модуля функції обмеженого **L**-індексу за сукупністю змінних у бікрузі. Зокрема, встановлено аналог теореми Хеймана для цього класу функцій, яка має застосування в аналітичній теорії диференціальних рівнянь до аналітичних розв'язків в одиничній кулі та у полікрузі, а також до цілих розв'язків. Також формулюємо дві нерозв'язані задачі про оцінки зростання для цих функцій, а також їхнє застосування до систем рівнянь з частинним похідними.

Ключові слова: аналітичні функції, теорема Хеймана.

We investigate the characteristic properties of functions of bounded of **L**-index in joint variables which are analytic in $\mathbb{D}\times\mathbb{C}$. The obtained propositions are analogs of known criterion for analytic functions in the unit ball, in the polydisc and for entire functions of several variables. They described estimates maximum modulus of the function of bounded of **L**-index in joint variables in a bidisc. Particularly, we obtained analog of Hayman's Theorem for this functions class. The theorem has applications in analytic theory of differential equations to analytic solutions in the unit ball, in the polydisc and to entire solutions. Also we posed two unsolved problems about growth estimates for these functions and its applications to system of partial differential equations.

Keyword: analytic functions, Hayman's Theorem.

1. Definition and notations. We need some standard notation. Denote $\mathbb{R}_+ = (0,+\infty)$, $\mathbf{0} = (0,0)$, $\mathbf{1} = (1,1)$, $R = (r_1,r_2) \in \mathbb{R}^2_+$, $z = (z_1,z_2) \in \mathbb{D}\times\mathbb{C}$. For $A = (a_1,a_2) \in \mathbb{R}^2$, $B = (b_1,b_2) \in \mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions $AB = (a_1b_1,a_2b_2)$, $A/B = (a_1/b_1,a_2/b_2)$, $A^B = a_1^{b_1}a_2^{b_2}$. The notation $A < B$ means that $a_j < b_j$, $j \in \{1,2\}$; the relation $A \le B$ is defined similarly. For $K = (k_1,k_2) \in \mathbb{Z}^2_+$ denote $\|K\| = k_1 + k_2$, $K! = k_1!\cdot k_2!$.

For $z^0 \in \mathbb{C}^2$ we denote $\mathbb{D}^2(z^0,R) := \{z \in \mathbb{C}^2\colon\ |z_j - z_j^0| < r_j,\ j \in \{1,2\}\}$ the bidisc, its skeleton $\mathbb{T}^2(z^0,R) := \{z \in \mathbb{C}^2\colon\ |z_j - z_j^0| = r_j,\ j \in \{1,2\}\}$, and $\mathbb{D}^2[z^0,R] := \{z \in \mathbb{C}^2\colon\ |z_j - z_j^0| \le r_j,\ j \in \{1,2\}\}$ the closed bidisc, $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$. For $K = (k_1,k_2) \in \mathbb{Z}^2_+$, $z \in \mathbb{C}^2$ and the partial derivatives of function $F(z) = F(z_1,z_2)$ we use the following notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|}F(z)}{\partial z^K} = \frac{\partial^{k_1+k_2}F(z)}{\partial z_1^{k_1}\partial z_2^{k_2}}.$$

Let $\mathbf{L}(z) = (l_1(z),l_2(z))$, where $l_j(z)\colon \mathbb{D}\times\mathbb{C} \to \mathbb{R}_+$ is a continuous function.

In this article, we continue to investigate analytic functions in $\mathbb{D}\times\mathbb{C}$ of bounded **L**-index initiated in articles [10,11]. The concept of bounded index is useful in analytic theory of differential equations. It allows examining properties of analytic solutions of differential equations. If the solution has bounded index then we immediately deduce its growth estimates, local behavior of its derivatives, uniform zero distribution in a some sense and other properties concerned with regular behavior. Of course, there is another method in analytic theory of differential equations - so-called Wiman-Valiron method. But it is applicable mostly for entire solutions of differential equations. In multidimensional case the method requires many additional assumptions which are not always such clear and natural as for the concept of bounded index. Moreover, we do not know an implementation of Wiman-Valiron method for the partial differential equations with coefficients which are analytic in $\mathbb{D}\times\mathbb{C}$. So, in the paper we want to deduce criterion of index boundedness for this class of analytic functions. Particularly, we prove analog of Hayman's Theorem. The theorem is convenient to investigate index boundedness of analytic solutions of linear higher-order differential equations.

An analytic function $F\colon \mathbb{D}\times\mathbb{C}\to\mathbb{C}$ is called ([10, 11]) a function of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0\in\mathbb{Z}_+$ such that for all $z\in\mathbb{D}\times\mathbb{C}$ and for all $J\in\mathbb{Z}_+^2$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}\le \max_{\|K\|\le n_0}\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}. \qquad (1)$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F,\mathbf{L},\mathbb{D}\times\mathbb{C})=n_0$. It is an analog of the definition of an analytic function of bounded $\mathbf{L}$-index in joint variables (see definitions for various classes of analytic functions in [2, 4, 5, 7, 14, 15, 17]).

By $Q(\mathbb{D}\times\mathbb{C})$ we denote the class of functions $\mathbf{L}$ which satisfy the conditions

$$(\forall z\in\mathbb{D}\times\mathbb{C})\colon\ l_1(z)>\beta/(1-|z_2|),$$
$$(\forall r_1\in[0,\beta],\ \forall r_2\in(0,+\infty))\colon$$
$$0<\lambda_{1,j}(R)\le\lambda_{2,j}(R)<+\infty,$$

where $\beta>1$ is a some constant, and

$$\lambda_{1,j}(z_0,R)=\inf_{z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]} l_j(z)/l_j(z^0)$$
$$\lambda_{2,j}(z_0,R)=\sup_{z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]} l_j(z)/l_j(z^0),$$
$$\lambda_{1,j}(R)=\inf_{z^0\in\mathbb{D}\times\mathbb{C}}\lambda_{1,j}(z_0,R),$$
$$\lambda_{2,j}(R)=\sup_{z^0\in\mathbb{D}\times\mathbb{C}}\lambda_{2,j}(z_0,R),\ j\in\{1,2\}.$$

A similar condition was used for other classes of analytic functions of bounded index as one so several variables [13, 19, 21].

2. Main criteria.

Denote $\mathcal{B}^2=(0,\beta]\times(0,+\infty)$, $\boldsymbol{\beta}:=(\beta,\beta)$. We have proved such theorem.

Theorem 1 ([10]). *Let* $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. *An analytic function* $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ *has bounded* $\mathbf{L}$*-index in joint variables if and only if for each* $R\in\mathcal{B}^2$ *there exist* $n_0\in\mathbb{Z}_+$, $p_0>0$ *such that for every* $z^0\in\mathbb{D}\times\mathbb{C}$ *there exists* $K^0\in\mathbb{Z}_+^2$, $\|K^0\|\le n_0$, *and*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le n_0,\right.$$
$$\left. z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}\le p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \qquad (2)$$

In the proofs of the following statements we will use methods developed for entire functions and for analytic functions in a polydisc and in the unit ball [1–5, 7, 8].

