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ANALYTIC FUNCTIONS IN THE UNIT BALL AND SUFFICIENT SETS OF BOUNDEDNESS OF L-INDEX IN DIRECTION

Вивчається взаємозв'язок між аналітичною в одиничній кулі функцією F обмеженого L-індексу за напрямком та функцією зрізки $g_z(t) = F(z+t\mathbf{b})$. Отримано умови на множину A, які забезпечують рівність $N_{\mathbf{b}}(F,L) = \max\{N(g_z,l_z) : z \in A\}$, де $l_z(t) = L(z+tb)$, $L : \mathbb{B}^n \to \mathbb{R}_+$ — неперервна функція, $\mathbb{B}^n = \{z \in \mathbb{C}^n : \ |z| < 1\}$. Ці результати є узагальненням відомих тверджень для цілих функцій декількох змінних.

Ключові слова: аналітична функція, одинична куля, функція зрізки, обмежений L-індекс за напрямком.

We study a relationship between analytic function F in the unit ball of bounded L-index in the direction and the slice function $g_z(t) = F(z+t\mathbf{b})$. There are obtained the conditions on a set A providing the equality $N_{\mathbf{b}}(F,L) = \max\{N(g_z,l_z) : z \in A\}$, where $l_z(t) = L(z+tb)$, $L : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function, $\mathbb{B}^n = \{z \in \mathbb{C}^n : \ |z| < 1\}$. The results are generalizations of known propositions for entire functions of several variables.

Keywords: analytic function, unit ball, slice function, bounded L-index in direction.

The paper is devoted to analytic functions in the unit ball. This class of analytic function of several variables is very important in complex analysis [16, 20].

Let $\mathbf{0} = (0,\ldots,0)$, $\mathbf{b} = (b_1,\ldots,b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+ = (0,+\infty)$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $L : \ \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}, \ \beta = \text{const} > 1. \qquad (1)$$

For a given $z \in \mathbb{B}^n$ we denote $S_z = \{t \in \mathbb{C} : z + t\mathbf{b} \in \mathbb{B}^n\}$.

Analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b}$ [7] if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{B}^n$ the following inequality is valid

$$\frac{|\partial^m_{\mathbf{b}} F(z)|}{m! L^m(z)} \le \max_{0\le k\le m_0} \frac{|\partial^k_{\mathbf{b}} F(z)|}{k! L^k(z)}, \qquad (2)$$

where $\partial^0_{\mathbf{b}} F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial^k_{\mathbf{b}} F(z) = \partial_{\mathbf{b}}\Big(\partial^{k-1}_{\mathbf{b}} F(z)\Big)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function F and is denoted by* $N_{\mathbf{b}}(F,L) = m_0$. If $n = 1$, $\mathbf{b} = 1$, $L = l$, $F = f$, then $N(f,l) \equiv N_1(f,l)$ is called the l-index of the function f. In the case $n = 1$ and $\mathbf{b} = 1$ we obtain the definition of an analytic function in the unit disc of bounded l-index [13, 21]. The definition is a generalization of concept of bounded L-index in direction introduced and considered for entire functions of several variables in [2, 5, 9]. The primary definition of bounded index for entire function of one variable was supposed by B. Lepson [14].

There was proved some interesting properties of entire functions from this class [2, 3]. Namely, an entire function F has bounded L-index in the direction $\mathbf{b}$ if and only if the slice function $g_z(t) = F(z+t\mathbf{b})$ has bounded l_z-index as a function of variable $t \in \mathbb{C}$ and there exists $M > 0$ such that $N(g_z,l_z) \le M$ for all $z \in \mathbb{C}^n$ ($l_z(t) = L(z+t\mathbf{b})$ and $N_{\mathbf{b}}(F,L) = \max\{N(g_z,l_z) : z \in \mathbb{C}^n\}$. This proposition shows a deep connection between the entire function F of bounded L-index in the direction $\mathbf{b}$ and the corresponding slice function $g_z(t) = F(z+t\mathbf{b})$. Clearly, the set $\mathbb{C}^n$ is very large in this proposition. Therefore, there are few known theorems [3] with lesser sets A

instead C^n in $N_{\mathbf{b}}(F,L)=\max\{N(g_z,l_z): z\in\mathbb{C}^n\}$. Moreover, there is an open problem [4]: *what is the least set A such that $N_{\mathbf{b}}(F,L)=\max\{N(g_z,l_z): z\in A\}$?*

In this paper, we will state few statements that contain the basic properties of analytic functions in the unit ball of bounded L-index in a direction. They demonstrate a relationship between analytic functions of several variables with $N_{\mathbf{b}}(F,L)<\infty$ and the corresponding slice function of one variable. Note that for a concept of bounded $\mathbf{L}$-index in joint variables similar propositions have not a sense (see definition and properties for various classes of analytic functions in [1,6,8,13,15,15,17]).

Denote $l_z(t)=L(z+t\mathbf{b})$, $g_z(t)=F(z+t\mathbf{b})$ for a given $z\in\mathbb{C}^n$.