Theorem 2. *Let* $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. *In order that an analytic function* $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ *be of bounded* $\mathbf{L}$*-index in joint variables it is necessary that for every* $R\in\mathcal{B}^2$ $\exists n_0\in\mathbb{Z}_+$ $\exists p\ge 1$ $\forall z^0\in\mathbb{D}\times\mathbb{C}$ $\exists K^0\in\mathbb{Z}_+^2$, $\|K^0\|\le n_0$, *and*

$$\max\left\{|F^{(K^0)}(z)|\colon z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}\le$$
$$\le p|F^{(K^0)}(z^0)| \qquad (3)$$

and it is sufficient that for every $R\in\mathcal{B}^2$ $\exists n_0\in\mathbb{Z}_+$ $\exists p\ge 1$ $\forall z^0\in\mathbb{D}\times\mathbb{C}$ $\exists$ $K_1^0=(k_1^0,0)$ *and* $K_2^0=(0,k_2^0)$ *such that* $k_1^0\le n_0$, $k_2^0\le n_0$ *and*

$$\max\left\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}\le$$
$$\le p|F^{(K_j^0)}(z^0)|\ \forall j\in\{1,2\}, \qquad (4)$$

Proof. The proof of Theorem 1 implies that the inequality (2) is true for some K^0. Therefore, we have

$$\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}\ge$$
$$\ge\max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z)}\colon z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}=$$
$$=\max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z^0)\mathbf{L}^{K^0}(z)}\colon\right.$$
$$\left. z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}\ge$$
$$\ge\max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\prod_{j=1}^2(\lambda_{2,j}(R))^{-n_0}}{\mathbf{L}^{K^0}(z^0)}\colon\right.$$
$$\left. z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}.$$

This inequality implies

$$\frac{p_0\prod_{j=1}^{2}(\lambda_{2,j}(R))^{n_0}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}\geq$$

$$\geq\max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z^0)}\colon z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}. \tag{5}$$

From (5) we obtain inequality (3) with $p=p_0\prod_{j=1}^{2}\left(\lambda_{2,j}(R)\right)^{n_0}$. The necessity of condition (3) is proved.

Now we prove the sufficiency of (4). We will use methods which are developed for analytic functions in the bidisc [8] and in the unit ball [1]. Suppose that for every $R\in\mathcal{B}^2$ $\exists n_0\in\mathbb{Z}_+, p>1$ such that $\forall z_0\in\mathbb{D}\times\mathbb{C}$ and some $K_J^0\in\mathbb{Z}_+^2$ with $k_j^0\leq n_0$ the inequality (4) holds.

We write Cauchy's formula as following $\forall z^0\in\mathbb{D}\times\mathbb{C}$ $\forall s\in\mathbb{Z}_+^2$

$$\frac{F^{(K_J^0+S)}(z^0)}{S!}=$$

$$=\frac{1}{(2\pi i)^2}\int\limits_{\mathbb{T}^2(z^0,R/\mathbf{L}(z^0))}\frac{F^{(K_J^0)}(z)}{(z-z^0)^{S+\mathbf{1}}}dz.$$

This yields

$$\frac{|F^{(K_j^0+S)}(z^0)|}{S!}\leq$$

$$\leq\frac{1}{(2\pi)^2}\int\limits_{\mathbb{T}^2(z^0,R/\mathbf{L}(z^0))}\frac{|F^{(K_j^0)}(z)|}{|z-z^0|^{S+\mathbf{1}}}|dz|\leq$$

$$\leq\frac{1}{(2\pi)^2}\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]\}\times$$

$$\times\int\limits_{\mathbb{T}^2(z^0,R/\mathbf{L}(z^0))}\frac{\mathbf{L}^{S+\mathbf{1}}(z^0)}{R^{S+\mathbf{1}}}|dz|=$$

$$=\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]\}\frac{\mathbf{L}^S(z^0)}{R^S}.$$

Now we put $R=\boldsymbol{\beta}$ and use (4)

$$\frac{|F^{(K_j^0+S)}(z^0)|}{S!}\leq\frac{\mathbf{L}^S(z^0)}{\beta^{\|S\|}}\times$$

$$\times\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]\}\leq$$

$$\leq\frac{p\mathbf{L}^S(z^0)}{\beta^{\|S\|}}|F^{(K_j^0)}(z^0)|. \tag{6}$$

We choose $S\in\mathbb{Z}_+^2$ such that $\|S\|\geq s_0$, where $\frac{p}{\beta^{s_0}}\leq 1$. Therefore (6) implies that for all $j\in\{1,2\}$ and $k_j^0\leq n_0$

$$\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!}\leq$$

$$\leq\frac{p}{\beta^{\|S\|}}\frac{S!K_j^0!}{(S+K_j^0)!}\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}\leq$$

$$\leq\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}.$$

Consequently, $N(F,\mathbf{L},\mathbb{D}\times\mathbb{C})\leq n_0+s_0$.

Denote $\widetilde{\mathbf{L}}(z)=(\widetilde{l}_1(z),\widetilde{l}_2(z))$. $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ means that there exist $\Theta_1=(\theta_{1,j},\theta_{1,2})\in\mathbb{R}_+^2$, $\Theta_2=(\theta_{2,j},\theta_{2,2})\in\mathbb{R}_+^2$ such that $\forall z\in\mathbb{D}\times\mathbb{C}$ $\theta_{1,j}\widetilde{l}_j(z)\leq l_j(z)\leq\theta_{2,j}\widetilde{l}_j(z),\ \ j\in\{1,2\}$.

Theorem 3. *Let $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$. An analytic function $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if it has bounded $\mathbf{L}$-index in joint variables.*

Proof. It is easy to prove that if $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ then $\widetilde{\mathbf{L}}\in Q(\mathbb{D}\times\mathbb{C})$.

Let $N(F,\widetilde{\mathbf{L}},\mathbb{D}\times\mathbb{C})=\widetilde{n}_0<+\infty$. Then by Theorem 1 for every $\widetilde{R}=(\widetilde{r}_1,\ldots,\widetilde{r}_n)\in\mathcal{B}^2$ there exists $\widetilde{p}\geq 1$ such that for each $z^0\in\mathbb{D}\times\mathbb{C}$ and some K^0 with $\|K^0\|\leq\widetilde{n}_0$, the inequality (2) holds with $\widetilde{\mathbf{L}}$ and $\widetilde{R}$ instead of $\mathbf{L}$ and R. Hence

$$\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}=\frac{\widetilde{p}}{K^0!}\frac{\Theta_2^{K^0}|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\mathbf{L}^{K^0}(z^0)}\geq$$

$$\geq\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\widetilde{\mathbf{L}}^{K^0}(z^0)}\geq$$

$$\geq\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{L}^K(z)}\colon\|K\|\leq\widetilde{n}_0,\right.$$

$$\left.z\in\mathbb{D}^2\left[z^0,\widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\}\geq$$

$$\geq\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{\Theta_1^K|F^{(K)}(z)|}{K!L^K(z)}\colon\|K\|\leq\widetilde{n}_0,\right.$$

$$\left.z\in\mathbb{D}^2\left[z^0,\widetilde{R}/\widetilde{\mathbf{L}}(z)\right]\right\}\geq$$

$$\geq\frac{\min\limits_{0\leq\|K\|\leq n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!L^K(z)}\colon\|K\|\leq\widetilde{n}_0,\right.$$

$$z\in\mathbb{D}^2\Big[z^0,\widetilde{R}/\widetilde{\mathbf{L}}(z)\Big]\Big\}.$$

In view of Theorem 1 we obtain that the function F has bounded $\mathbf{L}$-index.

Theorem 4. *Let $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. An analytic function $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $R\in\mathcal{B}^2$, $n_0\in\mathbb{Z}_+$, $p_0>1$ such that for each $z^0\in\mathbb{D}^2(z^0,R)$ and for some $K^0\in\mathbb{Z}_+^2$ with $\|K^0\|\le n_0$ the inequality* (2) *holds.*

Proof. The necessity of this theorem follows from the necessity of Theorem 1. We prove the sufficiency. The proof of Theorem 1 with $R=\boldsymbol{\beta}$ implies that $N(F,\mathbf{L},\mathbb{D}\times\mathbb{C})<+\infty$.

The formulation of the theorem uses idea about replacing of universal quantifier by existential quantifier. The replacement relaxes the sufficient conditions of index boundedness (see its implementation for entire functions in [9]). Let $\mathbf{L}^*(z)=\frac{R_0\mathbf{L}(z)}{R}$, $R^0=\boldsymbol{\beta}$. In general case from validity of (2) for F and $\mathbf{L}$ with $R=(r_1,\dots,r_n)$, $r_j<\beta$, $j\in\{1,2\}$ we obtain

$$\max\left\{\frac{|F^{(K)}(z)|}{K!(R_0\mathbf{L}(z)/R)^K}\colon \|K\|\le n_0,\right.$$
$$\left. z\in\mathbb{D}^2\left[z^0,R_0/\mathbf{L}^*(z^0)\right]\right\}\le$$
$$\le\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\le n_0,\right.$$
$$\left. z\in\mathbb{D}^2\left[z^0,R/\mathbf{L}(z^0)\right]\right\}\le$$
$$\le\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)}=\frac{\beta^{\|K^0\|}p_0}{R^{K^0}K^0!}\frac{|F^{(K^0)}(z)|}{(R_0\mathbf{L}(z)/R)^{K^0}}<$$
$$<\frac{p_0\beta^{nn_0}}{\prod_{j=1}^2 r_j^{n_0}}\frac{|F^{(K^0)}(z)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.$$

i. e. (2) holds for F, $\mathbf{L}^*$ and $R_0=(\beta,\beta)$. Now as above we apply Theorem 1 to the function $F(z)$ and $\mathbf{L}^*(z)=R_0\mathbf{L}(z)/R$. This implies that F is of bounded $\mathbf{L}^*$-index in joint variables. Therefore, by Theorem 3 the function F has bounded $\mathbf{L}$-index in joint variables.

3. Estimate of maximum modulus.

For an analytic function $F(z)$ we put $M(R,z^0,F)=\max\{|F(z)|\colon z\in\mathbb{T}^2(z^0,R)\}$.

The following Theorems 5 and 7 are given in article [11] without proof.

Theorem 5. *Let $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. If an analytic function $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables then for any $R',R''\in\mathcal{B}^2$, $R'<R''$, there exists $p_1=p_1(R',R'')\ge1$ such that for each $z^0\in\mathbb{D}\times\mathbb{C}$*

$$M\left(R''/\mathbf{L}(z^0),z^0,F\right)\le p_1M\left(R'/\mathbf{L}(z^0),z^0,F\right). \tag{7}$$

Proof. Let $N(F,L,\mathbb{D}\times\mathbb{C})=N<+\infty$. Suppose that inequality (7) does not hold, i.e. there exist R', $R''\in\mathcal{B}^2$, $R'<R''$, such that for each $p_*\ge1$ and some $z^0=z^0(p_*)$

$$M\left(R''/\mathbf{L}(z^0),z^0,F\right)>p_*M\left(R'/\mathbf{L}(z^0),z^0,F\right). \tag{8}$$

By Theorem 2 there exists a number $p_0=p_0(R'')\ge1$ such that for every $z^0\in\mathbb{D}\times\mathbb{C}$ and for some $k^0\in\mathbb{Z}_+^2$, $k_1^0+k_2^0\le N$, one has

$$M\left(R''/\mathbf{L}(z^0),z^0,F^{(k_1^0,k_2^0)}\right)\le p_0|F^{(k_1^0,k_2^0)}(z^0)|. \tag{9}$$

We put

$$b_1=p_0N!\Big(\frac{r_1''r_2''}{r_1'r_2'}\Big)^N\lambda_{2,1}^N(R'')\lambda_{2,2}^N(R'')\sum_{j=1}^N\frac{(N-j)!}{(r_1'')^j},$$

$$b_2=p_0\lambda_{2,2}^N(R'')\sum_{j=1}^N\frac{(N-j)!}{(r_2'')^j}\max\left\{(r_1'')^{-N},1\right\},$$

$$p_*=p_0(N!)^2\left(\frac{r_1''r_2''}{r_1'r_2'}\right)^N+b_1+b_2+1.$$

Let $z^0=z^0(p_*)$ be a point for which inequality (8) holds and k^0 is such for which (9) holds. We choose z^* and $z^*_{(j_1,j_2)}$ such that

$$M\left(R'/\mathbf{L}(z^0),z^0,F\right)=|F(z^*)|,$$
$$M\left(R''/\mathbf{L}(z^0),z^0,F^{(j_1,j_2)}\right)=|F^{(j_1,j_2)}(z^*_{(j_1,j_2)})|$$

for every $j=(j_1,j_2)\in\mathbb{Z}_+^2$, $j_1+j_2\le N$. We apply Cauchy's inequality

$$|F^{(j_1,j_2)}(z^0)|\le j_1!j_2!\left(\frac{l_1(z^0)}{r_1'}\right)^{j_1}\times$$
$$\times\left(\frac{l_2(z^0)}{r_2'}\right)^{j_2}|F(z^*)| \tag{10}$$

for estimate the difference

$$|F^{(j_1,j_2)}(z^*_{j,1},z^*_{j,2})-F^{(j_1,j_2)}(z^0_1,z^*_{j,2})|=$$

$$=\left|\int_{z_1^0}^{z^*_{j,1}} F^{(j_1+1,j_2)}(\zeta,z^*_{j,2})d\zeta\right|\le$$

$$\le\max\left\{|F^{(j_1+1,j_2)}(\zeta,z^*_{j,2})|:|\zeta-z^0_1|=\frac{r''_1}{l_1(z^0)}\right\}\times$$

$$\times\int_{z_1^0}^{z^*_{j,1}}|d\zeta|=|F^{(j_1+1,j_2)}(z^*_{(j_1+1,j_2)})|\frac{r''_1}{l_1(z^0)}. \tag{11}$$

Since $(z^0_1,z^*_{j,2})\in\mathbb{D}^2\left[z^0,R''/\mathbf{L}(z^0)\right]$, for $k\in\{1,2\}$ we have $|z^*_{j,k}-z^0_k|=\frac{r''_k}{l_k(z^0)}$ and $l_k(z^0_1,z^*_{j,2})\le\lambda_{2,k}(R'')l_k(z^0)$. Putting $j=k^0$ in (10), by Theorem 1 we obtain

$$|F^{(j_1,j_2)}(z^0_1,z^*_{j,2})|\le j_1!j_2!p_0|F^{(k^0)}(z^0)|\times$$
$$\times\frac{l_1^{j_1}(z^0_1,z^*_{j,2})l_2^{j_2}(z^0_1,z^*_{j,2})}{k^0_1!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}\le$$
$$\le\frac{j_1!j_2!\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')l_1^{j_1}(z^0)l_2^{j_2}(z^0)}{k^0_1!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}p_0k^0_1!k^0_2!\times$$
$$\times\left(\frac{l_1(z^0)}{r'_1}\right)^{k^0_1}\left(\frac{l_2(z^0)}{r'_2}\right)^{k^0_2}|F(z^*)|\le$$
$$\le j_1!j_2!\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')p_0\frac{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}{(r'_1)^{k^0_1}(r'_2)^{k^0_2}}|F(z^*)|. \tag{12}$$