Lemma 1. *If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ has bounded L-index $N_{\mathbf{b}}(F,L)$ in the direction $\mathbf{b}$, then for every $z^0\in\mathbb{B}^n$ the analytic function $g_{z^0}(t)$, $t\in S_{z^0}$, is of bounded l_{z^0}-index and $N(g_{z^0},l_{z^0})\le N_{\mathbf{b}}(F,L)$.*

Proof. Let $z^0\in\mathbb{B}^n$, $g(t)\equiv g_{z^0}(t)$, $l(t)\equiv l_{z^0}(t)$. Since for every $p\in\mathbb{N}$

$$g^{(p)}(t)=\partial_{\mathbf{b}}^{p}F(z^0+t\mathbf{b}), \tag{3}$$

by the definition of bounded L-index in the direction $\mathbf{b}$ for all $t\in S_{z^0}$ and for $p\in\mathbb{Z}_+$ we obtain

$$\frac{|g^{(p)}(t)|}{p!l^p(t)}=\frac{|\partial_{\mathbf{b}}^{p}F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N_{\mathbf{b}}(F,L)\right\}=$$
$$=\max\left\{\frac{|g^{(k)}(t)|}{k!l^k(t)}:0\le k\le N_{\mathbf{b}}(F,L)\right\}.$$

Hence, $g(t)$ is a function of bounded l-index and $N(g,l)\le N_{\mathbf{b}}(F,L)$. Lemma 1 is proved.

Equality (3) implies the following proposition.

Lemma 2. *If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n$ then $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}): z^0\in\mathbb{B}^n\}$.*

However, maximum can be calculated on a set A with a property $\bigcup_{z^0\in A}\{z^0+t\mathbf{b}: t\in S_{z^0}\}=\mathbb{B}^n$. Thus, the following assertion is valid.

Lemma 3. *If an analytic function $F:\mathbb{B}^n\to\mathbb{C}$ has bounded L-index in the direction $\mathbf{b}$ and j_0 is chosen with $b_{j_0}\ne 0$ then $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}): z^0\in\mathbb{C}^n, z^0_{j_0}=0\}$ and if $\sum_{j=1}^n b_j\ne 0$ then $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}): z^0\in\mathbb{C}^n, \sum_{j=1}^n z_j^0=0\}$.*

Proof. We prove that for every $z\in\mathbb{B}^n$ there exist $z^0\in\mathbb{C}^n$ and $t\in S_{z^0}$ with $z=z^0+t\mathbf{b}$ and $z^0_{j_0}=0$. Put $t=z_{j_0}/b_{j_0}$, $z_j^0=z_j-t\mathbf{b}_j$, $j\in\{1,2,\ldots,n\}$. Clearly, $z^0_{j_0}=0$ for this choice.

However, the point z^0 may not be contained in $\mathbb{B}^n$. But there exists $t\in\mathbb{C}$ that $z^0+t\mathbf{b}\in\mathbb{B}^n$. Let $z^0\notin\mathbb{B}^n$ and $|z|=R_1<1$. Therefore, $|z^0+t\mathbf{b}|=|z-\frac{z_{j_0}}{b_{j_0}}\mathbf{b}+t\mathbf{b}|=|z+(t-\frac{z_{j_0}}{b_{j_0}})\mathbf{b}|\le |z|+|t-\frac{z_{j_0}}{b_{j_0}}|\cdot|\mathbf{b}|\le R_1+|t-\frac{z_{j_0}}{b_{j_0}}|\cdot|\mathbf{b}|<1$. Thus, $|t-\frac{z_{j_0}}{b_{j_0}}|<\frac{1-R_1}{|\mathbf{b}|}$.

In the second part we prove that for every $z\in\mathbb{B}^n$ there exist $z^0\in\mathbb{C}^n$ and $t\in S_{z^0}$ such that $z=z^0+t\mathbf{b}$ and $\sum_{j=1}^n z_j^0=0$. Put $t=\frac{\sum_{j=1}^n z_j}{\sum_{j=1}^n b_j}$ and $z_j^0=z_j-t\mathbf{b}_j$, $1\le j\le n$. Thus, the following equality is valid $\sum_{j=1}^n z_j^0=\sum_{j=1}^n(z_j-tb_j)=\sum_{j=1}^n z_j-\sum_{j=1}^n b_jt=0$.
Lemma 3 is proved.

Note that for a given $z\in\mathbb{B}^n$ we can pick uniquely $z^0\in\mathbb{C}^n$ and $t\in S_{z^0}$ such that $\sum_{j=1}^n z_j^0=0$ and $z=z^0+t\mathbf{b}$.

Remark 1. *If for some $z^0\in\mathbb{C}^n$ $\{z^0+t\mathbf{b}: t\in\mathbb{C}\}\bigcap\mathbb{B}^n=\emptyset$ then we put $N(g_{z^0},l_{z^0})=0$.*

Lemmas 1–3 imply the following proposition.

Theorem 1. *An analytic function $F(z):\mathbb{B}^n\to\mathbb{C}$ is a function of bounded L-index in the direction $\mathbf{b}$ if and only if there exists number $M>0$ such that for every $z^0\in\mathbb{B}^n$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0},l_{z^0})\le M<+\infty$, as a function of variable $t\in S_{z^0}$ and $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}): z^0\in\mathbb{B}^n\}$.*

Proof. The necessity follows from Lemma 1.
Sufficiency. Since $N(g_{z^0},l_{z^0})\le M$, there exists $\max\{N(g_{z^0},l_{z^0}): z^0\in\mathbb{B}^n\}$. We denote this maximum by $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}): z^0\in\mathbb{B}^n\}<\infty$. Suppose that $N_{\mathbf{b}}(F)$ is not the