From inequalities (11) and (12) it follows that

$$|F^{(j_1+1,j_2)}(z^*_{(j_1+1,j_2)})|\ge$$
$$\ge\frac{l_1(z^0)}{r''_1}(|F^{(j_1,j_2)}(z^*_{j,1},z^*_{j,2})|-|F^{(j_1,j_2)}(z^0_1,z^*_{j,2})|)\ge$$
$$\ge\frac{l_1(z^0)}{r''_1}(|F^{(j_1,j_2)}(z^*_{j,1},z^*_{j,2})|-$$
$$-j_1!j_2!\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')p_0\frac{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}{(r'_1)^{k^0_1}(r'_2)^{k^0_2}}|F(z^*)|)=$$
$$=\frac{l_1(z^0)}{r''_1}|F^{(j_1,j_2)}(z^*_{j,1},z^*_{j,2})|-j_1!j_2!\lambda^{j_1}_{2,1}(R'')\times$$
$$\times\lambda^{j_2}_{2,2}(R'')p_0l_1(z^0)\frac{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}{r''_1(r'_1)^{k^0_1}(r'_2)^{k^0_2}}|F(z^*)|.$$

We choose $j=(j_1,j_2)=(k^0_1,k^0_2)$ and deduce

$$|F^{(k^0_1,k^0_2)}(z^*_{k^0})|\ge\frac{l_1(z^0)}{r''_1}|F^{(k^0_1-1,k^0_2)}(z^*_{(k^0_1-1,k^0_2)})|-$$
$$-\frac{p_0(k^0_1-1)!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{r''_1(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\times$$
$$\times\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)|\ge$$
$$\ge\frac{l_1^2(z^0)}{(r''_1)^2}|F^{(k^0_1-2,k^0_2)}(z^*_{(k^0_1-2,k^0_2)})|-$$
$$-\frac{p_0(k^0_1-2)!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r''_1)^2(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\times$$
$$\times\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)|-$$
$$-\frac{p_0(k^0_1-1)!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{r''_1(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\lambda^{j_1}_{2,1}(R'')\times$$
$$\times\lambda^{j_2}_{2,2}(R'')|F(z^*)|\ge$$
$$\ge\ldots\ge\frac{l_1^{k^0_1}(z^0)}{(r''_1)^{k^0_1}}|F^{(0,k^0_2)}(z^*_{0,k^0_2})|-$$
$$-\frac{p_0k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r''_1)^{k^0_1}(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)|-$$
$$-\ldots-\frac{p_0(k^0_1-2)!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r''_1)^2(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\lambda^{j_1}_{2,1}(R'')\times$$
$$\times\lambda^{j_2}_{2,2}(R'')|F(z^*)|-\frac{p_0(k^0_1-1)!k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{r''_1(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\times$$
$$\times\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)|=$$
$$=\frac{l_1^{k^0_1}(z^0)}{(r''_1)^{k^0_1}}|F^{(0,k^0_2)}(z^*_{(0,k^0_2)})|-\frac{p_0k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\times$$
$$\times\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)|\sum_{j_1=1}^{k^0_1}\frac{(k^0_1-j_1)!}{(r''_1)^{j_1}}\ge$$
$$\ge\frac{l_1^{k^0_1}(z^0)}{(r''_1)^{k^0_1}}\frac{l_2^{k^0_2}(z^0)}{(r''_2)^{k^0_2}}|F(z^*_{(0,0)})|-\frac{p_0k^0_2!l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r'_1)^{k^0_1}(r'_2)^{k^0_2}}\times$$
$$\times\lambda^{k^0_1}_{2,1}(R'')\lambda^{k^0_2}_{2,2}(R'')|F(z^*)|\sum_{j_1=1}^{k^0_1}\frac{(k^0_1-j_1)!}{(r''_1)^{j_1}}-$$
$$-\frac{p_0l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r'_1)^{k^0_1}(r'_2)^{k^0_2}(r''_1)^{k^0_1}}\lambda^{k^0_2}_{2,2}(R'')|F(z^*)|\sum_{j_2=1}^{k^0_2}\frac{(k^0_2-j_2)!}{(r''_2)^{j_2}}=$$
$$=\frac{l_1^{k^0_1}(z^0)l_2^{k^0_2}(z^0)}{(r''_1)^{k^0_1}(r''_2)^{k^0_2}}|F(z^*_{(0,0)})|-|F(z^*)|(\widetilde{b}_1+\widetilde{b}_2), \tag{13}$$

where

$$\widetilde{b}_1=\frac{p_0k_2^0!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1')^{k_1^0}(r_2')^{k_2^0}}\lambda_{2,1}^{k_1^0}(R'')\lambda_{2,2}^{k_2^0}(R'')\times$$

$$\times\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}=p_0k_2^0!\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}\frac{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}{(r_1')^{k_1^0}(r_2')^{k_2^0}}\times$$

$$\times\lambda_{2,1}^{k_1^0}(R'')\lambda_{2,2}^{k_2^0}(R'')\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}\leq$$

$$\leq p_0N!\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}\left(\frac{r_1''r_2''}{r_1'r_2'}\right)^N\lambda_{2,1}^N(R'')\lambda_{2,2}^N(R'')\times$$

$$\times\sum_{j=1}^{N}\frac{(N-j)!}{(r_1'')^j}=\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}b_1,$$

$$\widetilde{b}_2=\frac{p_0}{(r_1'')^{k_1^0}}\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}\lambda_{2,2}^{k_2^0}(R'')\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}}\leq$$

$$\leq p_0\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}\lambda_{2,2}^N(R'')\sum_{j=1}^{N}\frac{(N-j)!}{(r_2'')^j}\times$$

$$\times\max\left\{\frac{1}{(r_1'')^N},1\right\}=\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}b_2. \quad (14)$$

Inequality (13) implies that

$$|F^{(k_1^0,k_2^0)}(z^*_{(k_1^0,k_2^0)})|\geq\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}|F(z^*)|\times$$

$$\times\left(\frac{|F(z^*_{(0,0)})|}{|F(z^*)|}-(b_1+b_2)\right). \quad (15)$$

In view of (8) we have that $\frac{|F(z^*_{(0,0)})|}{|F(z^*)|}\geq p_*>b_1+b_2$. Hence, applying (10) and (9) to (15), we deduce

$$|F^{(k_1^0,k_2^0)}(z^*_{(k_1^0,k_2^0)})|\geq\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}|F(z^*)|\times$$

$$\times(p_*-(b_1+b_2))\geq$$

$$\geq\frac{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}(p_*-(b_1+b_2))\times$$

$$\times\frac{|F^{(k_1^0,k_2^0)}(z^0)|(r_1')^{k_1^0}(r_2')^{k_2^0}}{k_1^0!k_2^0!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}\geq$$

$$\geq\left(\frac{r_1'r_2'}{r_1''r_2''}\right)^N(p_*-(b_1+b_2))\frac{|F^{(k_1^0,k_2^0)}(z^*_{(k_1^0,k_2^0)})|}{p_0(N!)^2}.$$

Therefore, $p_*\leq p_0(N!)^2\left(\frac{r_1''r_2''}{r_1'r_2'}\right)^N+b_1+b_2$, but it contradicts of choice $p_*=p_0(N!)^2\left(\frac{r_1''r_2''}{r_1'r_2'}\right)^N+b_1+b_2+1$. Thus, inequality (7) is valid.

Theorem 6. *Let* $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$, F *be an analytic in* $\mathbb{D}\times\mathbb{C}$ *function. If there exist* $R',R''\in\mathcal{B}^2$, $R'<R''$, *and* $p_1\geq 1$ *such that for each* $z^0\in\mathbb{D}\times\mathbb{C}$ *inequality* (7) *holds, then the function* F *has bounded* $\mathbf{L}$*-index in joint variables.*

Proof. At first, we assume that $R'<\mathbf{1}<R''$. Let $z^0\in\mathbb{D}\times\mathbb{C}$ be an arbitrary point. We expand a function F in power series in $\mathbb{D}^2(z^0,R)$

$$F(z)=\sum_{k\geq\mathbf{0}}b_k(z-z^0)^k=$$

$$=\sum_{k_1\geq0,k_2\geq0}b_{k_1,k_2}(z_1-z_1^0)^{k_1}(z_2-z_2^0)^{k_2}, \quad (16)$$

where $k=(k_1,k_2)$, $b_k=b_{k_1,k_2}=\frac{F^{(k_1,k_2)}(z_1^0,z_2^0)}{k_1!k_2!}$, $R=(r_1,r_2)$.

Let $\mu(R,z^0,F)=\max\{|b_k|R^k\colon\ k\geq\mathbf{0}\}=\max\{|b_{k_1,k_2}|r_1^{k_1}r_2^{k_2}\colon\ k_1\geq0,k_2\geq0\}$ be a maximal term of series (16) and $\nu(R)=\nu(R,z^0,F)=(\nu_1(R),\nu_2(R))$ be a set of indices such that

$$\mu(R,z^0,F)=|b_{\nu(R)}|R^{\nu(R)},$$
$$\|\nu(R)\|=\nu_1(R)+\nu_2(R)=\max\{k_1+k_2\colon k_1\geq0,$$
$$k_2\geq0,|b_k|R^k=\mu(R,z^0,F)\}.$$

We apply Cauchy's inequality

$$\forall R=(r_1,r_2),0<r_j<1,j\in\{1,2\}\colon$$
$$\mu(R,z^0,F)\leq M(R,z^0,F).$$

For given R' and R'', such that $0<r_j'<1$, $1<r_j''<\beta$, we conclude

$$M(R'R,z^0,F)\leq\sum_{k\geq\mathbf{0}}|b_k|(R'R)^k\leq$$

$$\leq\sum_{k\geq\mathbf{0}}\mu(R,z^0,F)(R')^k=\mu(R,z^0,F)\sum_{k\geq\mathbf{0}}(R')^k=$$

$$=\prod_{j=1}^{2}\frac{1}{1-r_j'}\mu(R,z^0,F).$$

Besides,

$$\ln\mu(R,z^0,F)=\ln\{|b_{\nu(R)}|R^{\nu(R)}\}=$$
$$=\ln\left\{|b_{\nu(R)}|(RR'')^{\nu(R)}\frac{1}{(R'')^{\nu(R)}}\right\}=$$
$$=\ln\{|b_{\nu(R)}|(RR'')^{\nu(R)}\}+\ln\left\{\frac{1}{(R'')^{\nu(R)}}\right\}\le$$
$$\le\ln\mu(R''R,z^0,F)-\|\nu(R)\|\ln\min\{r_1'',r_2''\}.$$

This implies that

$$\|\nu(R)\|\le\frac{1}{\ln\min\{r_1'',r_2''\}}(\ln\mu(R''R,z^0,F)-$$
$$-\ln\mu(R,z^0,F))\le$$
$$\le\frac{1}{\ln\min\{r_1'',r_2''\}}(\ln M(R''R,z^0,F)-$$
$$-\ln((1-r_1')(1-r_2')M(R'R,z^0,F)))\le$$
$$\le\frac{1}{\ln\min\{r_1'',r_2''\}}(\ln M(R''R,z^0,F)-$$
$$-\ln M(R'R,z^0,F)))-\frac{\sum_{j=1}^2\ln(1-r_j')}{\ln\min\{r_1'',r_2''\}}=$$
$$=\frac{1}{\ln\min\{r_1'',r_2''\}}\ln\frac{M(R''R,z^0,F)}{M(R'R,z^0,F)}-$$
$$-\frac{\sum_{j=1}^2\ln(1-R_j)}{\ln\min\{r_1'',r_2''\}}. \quad (17)$$

Put $R=\frac{\mathbf{1}}{\mathbf{L}(z^0)}$. Now let $N(F,z^0,\mathbf{L})$ be a **L**-index of the function F in joint variables at point z^0 i. e. it is the least integer for which inequality (1) holds at the point z^0. Clearly that

$$N(F,z^0,\mathbf{L})\le\nu\left(\frac{1}{\mathbf{L}(z^0)},z^0,F\right)=\nu(R,z^0,F). \quad (18)$$

But

$$M\left(R''/\mathbf{L}(z^0),z^0,F\right)\le$$
$$\le p_1(R',R'')M\left(R'/\mathbf{L}(z^0),z^0,F\right). \quad (19)$$

Therefore, from (17), (18), (19) we obtain that $\forall z^0\in\mathbb{D}\times\mathbb{C}$

$$N(F,z^0,\mathbf{L})\le$$
$$\le\frac{-\sum_{j=1}^2\ln(1-r_j')}{\ln\min\{r_1'',r_2''\}}+\frac{\ln p_1(R',R'')}{\ln\min\{r_1'',r_2''\}}.$$

This means that F has bounded **L**-index in joint variables, if $R'<\mathbf{1}<R''$ anf $R',R''\in\mathcal{B}^2$.

Now we will prove the theorem for any $R'<R''$. For entire functions of several variables the constraint $R'<\mathbf{1}<R$ was first removed in [6]. From (7) with $R_1<R_2$ it follows that

$$\max\Big\{|F(z)|:z\in\mathbb{T}^2\big(z^0,\frac{2R''}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z^0)}\big)\Big\}\le$$
$$\le P_1\max\Big\{|F(z)|:z\in\mathbb{T}^2\big(z^0,\frac{2R'}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z^0)}\big)\Big\}.$$

Denoting $\widetilde{\mathbf{L}}(z)=\frac{2\mathbf{L}(z)}{R'+R''}$, we obtain

$$\max\Big\{|F(z)|:z\in\mathbb{T}^2\big(z^0,\frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\big)\Big\}\le$$
$$\le P_1\max\Big\{|F(z)|:z\in\mathbb{T}^2\big(z^0,\frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\big)\Big\},$$

where $\frac{2R'}{R'+R''}<\mathbf{1}<\frac{2R''}{R'+R''}$. Taking into account the first part of the proof, we conclude that the function F has bounded $\widetilde{\mathbf{L}}$-index in joint variables. By Theorem 3, the function F is of bounded **L**-index in joint variables.

4. Analog of Hayman's Theorem. Theorem 7 is an analogue of known Theorem of Hayman, which was established for entire functions of one complex variable (see [12]). Its applications to differential equations are considered for analytic functions in the unit bidisc [16], in the unit ball [3].