L-index in the direction $\mathbf{b}$ of the function $F(z)$. It means that there exists $n^* > N_{\mathbf{b}}(F, L)$ and $z^* \in \mathbb{B}^n$ for which

$$\frac{\left|\partial_{\mathbf{b}}^{n^*} F(z^*)\right|}{n^*! L^{n^*}(z^*)} > \max_{0 \le k \le N_{\mathbf{b}}(F,L)} \frac{\left|\partial_{\mathbf{b}}^{k} F(z^*)\right|}{k! L^{k}(z^*)}. \qquad (4)$$

Since for $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ we have $g_{z^0}^{(p)}(t) = \frac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}$, inequality (4) can be rewritten as $\frac{\left|g_{z^*}^{(n^*)}(0)\right|}{n^*! l_{z^*}^{n^*}(0)} > \max\left\{\frac{|g_{z^*}^{(k)}(0)|}{k! l_{z^*}^{k}(0)} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\}$. It contradicts that all l_{z^0}-indices $N(g_{z_0}, l_{z^0})$ are not greater than $N_{\mathbf{b}}(F)$. As follows $N_{\mathbf{b}}(F)$ is the L-index in the direction $\mathbf{b}$ of the function $F(z)$. Theorem 1 is proved.

From Lemma 3 the following condition is sufficient in Theorem 1: *there exists $M < +\infty$ such that $N(g_{z^0}, l_{z^0}) \le M$ for every $z^0 \in \mathbb{C}^n$ with $\sum_{j=1}^n z_j^0 = 0$.*

In connection with Lemma 3 and 1 there is a natural *question:* what is the least set A for which $N_{\mathbf{b}}(F, L) = \max\limits_{z^0 \in A} N(g_{z^0}, l_{z^0})$. Below we prove propositions which give a partial answer to the question. A solution is partial because it is unknown whether our sets are the least which satisfy the mentioned equality.

Theorem 2. *Let $A_0 \subset \mathbb{C}^n$ be such that $\bigcup_{z \in A_0}\{z + t\mathbf{b} : t \in S_z\} = \mathbb{B}^n$. An analytic function $F(z): \mathbb{B}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of variable $t \in S_{z^0}$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

Proof. By Theorem 1 the analytic function F is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists number $M > 0$ such that for every $z^0 \in \mathbb{B}^n$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of variable $t \in S_{z^0}$. But in view of property of the set A_0 for every $z^0 + t\mathbf{b}$ there exist $\widetilde{z}^0 \in A_0$ and $\widetilde{t} \in \mathbb{B}_{\widetilde{z}^0}$ such that $z^0 + t\mathbf{b} = \widetilde{z}^0 + \widetilde{t}\mathbf{b}$. In other words, for all $p \in \mathbb{Z}_+$ $(g_{z_0}(t))^{(p)} = (g_{\widetilde{z}_0}(\widetilde{t}))^{(p)}$. But $\widetilde{t}$ depends on t. Thus, the condition that $g_{z^0}(t)$ is of bounded l_{z^0}-index for all $z^0 \in \mathbb{B}^n$ is equivalent to the condition $g_{\widetilde{z}^0}(t)$ is of bounded $l_{\widetilde{z}^0}$-index for all $\widetilde{z}^0 \in A_0$.

Remark 2. *An intersection of arbitrary hyperplane $H = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\}$ and the set $\mathbb{B}_{\mathbf{b}}^n = \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} : z \in \mathbb{B}^n\}$, where $\langle \mathbf{b}, c\rangle \ne 0$, satisfies conditions of Theorem 2.*

We prove that for every $w \in \mathbb{B}^n$ there exist $z \in H \bigcap \mathbb{B}_{\mathbf{b}}^n$ and $t \in \mathbb{C}$ such that $w = z + t\mathbf{b}$.

Choosing $z = w + \frac{1-\langle w,c\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \in H \bigcap \mathbb{B}_{\mathbf{b}}^n$, $t = \dfrac{\langle w, c\rangle - 1}{\langle \mathbf{b}, c\rangle}$, we obtain

$$z + t\mathbf{b} = w + \frac{1 - \langle w, c\rangle}{\langle \mathbf{b}, c\rangle}\mathbf{b} + \frac{\langle w, c\rangle - 1}{\langle \mathbf{b}, c\rangle}\mathbf{b} = w.$$

Theorem 3. *Let A be a dense set in $\mathbb{B}^n$. An analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists $M > 0$ such that for every $z^0 \in A$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of $t \in S_{z^0}$, and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.*

Proof. The necessity follows from Theorem 1.

Sufficiency. Let $\overline{A}$ be a closure of the set A. Since $\overline{A} = \mathbb{B}^n$, for every $z^0 \in \mathbb{B}^n$ there exists a sequence (z^m) such that $z^{(m)} \to z^0$ as $m \to +\infty$ and $z^{(m)} \in A$ for all $m \in \mathbb{N}$. But $F(z + t\mathbf{b})$ is of bounded l_z-index for all $z \in \overline{A} \cap \mathbb{B}^n$ as a function of variable t. Therefore, in view of definition of bounded l_z-index there exists $M > 0$ such that for all $z \in A$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$ $\frac{|g_z^{(p)}(t)|}{p! l^p(t)} \le \max\left\{\frac{|g_z^{(k)}(t)|}{k! l_z^k(t)} : 0 \le k \le M\right\}$.