Theorem 7. *Let* $\mathbf{L}\in Q(\mathbb{D}\times\mathbb{C})$. *An analytic function* $F:\mathbb{D}\times\mathbb{C}\to\mathbb{C}$ *has bounded* **L***-index in joint variables if and only if there exist* $p\in\mathbb{Z}_+$ *and* $c\in\mathbb{R}_+$ *such that for each* $z\in\mathbb{D}\times\mathbb{C}$ *the inequality*

$$\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)}:j_1+j_2=p+1\right\}\le$$
$$\le c\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)}:k_1+k_2\le p\right\} \quad (20)$$

holds.

Proof. Let $N=N(F,\mathbf{L},\mathbb{C}\times\mathbb{D})<+\infty$. We obtain immediately the necessity from the definition of the boundedness of **L**-index in joint variables with $p=N$ and $c=((N+1)!)^2$. We prove the sufficiency. If $F\equiv0$ then theorem is obvious. Thus, we suppose that $F\not\equiv0$. Let (20) holds, $z^0\in\mathbb{D}\times\mathbb{C}$, $z\in$

$\mathbb{T}^2\left(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right)$. For all $j = (j_1, j_2) \in \mathbb{Z}_+^2$, $j_1 + j_2 \le p + 1$ we have

$$\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} \le \frac{|F^{(j_1,j_2)}(z)|l_1^{j_1}(z)l_2^{j_2}(z)}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)l_1^{j_1}(z)l_2^{j_2}(z)} \le$$
$$\le \lambda_{1,1}^{j_1}(\beta)\lambda_{1,2}^{j_2}(\beta)\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)} \le$$
$$\le \lambda_{1,1}^{j_1}(\beta)\lambda_{1,2}^{j_2}(\beta)c\max_{0\le k_1+k_2\le p}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} =$$
$$= \lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)c\times$$
$$\times\max_{0\le k_1+k_2\le p}\frac{l_1^{k_1}(z^0)l_2^{k_2}(z^0)|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)l_1^{k_1}(z)l_2^{k_2}(z)} \le$$
$$\le \lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)c\times$$
$$\times\max_{0\le k_1+k_2\le p}\frac{1}{\lambda_{1,1}^{k_1}(\beta)\lambda_{1,2}^{k_2}(\beta)}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)} \le$$
$$\le c\max\{\lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)\colon j_1+j_2\le p+1\}\times$$
$$\times\max\left\{\frac{1}{\lambda_{1,1}^{k_1}(\beta)\lambda_{1,2}^{k_2}(\beta)}\colon k_1+k_2\le p\right\}\times$$
$$\times\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}\colon k_1+k_2\le p\right\} = B\cdot G(z),$$

where

$$B = c\max\{\lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)\colon j_1+j_2\le p+1\}\times$$
$$\times\max\left\{\lambda_{1,1}^{-k_1}(\beta)\lambda_{1,2}^{-k_2}(\beta)\colon k_1+k_2\le p\right\},$$
$$G(z) = \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}\colon k_1+k_2\le p\right\}.$$

We choose $z^{(1)} = (z_1^{(1)}, z_2^{(1)}) \in \mathbb{T}^2(z^0, \mathbf{1}/(2\beta\mathbf{L}(z^0)))$ and $z^{(2)} = (z_1^{(2)}, z_2^{(2)}) \in \mathbb{T}^2(z^0, \boldsymbol{\beta}/\mathbf{L}(z^0))$ such that $F(z^{(1)}) \ne 0$ and

$$|F(z^{(2)})| = \\ = \max\left\{|F(z)|\colon z\in\mathbb{T}^2\left(z^0, \boldsymbol{\beta}/\mathbf{L}(z^0)\right)\right\} \ne 0. \tag{21}$$

These points exist, otherwise if $F(z) \equiv 0$ on skeleton $\mathbb{T}^2(z^0, \mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)))$ or $\mathbb{T}^2(z^0, \boldsymbol{\beta}/\mathbf{L}(z^0))$ then by the uniqueness theorem $F \equiv 0$ in $\mathbb{D}\times\mathbb{C}$. We connect the points $z^{(1)}$ and $z^{(2)}$ with plane

$$\alpha\colon \quad z_2 = k_2z_1 + c_2$$
$$\frac{z_2 - z_2^{(1)}}{z_2^{(2)} - z_2^{(1)}} = \frac{z_1 - z_1^{(1)}}{z_1^{(2)} - z_1^{(1)}}, k_2 = \frac{z_2^{(2)} - z_2^{(1)}}{z_1^{(2)} - z_1^{(1)}},$$
$$c_2 = \frac{z_2^{(1)}z_1^{(2)} - z_1^{(1)}z_2^{(2)}}{z_1^{(2)} - z_1^{(1)}}.$$

Let $\widetilde{G}(z_1) = G(z)|_\alpha$ be a restriction of the function G onto α. All functions $F^{(k_1,k_2)}|_\alpha$ are analytic functions of variable z_1 and $\widetilde{G}(z_1^{(1)}) = G(z^{(1)})|_\alpha \ne 0$, because $F(z^{(1)}) \ne 0$. That's why zeros of the function $\widetilde{G}(z_1)$ are isolated as zeros of a function of one variable. Therefore we can choose piecewise analytic curve γ onto α :

$$z = z(t) = (z_1(t), k_2z_1(t) + c_2), t\in[0,1],$$

which connect the points $z^{(1)}$, $z^{(2)}$ and such that $G(z(t)) \ne 0$ and $\int_0^1 |z_1'(t)|dt \le \frac{2\beta}{l_1(z^0)}$. For a construction of the curve we connect $z^{(1)}$ and $z^{(2)}$ by a line $z_1^*(t) = (z_1^{(2)} - z_1^{(1)})t + z_1^{(1)}, t \in [0,1]$. The curve γ can cross points z_1 at which the function $\widetilde{G}(z_1) = 0$. The number of such points $m = m(z^{(1)}, z^{(2)})$ is finite. Let $(z_{1,k}^*)$ be a sequence of these points in ascending order of the value $|z_1^{(1)} - z_{1,k}^*|$, $k \in \{1, 2, ..., m\}$. We choose $r < \min_{1\le k\le m-1}\{|z_{1,k}^* - z_{1,k+1}^*|, |z_{1,1}^* - z_1^{(1)}|, |z_{1,m}^* - z_1^{(2)}|, \frac{2\beta^2-1}{2\pi\beta l_1(z^0)}\}$. Now we construct circles with centre at the points $z_{1,k}^*$ and corresponding radii $r_k' < \frac{r}{2^k}$ such that $\widetilde{G}(z_1) \ne 0$ for all z_1 on the circles. It is possible, because $F \not\equiv 0$. Every such circle is divided onto two semicircles by the line $z_1^*(t)$. The required piecewise-analytic curve consists with arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with the points $z_1^{(1)}, z_1^{(2)}$. The length of $z_1(t)$ in $\mathbb{C}$ is less than $\frac{\beta}{l_1(z^0)} + \frac{1}{2\beta l_1(z^0)} + \pi r \le \frac{2\beta}{l_1(z^0)}$.