Substituting instead of z a sequence $z^{(m)} \in A$ and $z^{(m)} \to z^0$, for each $m \in \mathbb{N}$ we obtain $\frac{|g_{z^m}^{(p)}(t)|}{p! l_{z^m}^p(t)} \le \max\left\{\frac{|g_{z^m}^{(k)}(t)|}{k! l_{z^m}^k(t)} : 0 \le k \le M\right\}$. In other words, we have

$$\frac{\left|\partial_{\mathbf{b}}^{p} F(z^m + t\mathbf{b})\right|}{p! L^{p}(z^m + t\mathbf{b})} \le \max_{0 \le k \le M} \frac{\left|\partial_{\mathbf{b}}^{k} F(z^m + t\mathbf{b})\right|}{k! L^{k}(z^m + t\mathbf{b})}. \qquad (5)$$

Remind that F is an analytic function in $\mathbb{B}^n$ and L is a positive continuous function. Therefore, we calculate a limit in (5) as $m \to +\infty$ $(z^m \to z^0)$. Then for all $z^0 \in \mathbb{B}^n$, $t \in S_{z^0}$,

$m \in \mathbb{Z}_+$

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \le \max_{0\le k\le M} \frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}.$$

This inequality implies that for every given $z^0 \in \mathbb{B}^n$ $F(z^0+t\mathbf{b})$ is of bounded $L(z^0+t\mathbf{b})$-index as a function of variable t. Applying Theorem 1 we obtain the desired conclusion. Theorem 3 is proved.

Remark 2 and Theorem 3 yield the following corollary.

Corollary 1. *Let A_0 be such that its closure is $\overline{A}_0 = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\} \bigcap \mathbb{B}^n_{\mathbf{b}}$, where $\langle c, \mathbf{b}\rangle \ne 0$, $\mathbb{B}^n_{\mathbf{b}} = \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \colon z \in \mathbb{B}^n\}$. An analytic function $F(z) : \mathbb{B}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists number $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0}, l_{z^0}) \le M < +\infty$, as a function of variable $t \in S_{z^0}$. And $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

Proof. In view of Remark 2 in Theorem 2 we can take an arbitrary hyperplane $B_0 = \{z \in \mathbb{B}^n : \langle z, c\rangle = 1\}$, where $\langle c, \mathbf{b}\rangle \ne 0$. Let A_0 be a dense set in B_0, $\overline{A_0} = B_0$. Repeating considerations of Theorem 3, we obtain the desired conclusion.

Indeed, the necessity follows from Theorem 1 (in this theorem same condition is satisfied for all $z^0 \in \mathbb{C}^n$, and we need this condition for all $z^0 \in A_0$ such that $\overline{A}_0 \cap \mathbb{B}^n = \{z \in \mathbb{B}^n : \langle z, c\rangle = 1\}$).

To prove the sufficiency, we use the density of the set A_0. Obviously, for every $z^0 \in B_0$ there exists a sequence $z^{(m)} \to z^0$ and $z^{(m)} \in A_0$. But $g_z(t)$ is of bounded l_z-index for all $z \in A_0$. Taking the conditions of Corollary 1 into account, for some $M > 0$ and for all $z \in A_0$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$ the following inequality holds $\frac{g_z^{(p)}(t)}{p!l_z^p(t)} \le \max\left\{\frac{|g_z^{(k)}(t)|}{k!l_z^k(t)} : 0 \le k \le M\right\}$.

Substituting an arbitrary sequence $z^{(m)} \in A$, $z^{(m)} \to z^0$ instead of $z \in A^0$, we have $\frac{|g^{(p)}_{z^{(m)}}(t)|}{p!l^p_{z^{(m)}}(t)} \le \max\left\{\frac{|g^{(k)}_{z^{(m)}}(t)|}{k!l^k_{z^{(m)}}(t)} : 0 \le k \le M\right\}$, that is

$$\frac{\left|\partial_{\mathbf{b}}^p F(z^{(m)}+t\mathbf{b})\right|}{L^p(z^{(m)}+t\mathbf{b})} \le \max_{0\le k\le M} \frac{\left|\partial_{\mathbf{b}}^k F(z^{(m)}+t\mathbf{b})\right|}{k!L^k(z^{(m)}+t\mathbf{b})}.$$

However, F is an analytic function in $\mathbb{B}^n$, L is a positive continuous. So we calculate a limit as $m \to +\infty$ ($z^m \to z$). For all $z^0 \in B_0$, $t \in S_{z^0}$, $m \in \mathbb{Z}_+$ we have

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{L^p(z^0+t\mathbf{b})} \le \max_{0\le k\le M} \frac{\left|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})\right|}{k!L^k(z^0+t\mathbf{b})}.$$

Therefore, $F(z^0+t\mathbf{b})$ is of bounded $L(z^0+t\mathbf{b})$-index as a function of t at each $z^0 \in B^n$. By Theorem 3 and Remark 2 F is of bounded L-index in the direction $\mathbf{b}$.

Remark 3. *Let $H = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\}$. The condition $\langle c, \mathbf{b}\rangle \ne 0$ is essential. If $\langle c, \mathbf{b}\rangle = 0$ then for all $z^0 \in H$ and for all $t \in \mathbb{C}$ the point $z^0 + t\mathbf{b} \in H$ because $\langle z^0 + t\mathbf{b}, c\rangle = \langle z^0, c\rangle + t\langle \mathbf{b}, c\rangle = 1$. Thus, this line $z^0 + t\mathbf{b}$ does not describe points outside the hyperplane H.*

We consider $F(z_1, z_2) = \exp(-z_1^2 + z_2^2)$, $\mathbf{b} = (1, 1)$, $c = (-1, 1)$. On the hyperplane $-z_1 + z_2 = 1$ function $F(z_1, z_2)$ takes a look

$$F(z^0 + t\mathbf{b}) = F(z_1^0 + t, z_2^0 + t) =$$
$$= \exp(-(z_1^0 + t)^2 + (1 + z_1^0 + t)^2) =$$
$$= \exp(1 + 2z_1^0 + 2t).$$

Using the definition of l-index boundedness and evaluating corresponding derivatives it is easy to show that $\exp(1+2z_1^0+2t)$ is of bounded index with $l(t) = 1$ and $N(g, l) = 4$.

Thus, F is of unbounded index in the direction $\mathbf{b}$. On the contrary, we assume $N_{\mathbf{b}}(F) = m$ and calculate directional derivatives

$$\partial_{\mathbf{b}}^p F = 2^p(-z_1 + z_2)^p \exp(-z_1 + z_2),\ p \in \mathbb{N}.$$

By the definition of bounded index, an inequality holds $\forall\, p \in \mathbb{N}$ $\forall z \in \mathbb{C}^n$

$$2^p|-z_1+z_2|^p|\exp(-z_1+z_2)| \le$$
$$\le \max_{0\le k\le m} 2^k|-z_1+z_2|^k|\exp(-z_1+z_2)|. \quad (6)$$

Let $p > m$ and $|-z_1+z_2| = 2$. Dividing equation (6) by $2^p|\exp(-z_1+z_2)|$, we get $2^{2p} \le 2^{2m}$. It is impossible. Therefore, $F(z)$ is of unbounded index in the direction $\mathbf{b}$.

Using calculated derivatives it is easy to prove that the function $F(z_1, z_2)$ is of bounded

L-index in the direction $\mathbf{b}$ with $L(z_1, z_2) = 2|-z_1+z_2|+1$ and $N_{\mathbf{b}}(F,L)=0$.

Now we consider another function

$$F(z) = (1+\langle z,d\rangle)\prod_{j=1}^{\infty}(1+\langle z,c\rangle\cdot 2^{-j})^j,\ c\neq d.$$

The multiplicity of zeros of the function F increases to infinity. In view of definition of L-index in the direction it means that $F(z)$ is of unbounded L-index in any direction $\mathbf{b}$ ($\langle \mathbf{b}, c\rangle \neq 0$) and for any positive continuous function L.

We select $\mathbf{b}\in\mathbb{C}^n$ that $\langle \mathbf{b}, d\rangle = 0$. Let $H = \{z\in\mathbb{C}^n : \langle z, d\rangle = -1\}$. But for $z^0\in H$ we have

$$F(z^0+t\mathbf{b}) = (1+\langle z^0,d\rangle + t\langle \mathbf{b},d\rangle)\times$$
$$\times\prod_{j=1}^{\infty}(1+\langle z^0,c\rangle 2^{-j} + t\langle \mathbf{b},c\rangle 2^{-j})^j \equiv 0.$$

Thus, $F(z^0+t\mathbf{b})$ is of bounded index as a function of variable t.

Theorem 4. *Let (r_p) be a positive sequence such that $r_p\to 1$ as $p\to\infty$, $D_p = \{z\in\mathbb{C}^n : |z| = r_p\}$, A_p be a dense set in D_p (i.e. $\overline{A}_p = D_p$) and $A = \bigcup_{p=1}^{\infty} A_p$. An analytic function F in $\mathbb{B}^n$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exists number $M>0$ such that for all $z^0\in A$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0})\le M < +\infty$, as a function of variable $t\in S_{z^0}$. And $N_{\mathbf{b}}(F,L) = \max\{N(g_{z^0}, l_{z^0}) : z^0\in A\}$.*

Proof. Theorem 1 implies the necessity of this theorem.

Sufficiency. It is easy to prove $\{z+t\mathbf{b} : t\in S_z,\ z\in A\} = \mathbb{B}^n$. Further, we repeat arguments with the proof of sufficiency in Theorem 3 and obtain the desired conclusion.

Auxiliary class $Q^n_{\mathbf{b}}$

The positivity and continuity of function L and condition (1) are not sufficient to explore the behavior of analytic function of bounded L-index in direction. Below we impose an extra condition that function L does not vary as soon. Similar proposition for $L:\mathbb{C}^n\to\mathbb{R}_+$ are obtained in [9]. For $\eta\in[0,\beta]$, $z\in\mathbb{B}^n$, we define $\lambda_1^{\mathbf{b}}(z,\eta,L) = \inf\left\{\frac{L(z+t\mathbf{b})}{L(z)} : |t|\le\frac{\eta}{L(z)}\right\}$, $\lambda_2^{\mathbf{b}}(z,\eta,L) = \sup\left\{\frac{L(z+t\mathbf{b})}{L(z)} : |t|\le\frac{\eta}{L(z)}\right\}$, $\lambda_1^{\mathbf{b}}(\eta,L) = \inf\{\lambda_1^{\mathbf{b}}(z,\eta,L) : z\in\mathbb{B}^n\}$, $\lambda_2^{\mathbf{b}}(\eta,L) = \sup\{\lambda_2^{\mathbf{b}}(z,\eta,L) : z\in\mathbb{B}^n\}$.