Then

$$\int_0^1 |z_2'(t)|dt = |k_2|\int_0^1 |z_1'(t)|dt \le$$
$$\le \frac{|z_2^{(2)} - z_2^{(1)}|}{|z_1^{(2)} - z_1^{(1)}|}\cdot\frac{2\beta}{l_1(z^0)} \le \frac{2\beta^2+1}{2\beta l_2(z^0)}\cdot\frac{2\beta l_1(z^0)}{2\beta^2-1}\times$$

$$\times\frac{2\beta}{l_1(z^0)}=\frac{2\beta(2\beta^2+1)}{(2\beta^2-1)l_2(z^0)}.$$

Hence,

$$\int_0^1\sum_{i=1}^2 l_i(z^0)|z_i^{'}(t)|dt=\int_0^1 l_1(z^0)|z_1^{'}(t)|dt+$$
$$+\int_0^1 l_2(z^0)|z_2^{'}(t)|dt\leq$$
$$\leq l_1(z^0)\cdot\frac{2\beta}{l_1(z^0)}+l_2(z^0)\cdot\frac{2\beta(2\beta^2+1)}{(2\beta^2-1)l_2(z^0)}=$$
$$=\frac{8\beta^3}{2\beta^2-1}=S. \tag{22}$$

Since the function $z=z(t)$ is piece-wise analytic on $[0,1]$, then for arbitrary $k\in\mathbb{Z}_+^2, j\in\mathbb{Z}_+^2, \|k\|\leq p, \|j\|\leq p, k\neq j$ either

$$\frac{|F^{(k_1,k_2)}(z_1(t),z_2(t))|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}\equiv\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} \tag{23}$$

or the equality

$$\frac{|F^{(k_1,k_2)}(z_1(t),z_2(t))|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}=\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} \tag{24}$$

holds only for a finite set of points $t_k\in[0;1]$. Then for function $G(z(t))$ as maximum of such expressions $\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}$ by all $\|j\|\leq p$ two cases are possible:

1. In some interval of analyticity of the curve γ the function $G(z(t))$ identically equals simultaneously to some derivatives, that is (23) holds. It means that $G(z(t))\equiv\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}$ for some $\|j\|\leq p$. Clearly, the function $F^{(j_1,j_2)}(z_1(t),z_2(t))$ is analytic. Then $|F^{(j_1j_2)}(z_1(t),z_2(t))|$ is continuously differentiable function on the interval of analyticity except points where this partial derivative equals zero $|F^{(j_1,j_2)}(z_1(t),z_2(t))|=0$. However, there are not the points, because in the opposite case $G(z(t))=0$. But it contradicts the construction of the curve γ.

2. In some interval of analyticity of the curve γ the function $G(z(t))$ equals simultaneously to some derivatives at a finite number of points t_k, that is (24) holds. Then the points t_k divide interval of analyticity onto a finite number of segments, in which of them $G(z(t))$ equals to one of partial derivatives, i. e. $G(z(t))\equiv\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}$ for some j, $\|j\|\leq p$. As above, in each from these segments the functions $|F^{(j_1,j_2)}(z_1(t),z_2(t))|$ and $G(z(t))$ are continuously differentiable except the points t_k.

Taking into account (2) and using the inequality $\frac{d}{dx}|\varphi(x)|\leq\left|\frac{d}{dx}\varphi(x)\right|$, which holds for complex-valued functions of real argument outside a countable set of points, we have

$$\frac{d}{dt}G(z(t))\leq$$
$$\leq\max\left\{\frac{1}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\left|\frac{d}{dt}F^{(j_1,j_2)}(z_1(t),z_2(t))\right|:\right.$$
$$\left.j_1+j_2\leq p\right\}\leq$$
$$\leq\max_{j_1+j_2\leq p}\left\{\frac{|F^{(j_1+1,j_2)}(z_1(t),z_2(t))|\cdot|z_1^{'}(t)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}+\right.$$
$$\left.+\frac{|F^{(j_1,j_2+1)}(z_1(t),z_2(t))|\cdot|z_2^{'}(t)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\right\}=$$
$$=\max_{j_1+j_2\leq p}\left\{|F^{(j_1+1,j_2)}(z(t))|\frac{|z_1^{'}(t)|l_1(z^0)}{l_1^{j_1+1}(z^0)l_2^{j_2}(z^0)}+\right.$$
$$\left.+|F^{(j_1,j_2+1)}(z(t))|\frac{|z_2^{'}(t)|l_2(z^0)}{l_1^{j_1}(z^0)l_2^{j_2+1}(z^0)}\right\}\leq$$
$$\leq(|z_1^{'}(t)|l_1(z^0)+|z_2^{'}(t)|l_2(z^0))\times$$
$$\times\max_{j_1+j_2\leq p+1}\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\leq$$
$$\leq\left(\sum_{i=1}^2 l_i(z^0)|z_i^{'}(t)|\right)BG(z(t)).$$

Therefore, (22) yields

$$\left|\ln\frac{G(z^{(2)})}{G(z^{(1)})}\right|=\left|\int_0^1\frac{1}{G(z(t))}\frac{d}{dt}G(z(t))\right|\leq$$
$$\leq B\int_0^1\sum_{i=1}^2 l_i(z^0)|z_i^{'}(t)|dt\leq B\cdot S.$$

Using (21), we deduce

$$\max\left\{|F(z)|\colon z\in\mathbb{T}^2\left(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0)\right)\right\}=$$
$$=|F(z^{(2)})|\leq G(z^{(2)})\leq G(z^{(1)})\cdot\exp(BS).$$

Since $z^{(1)}\in\mathbb{T}^2(z^0,\mathbf{1}/(2\beta\mathbf{L}(z^0)))$, the Cauchy inequality holds

$$\frac{|F^{(j)}(z^{(1)})|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} \leq$$
$$\leq j_1!j_2!(2\beta)^{j_1+j_2}M\left(\mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)), z^0, F\right)$$

for every $j \in \mathbb{Z}_+^2$. Therefore, for $j_1+j_2 \leq p$ we have

$$G(z^{(1)}) \leq (p!)^2(2\beta)^{2p}M(\mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)), z^0, F) \text{ and}$$

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(z^0, \boldsymbol{\beta}/\mathbf{L}(z^0)\right)\right\} \leq$$
$$\leq e^{BS}(p!)^2(2\beta)^{2p}\times$$
$$\times \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(z^0, \mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0))\right)\right\}.$$

Hence, by Theorem 6 F is a function of bounded $\mathbf{L}$-index in joint variables.

Theorem 8. *Let $\beta > 1$, $\mathbf{L} \in Q(\mathbb{D}\times\mathbb{C})$. An analytic function $F : \mathbb{D}\times\mathbb{C} \to \mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $c \in (0; +\infty)$ and $N_0 \in \mathbb{N}$ such that for each $z \in \mathbb{D}\times\mathbb{C}$ the inequality*

$$\sum_{k_1+k_2=0}^{N_0} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \geq$$
$$\geq c \sum_{k_1+k_2=N_0+1}^{+\infty} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \qquad (25)$$

holds.