If it will not cause misunderstandings, then $\lambda_1^{\mathbf{b}}(z,\eta)\equiv\lambda_1^{\mathbf{b}}(z,\eta,L)$, $\lambda_2^{\mathbf{b}}(z,\eta)\equiv\lambda_2^{\mathbf{b}}(z,\eta,L)$, $\lambda_1^{\mathbf{b}}(\eta)\equiv\lambda_1^{\mathbf{b}}(\eta,L)$, $\lambda_2^{\mathbf{b}}(z,\eta)\equiv\lambda_2^{\mathbf{b}}(\eta,L)$.

By $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$ we denote the class of all positive functions $L:\mathbb{B}^n\to\mathbb{R}_+$ satisfying (1) and $0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty$ for any $\eta\in[0,\beta]$

Let $\mathbb{D}\equiv\mathbb{B}^1$, $Q_\beta(\mathbb{D})\equiv Q_{1,\beta}(\mathbb{D})$.

The following lemma suggests possible approach to compose function with $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$.

Lemma 4. *Let $\overline{\mathbb{B}}^n = \{z\in\mathbb{C}^n : |z|\le 1\}$, $L:\overline{\mathbb{B}}^n\to\mathbb{R}_+$ be a continuous function, $m = \min\{L(z) : z\in\overline{\mathbb{B}}^n\}$. Then $\widetilde{L}(z) = \frac{\beta|\mathbf{b}|}{m}\cdot\frac{L(z)}{(1-|z|)^\alpha}\in Q^n_{\mathbf{b}}(\mathbb{B}^n)$ for every $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$, $\alpha\ge 1$.*

Proof. Using the definition of $Q^n_{\mathbf{b}}$ we have $\forall z\in\mathbb{B}^n$

$$\lambda_1^{\mathbf{b}}(z,\eta,\widetilde{L}) = \inf\left\{\frac{L(z+t\mathbf{b})}{(1-|z+t\mathbf{b}|)^\alpha}\frac{(1-|z|)^\alpha}{L(z)} :\right.$$
$$\left.|t|\le\frac{\eta m(1-|z|)^\alpha}{\beta|b|L(z)}\right\}\ge$$
$$\ge\inf\left\{\frac{L(z+t\mathbf{b})}{L(z)} : |t|\le\frac{\eta m(1-|z|)^\alpha}{\beta|b|L(z)}\right\}\times$$
$$\inf\left\{\left(\frac{1-|z|}{1-|z+t\mathbf{b}|}\right)^\alpha : |t|\le\frac{\eta m(1-|z|)^\alpha}{\beta|b|L(z)}\right\}$$

Notice that if $\eta\in[0,\beta]$, $z\in\mathbb{B}^n$ and $|t|\le\frac{\eta}{L(z)}$ then $z+t\mathbf{b}\in\mathbb{B}^n$. Indeed, we have $|z+t\mathbf{b}|\le|z|+|t\mathbf{b}|\le|z|+\frac{\eta|\mathbf{b}|}{L(z)}<|z|+\frac{\beta|\mathbf{b}|}{\frac{\beta|\mathbf{b}|}{1-|z|}}=1$. Therefore, the first infimum is not lesser than a some constant $K>0$ which is independent from z and t_0. Besides, we have $\forall z\in\mathbb{B}^n$ and $\forall t\in S_z$ $\frac{m}{L(z)}\le 1$. Thus, for the second infimum the following estimates are valid

$$\inf\left\{\left(\frac{1-|z|}{1-|z+t\mathbf{b}|}\right)^\alpha : |t|\le\frac{\eta m(1-|z|)^\alpha}{\beta|b|L(z)}\right\}\ge$$
$$\ge\inf\left\{\left(\frac{1-|z|}{1-|z+t\mathbf{b}|}\right)^\alpha : |t|\le\frac{\eta(1-|z|)^\alpha}{\beta|b|}\right\}=$$
$$=\left(\frac{1-|z|}{1-|z+t^*\mathbf{b}|}\right)^\alpha.$$

where $|t^*|\leq\frac{\eta(1-|z|)}{\beta|b|}$. Now we find a lower estimate for this fraction

$$\frac{1-|z|}{1-|z+t^*\mathbf{b}|}\geq\frac{1-|z|}{1-||z|-|t^*\mathbf{b}||}\geq$$
$$\geq\frac{1-|z|}{1-||z|-\frac{\eta(1-|z|)}{\beta}|}$$

Denoting $u=|z|\in[0;1)$, $\gamma=\frac{\eta}{\beta}\in[0,1]$, we consider a function of one real variable $s(u)=\frac{1-u}{1-|u-\alpha(1-u)|}=\frac{1-u}{1-|(1+\gamma)u-\gamma|}$. For $u\in[0,\frac{\gamma}{\gamma+1}]$ the function $s(u)$ strictly decreases and for $t\in[\frac{\gamma}{1+\gamma};1)$ the function $s(u)\equiv\frac{1}{1+\gamma}$. In fact, we proved that $\lambda_1^{\mathbf{b}}(z,\eta,\widetilde{L})\geq K\cdot\frac{1}{1+\frac{\eta}{\beta}}>0$. Hence, we have $\lambda_1^{\mathbf{b}}(\eta,\widetilde{L})>0$. By analogy, it can be proved that $\lambda_2^{\mathbf{b}}(\eta,\widetilde{L})<\infty$.

We often use the following properties $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$.