Proof. Let $\frac{1}{\beta} < \theta_j < 1$, $j \in \{1,2\}$. If the function F has bounded $\mathbf{L}$-index in joint variables then by Theorem 3 F has bounded $\widetilde{\mathbf{L}}$-index in joint variables, where $\widetilde{\mathbf{L}} = (\widetilde{l}_1(z), \widetilde{l}_2(z))$, $\widetilde{l}_j(z) = \theta_j l_j(z)$, $j \in \{1,2\}$. Denote $N = N(F, \widetilde{\mathbf{L}}, \mathbb{D}\times\mathbb{C})$. Therefore,

$$\max_{0\leq k_1+k_2\leq N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} =$$
$$= \max_{0\leq k_1+k_2\leq N} \frac{\theta_1^{k_1}\theta_2^{k_2}|F^{(k_1,k_2)}(z)|}{k_1!k_2!\widetilde{l}_1^{k_1}(z)\widetilde{l}_2^{k_2}(z)} \geq$$
$$\geq (\theta_1\theta_2)^N \max_{0\leq k_1+k_2\leq N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!\widetilde{l}_1^{k_1}(z)\widetilde{l}_2^{k_2}(z)} \geq$$
$$\geq (\theta_1\theta_2)^N \frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!\widetilde{l}_1^{j_1}(z)\widetilde{l}_2^{j_2}(z)} =$$
$$= \theta_1^{N-j_1}\theta_2^{N-j_2} \frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!l_1^{j_1}(z)l_2^{j_2}(z)}$$

for all $j_1 \geq 0, j_2 \geq 0$ and

$$\sum_{j_1+j_2=N+1}^{+\infty} \frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!l_1^{j_1}(z)l_2^{j_2}(z)} \leq$$
$$\leq \max_{0\leq k_1+k_2\leq N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}\times$$
$$\times \sum_{j_1+j_2=N+1}^{+\infty} \theta_1^{j_1-N}\theta_2^{j_2-N} =$$
$$= \frac{\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)} \max_{0\leq k_1+k_2\leq N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \leq$$
$$\leq \frac{\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)} \sum_{k_1+k_2=0}^{N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}.$$

Hence, we obtain (25) with $N_0 = N$ and $c = \frac{\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)}$. On the contrary, inequality (25) implies

$$\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!l_1^{j_1}(z)l_2^{j_2}(z)} : j_1+j_2 = N+1\right\} \leq$$
$$\leq \sum_{k_1+k_2=N+1}^{+\infty} \frac{F^{(k_1,k_2)}(z)}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \leq$$
$$\leq \frac{1}{c}\sum_{k_1+k_2=0}^{N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \leq$$
$$\leq \frac{(N+1)N}{2c} \max_{0\leq k_1+k_2\leq N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}$$

and by Theorem 7 F is of bounded $\mathbf{L}$-index in joint variables.

Analogs of Theorems 5 and 7 were used [3, 16] to obtain growth estimates analytic functions in the unit ball of bounded $\mathbf{L}$-index in joint variables and to deduce sufficient conditions of index boundedness for analytic solutions in the unit bidisc of some system of partial differential equations. It is naturally to pose the similar **questions** for analytic function in $\mathbb{D}\times\mathbb{C}$:

1. What are growth estimates analytic functions in $\mathbb{D}\times\mathbb{C}$ of bounded $\mathbf{L}$-index in joint variables?

2. What are sufficient conditions index boundedness for analytic solutions in $\mathbb{D}\times\mathbb{C}$ of linear higher-order system of partial differential equations?

Now we are not ready to give a full answer to this questions.

REFERENCES

1. Bandura, A., Skaskiv, O. (2017). Functions analytic in a unit ball of bounded **L**-index in joint variables: J. Math. Sci. **227**(1), 1–12. DOI: 10.1007/s10958-017-3570-6
2. Bandura, A., Skaskiv, O. (2017). Analytic functions in the unit Ball. Bounded L-index in joint variables and solutions of systems of PDE's. Beau-Bassin: LAP Lambert Academic Publishing.
3. Bandura, A.I., Skaskiv, O.B. (2017) Analytic functions in the unit ball of bounded **L**-index: asymptotic and local properties: Mat. Stud. **48**(1), 37–73. DOI: 10.15330/ms.48.1.37-73
4. Bandura, A.I., Bordulyak, M.T., Skaskiv, O.B. (2016). Sufficient conditions of boundedness of L-index in joint variables: Mat. Stud., **45**(1), 12–26. DOI: 10.15330/ms.45.1.12-26
5. Bandura, A., Skaskiv, O. (2016) Entire functions of several variables of bounded index. Lviv: Publisher I. E. Chyzhykov. http://chyslo.com.ua
6. Bandura, A., Skaskiv, O. (2017). Entire functions of bounded **L**-index: Its zeros and behavior of partial logarithmic derivatives: J. Complex Analysis **2017**, 1–10. Article ID 3253095. DOI: 10.1155/2017/3253095
7. Bandura, A.I., Petrechko, N.V., Skaskiv, O.B. (2016). Analytic in a polydisc functions of bounded **L**-index in joint variables: Mat. Stud. **46**(1), 72–80. DOI: 10.15330/ms.46.1.72-80
8. Bandura, A., Petrechko, N., Skaskiv, O. (2018). Maximum modulus in a bidisc of analytic functions of bounded **L**-index and an analogue of Hayman's theorem: Mat. Bohemica **143**(3) (2018) (in print). DOI: 10.21136/MB.2017.0110-16
9. Bandura, A., Skaskiv, O. (2017). Directional logarithmic derivative and the distribution of zeros of an entire function of bounded L-index along the direction: Ukr. Math. J. **69**(3), 500–508. DOI: 10.1007/s11253-017-1377-8
10. Bandura, A.I., Skaskiv, O.B., Tsvigun, V.L. (2018). Analytic functions in $\mathbb{D}\times\mathbb{C}$ of bounded index in joint variables: Prykarpatskyi Visn. Nauk. Tov. Im. Shevchenka. Chyslo. **42** (1) (in print). (in Ukrainian)
11. Bandura, A. I., Skaskiv, O. B., Tsvigun, V. L. (2018) Analytic functions in $\mathbb{D}\times\mathbb{C}$ of bounded **L**-index in joint variables: Mat. metody fiz.-mekh. polya. (in print).
12. Hayman, W.K. (1973). Differential inequalities and local valency: Pacific J. Math. **44**(1), 117–137.
13. Kushnir, V.O., Sheremeta, M.M. (1999). Analytic functions of bounded l-index: Mat. Stud., **12** (1), 59–66.
14. Lepson, B. (1968). Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index: Proc. Sympos. Pure Math., Amer. Math. Soc.: Providence, Rhode Island, **2**298–307.
15. Nuray, F., Patterson, R.F. (2015). Multivalence of bivariate functions of bounded index: Le Matematiche, **70** (2), 225–233: DOI:10.4418/2015.70.2.14
16. Petrechko, N. (2017). Bounded **L**-index in joint variables and analytic solutions of some systems of PDE's in bidisc: Visn. Lviv Univ. Ser. Mech. Math. Issue **83**, 100–108.
17. Salmassi, M. (1989). Functions of bounded indices in several variables: Indian J. Math., **31** (3), 249–257.
18. Shah, S.M. (1971). On entire functions of bounded index whose derivatives are of unbounded index: J. Lond. Math. Soc. (2) **4**(1), 127–139.
19. Sheremeta, M. (1999). Analytic functions of bounded index. Lviv: VNTL Publishers.
20. Sheremeta, M.N., Kuzyk, A.D. (1992). Logarithmic derivative and zeros of an entire function of bounded l-index: Sib. Math. J. **33**(2), 304–312. DOI: 10.1007/BF00971102
21. Strochyk, S.N., Sheremeta, M. M. (1993). Analytic in the unit disc functions of bounded index: Dopov. Akad. Nauk Ukr., 1, 19–22. (in Ukrainian)