Lemma 5. *1. If $L\in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$ then for every $\theta\in\mathbb{C}\setminus\{0\}$ $L\in Q_{\theta\mathbf{b},\beta/|\theta|}(\mathbb{B}^n)$ and $|\theta|L\in Q_{\theta\mathbf{b},\beta}(\mathbb{B}^n)$*

2. If $L\in Q_{\mathbf{b}_1,\beta}(\mathbb{B}^n)\bigcap Q_{\mathbf{b}_2,\beta}(\mathbb{B}^n)$ and for all $z\in\mathbb{B}^n$ $L(z)>\frac{\beta\max\{|\mathbf{b}_1|,|\mathbf{b}_2|,|\mathbf{b}_1+\mathbf{b}_2|\}}{1-|z|}$ then $\min\{\lambda_2^{\mathbf{b}_1}(\beta,L),\lambda_2^{\mathbf{b}_2}(\beta,L)\}L\in Q_{\mathbf{b}_1+\mathbf{b}_2,\beta}(\mathbb{B}^n)$.

Proof.

1. First, we prove that $(\forall\theta\in\mathbb{C}\setminus\{0\}):L\in Q_{\theta\mathbf{b},\beta}(\mathbb{B}^n)$. Indeed, we have by definition

$$\lambda_1^{\theta\mathbf{b}}(z,\eta,L)=\inf\left\{\frac{L(z+t\theta\mathbf{b})}{L(z)}:|t|\leq\frac{\eta}{L(z)}\right\}=$$
$$=\inf\left\{\frac{L(z+(t\theta)\mathbf{b})}{L(z)}:|\theta t|\leq\frac{|\theta|\eta}{L(z)}\right\}=$$
$$=\lambda_1^{\mathbf{b}}(z,|\theta|\eta,L).$$

Therefore, we get

$$\lambda_1^{\theta\mathbf{b}}(\eta,L)=\inf\{\lambda_1^{\theta\mathbf{b}}(z,\eta,L):z\in\mathbb{B}^n\}=$$
$$=\inf\{\lambda_1^{\mathbf{b}}(z,|\theta|\eta,L):z\in\mathbb{B}^n\}=\lambda_1^{\mathbf{b}}(|\theta|\eta,L)>0,$$

because $L\in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. Similarly, we prove that $\lambda_2^{\theta\mathbf{b}}(\eta,L)<+\infty$. But $|\theta|\eta\in[0,\beta]$. So $\eta\in[0,\beta/|\theta|]$. Thus, $L\in Q_{\theta\mathbf{b},\beta/|\theta|}(\mathbb{B}^n)$.

Let $L^*=|\theta|\cdot L$. Using definition of $\lambda_1^{\mathbf{b}}(z,\eta,L^*)$ we have

$$\lambda_1^{\theta\mathbf{b}}(z,\eta,L^*)=\inf\left\{\frac{L^*(z+t\theta\mathbf{b})}{L^*(z)}:|t|\leq\frac{\eta}{L^*(z)}\right\}=$$
$$=\inf\left\{\frac{|\theta|L(z+t\theta\mathbf{b})}{|\theta|L(z)}:|t|\leq\frac{\eta}{|\theta|L(z)}\right\}=$$
$$=\inf\left\{\frac{L(z+(t\theta)\mathbf{b})}{L(z)}:|\theta t|\leq\frac{\eta}{L(z)}\right\}=$$
$$=\lambda_1^{\mathbf{b}}(z,\eta,L).$$

Therefore, we obtain

$$\lambda_1^{\theta\mathbf{b}}(\eta,L^*)=\inf\{\lambda_1^{\theta\mathbf{b}}(z,\eta,L^*):z\in\mathbb{B}^n\}=$$
$$=\inf\{\lambda_1^{\mathbf{b}}(z,\eta,L):z\in\mathbb{B}^n\}=\lambda_1^{\mathbf{b}}(\eta,L)>0,$$

because $L\in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. Similarly, we prove that $\lambda_2^{\theta\mathbf{b}}(\eta,L^*)=\lambda_2^{\mathbf{b}}(\eta,L)<+\infty$. Thus, $L^*=|\theta|\cdot L\in Q_{\theta\mathbf{b},\beta}(\mathbb{B}^n)$.

2. It remains to prove a second part.

If $z^0\in\mathbb{B}^n$ and $|t|\leq\frac{\eta}{L(z^0)}$ then $z^0+t\mathbf{b}_1\in\mathbb{B}^n$ and $z^0+t\mathbf{b}_2\in\mathbb{B}^n$. Indeed, we have

$$|z^0+t\mathbf{b}_1|\leq|z^0|+|t|\cdot|\mathbf{b}_1|\leq|z^0|+\frac{\eta|\mathbf{b}_1|}{L(z^0))}<$$
$$<|z^0|+\frac{\beta|\mathbf{b}_1|}{\frac{\beta\max\{|\mathbf{b}_1|,|\mathbf{b}_2|,|\mathbf{b}_1+\mathbf{b}_2|\}}{1-|z^0|}}\leq1.$$

Thus, $z^0+t\mathbf{b}_1\in\mathbb{B}^n$.

Denote $L^*(z)=\min\{\lambda_2^{\mathbf{b}_1}(\beta,L),\lambda_2^{\mathbf{b}_2}(\beta,L)\}\cdot L(z)$. Assume that $\min\{\lambda_2^{\mathbf{b}_1}(\beta,L),\lambda_2^{\mathbf{b}_2}(\beta,L)\}=\lambda_2^{\mathbf{b}_2}(\beta,L)$. Using definitions of $\lambda_1^{\mathbf{b}}(\eta,L)$, $\lambda_2^{\mathbf{b}}(\eta,L)$ and $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$ we obtain that

$$\inf\left\{\frac{L^*(z^0+t(\mathbf{b}_1+\mathbf{b}_2))}{L^*(z^0)}:\ |t|\leq\frac{\eta}{L^*(z^0))}\right\}\geq$$
$$\geq\inf\left\{\frac{L^*(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{L^*(z^0+t\mathbf{b}_2)}:|t|\leq\frac{\eta}{L^*(z^0)}\right\}\times$$
$$\times\inf\left\{\frac{L^*(z^0+t\mathbf{b}_2)}{L^*(z^0)}:|t|\leq\frac{\eta}{L^*(z^0)}\right\}=$$
$$=\inf\left\{\frac{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{\lambda_2^{\mathbf{b}_1}(\beta,L)L(z^0+t\mathbf{b}_2)}:\right.$$
$$\left.|t|\leq\eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0))\right\}\times$$
$$\times\inf\{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+t\mathbf{b}_2)/\lambda_2^{\mathbf{b}_1}(\beta,L)L(z^0):$$
$$|t|\leq\eta/\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)\}=$$
$$=\inf\{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)/L(z^0+t\mathbf{b}_2):$$

$$|t-t_0|\le \eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+\mathbf{b}_2))\}\times$$
$$\times\inf\{L(z^0+t\mathbf{b}_2)/L(z^0):$$
$$|t|\le\eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0))\}\ge$$
$$\ge\inf\{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)/L(z^0+t\mathbf{b}_2):$$
$$|t|\le\eta/\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)\}\times$$
$$\times\inf\left\{\frac{L(z^0+t\mathbf{b}_2)}{L(z^0)}:|t-t_0|\le\frac{\eta}{L(z^0)}\right\}\ge$$
$$\ge\inf\{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)/L(z^0+t\mathbf{b}_2):$$
$$|t-t_0|\le\eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0))\}\lambda_1^{\mathbf{b}_2}(z^0,\eta,L)\ge$$
$$\ge\lambda_1^{\mathbf{b}_2}(\eta,L)\frac{L(z^0+\hat{t}\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{L(z^0+\hat{t}\mathbf{b}_2)} \qquad (7)$$

where $\hat{t}$ is a point at which infimum is attained

$$\frac{L(z^0+\hat{t}\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{L(z^0+\hat{t}\mathbf{b}_2)}=$$
$$=\inf\left\{\frac{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{L(z^0+t\mathbf{b}_2)}:|t|\le\frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)}\right\}.$$

But $L\in Q_{\mathbf{b}_2,\beta}(\mathbb{B}^n)$, then for all $\eta\in[0,\beta]$

$$\sup\left\{\frac{L(z^0+t\mathbf{b}_2)}{L(z^0)}:|t|\le\frac{\eta}{L(z^0)}\right\}\le\lambda_2^{\mathbf{b}_2}(\eta,L)<\infty.$$

Hence, $L(z^0+t\mathbf{b}_2)\le\lambda_2^{\mathbf{b}_2}(\eta,L)\cdot L(z^0)$, i.e. for $t=\hat{t}$ we have $L(z^0)\ge\frac{L(z^0+\hat{t}\mathbf{b}_2)}{\lambda_2^{\mathbf{b}_2}(\eta,L)}$. Using a proved inequality and (7), we obtain

$$\inf\{L^*(z^0+t(\mathbf{b}_1+\mathbf{b}_2))/L^*(z^0):|t|\le\eta/L^*(z^0)\}\ge$$
$$\ge\lambda_1^{\mathbf{b}_2}(\eta,L)\inf\left\{L(z^0+t\mathbf{b}_1+\hat{t}\mathbf{b}_2)/L(z^0+\hat{t}\mathbf{b}_2):\right.$$
$$\left.|t|\le\frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)}\right\}\ge\lambda_1^{\mathbf{b}_2}(\eta,L)\times$$
$$\times\inf\left\{L(z^0+t\mathbf{b}_1+\hat{t}\mathbf{b}_2)/L(z^0+\hat{t}\mathbf{b}_2):\right.$$
$$\left.|t|\le\frac{\eta\lambda_2^{\mathbf{b}_2}(\eta,L)}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0!+\hat{t}\mathbf{b}_2)}\right\}\ge$$
$$\ge\lambda_1^{\mathbf{b}_2}(\eta,L)\cdot\inf\{L(z^0+t\mathbf{b}_1+\hat{t}\mathbf{b}_2)/L(z^0+\hat{t}\mathbf{b}_2):$$
$$|t|\le\eta/L(z^0+\hat{t}\mathbf{b}_2)\}=\lambda_1^{\mathbf{b}_2}(\eta,L)\lambda_1^{\mathbf{b}_1}(z^0+\hat{t}\mathbf{b}_2,\eta,L)\ge$$
$$\ge\lambda_1^{\mathbf{b}_2}(\eta,L)\lambda_1^{\mathbf{b}_1}(\eta,L).$$

Therefore, $\lambda_1^{\mathbf{b}_1+\mathbf{b}_2}(\eta,L^*)\ge\lambda_1^{\mathbf{b}_2}(\eta,L)\lambda_1^{\mathbf{b}_1}(\eta,L)>0$. By analogy, we can prove that for all $\eta\in[0,\beta]$ $\lambda_2^{\mathbf{b}_1+\mathbf{b}_2}(\eta,L^*)<+\infty$. Thus, $L^*\in Q_{\mathbf{b}_1+\mathbf{b}_2,\beta}(\mathbb{B}^n)$.

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