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## Differential-Difference Games of Pursuit

Розглядається задача взаємодії переслідувача та втікача. У процесі гри кожен гравець обирає свої керування у вигляді деяких функцій. Мета переслідувача - вивести траєкторію процесу на термінальну множину за найкоротший час, мета втікача - відхилити траєкторію процесу від зустрічі з термінальною множиною на всьому напівнеперервному інтервалі часу $t \geq 0$ або, якщо це неможливо, то максимально відтягнути момент зустрічі.

Для переслідувача досліджені достатні умови на параметри процесу для приведення траєкторії на термінальну множину з будь-якого початкового положення за деякий гарантований час при будь-яких керуваннях втікача. Результати отримані Методом Розв'язуючих Функцій порівняно з Першим Прямим Методом Понтрягіна.

We consider the pursuit problem for 2-person conflict-controlled process with single pursuer and single evader. The problem is given by a system of differential-difference equations with time lag. The players pursuing their own goals and choose controls in the form of certain functions. The goal of the pursuer is to catch the evader in the shortest possible time. The goal of the evader is to avoid the meeting of the players' trajectories on a whole semiinfinite interval of time or if it is impossible to postpone maximally the moment of meeting. For such a conflict-controlled process we present conditions on its parameters and initial state, which are sufficient for the trajectories of the players to meet no later than a certain moment of time for any counteractions of the evader. Results obtained by the Method of Resolving Functions for such conflict-controlled process we also compare to Pontryagin's First Direct Method.

## 1 Introduction

A variety of interesting examples stimulated the development of the Dynamic Games Theory. Fundamental results in Differential Games Theory were obtained by Isaacs (1965), Pontryagin et al. (1962), and Krasovskii (1973). The basis for R.Isaacs' investigations was the Method of Dynamic Programming for Isaacs-Bellman equation. The classical Isaacs' scheme was later intensified by L.S.Pontryagin (Pontryagin 1965). Pontryagin's First Direct Method is the simplest and the most efficient method for solution of specific pursuit problems. This method afford conveniently checkable sufficient conditions for pursuit termination. Due to its versatility Pontryagin's First Direct Method gave rise to a number of extensions (Pshenichnyi and Chikrii (1977)).

Further development of Pontryagin's ideas resulted in the Method of Resolving Functions, one of the most powerful methods of dynamic game theory, which justifies, in particular, the rule of pursuit along straight ray and the classical rule of parallel approach, well-known to rocket designers and controllers (Chikrii 1992; Eidel'man and Chikrii 2000; Chikrii et al. 2007; Chikrii and Rappoport 2012; Chikrii 2014).

The essence of the Method of Resolving Functions is in the construction of some numeric resolving function of the known parameters of the process. The resolving function outlines the course of the process. At the moment at which its integral turns into unit, the trajectory of the process hits the terminal set.

It is necessary to apply ordinary differential equations to the investigation of mathematical models of different physical and technical objects. But this is not enough. Application of functional differential equations is more appropriate in such situations. The development of functional differential equation theory is connected with Bellman and Cooke (1963), Hale (1977), and Halanay (1966).

Baranovskaya (1999) obtained a general scheme of the Method of Resolving Functions for local convergence problems with fixed time, which are described by a system of differential-difference equations of delay-type

$$\dot{z}(t) = Az(t) + Bz(t-\tau) + \varphi(u,v),$$
$$z \in \mathbb{R}^n, u \in U, v \in V,$$

where $A, B$ are square matrices of order $n \times n$, $\varphi(u,v), \varphi : U \times V \to \mathbb{R}^n$ is a jointly continuous function, $U \in K(\mathbb{R}^n), V \in K(\mathbb{R}^n)$, $\tau =$ const $> 0$.

In this paper, we investigate the local convergence problem with fixed time, which is described by a system of differential-difference equations of delay-type in the form of Riemann-Stieltjes integrals. Necessary and sufficient conditions for solvability of such problems are established. Results obtained by the Method of Resolving Functions for such conflict-controlled process also compared to Pontryagin's First Direct Method.

Section 2 discusses some results from differential-difference equations theory and presents Cauchy's formula for a system of differential-difference equations with time lag.

Section 3 presents several auxiliary results from the theory of set-valued maps given in a form convenient for further implementation.

Section 4 is dedicated to solving of local convergence problem with fixed time for differential-difference games.

Section 5 presents a connection between results obtained by the Method of Resolving Functions with Pontryagin's First Direct Method.

## 2 Differential-Difference Equations with Time Delay

The facts exposed in this section deal with a situation with one actor, only. We need them later in the game setting. Let us consider a system of differential-difference equations of delay-type studied by Halanay (1966) (p. 362)

$$\dot{x}(t) = \int\limits_{-\infty}^{0} x(t+s)d_s\eta(t,s) \quad + \quad f(t), \quad (1)$$

where

(a) $\eta_{ij}(t,s)$ are defined for $t \geq 0$, $-\infty \leq s \leq \infty$, $\eta_{ij}(t,s) = 0$ for $s \geq 0$;

(b) there exist functions $\tau_{ij}(t) > 0, V_{ij}(t) > 0$, bounded for $t \geq 0$ such that

$$\eta_{ij}(t,s) \equiv \eta_{ij}(t, -\tau_{ij}(t)) \text{ for } s \leq -\tau_{ij}(t),$$
$$\vee_{s=-\tau_{ij}(t)}^{s=0} \eta_{ij}(t,s) \leq V_{ij}(t),$$

where, as usual, $\vee_{s=\alpha}^{s=\beta} f(s)$ means the total variation of function $f$ in $[\alpha, \beta]$. We set $\tau = sup_{i,j,t}\tau_{ij}(t)$;

(c) $\eta_{ij}(t,s)$ are continuous in t, uniformly with respect to s.

The adjoint system will be

$$\frac{d}{d\alpha}[y(\alpha) + \int\limits_{-\tau}^{0} \eta(\alpha-\beta,\beta)y(\alpha-\beta)d\beta] = 0. \quad (2)$$

System (2) may also be represented as

$$y(\alpha) + \int\limits_{-\tau}^{0} \eta(\alpha-\beta,\beta)y(\alpha-\beta)d\beta = const$$

or

$$y(\alpha) + \int\limits_{\alpha}^{\alpha+\tau} \eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma = const.$$

Then, for a fixed $\sigma$ we have

$$y(\alpha) + \int\limits_{\alpha}^{\alpha+\tau} \eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma =$$
$$= y(\sigma) + \int\limits_{\sigma}^{\sigma+\tau} \eta(\gamma,\sigma-\gamma)y(\gamma)d\gamma$$

or

$$y(\alpha) + \int\limits_{\alpha}^{\sigma} \eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma =$$
$$= y(\sigma) + \int\limits_{\sigma}^{\sigma+\tau} \eta(\gamma,\sigma-\gamma)y(\gamma)d\gamma -$$
$$- \int\limits_{\sigma}^{\alpha+\tau} \eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma.$$

If the function $y$ is defined on $[\sigma, \sigma+\tau]$, then for $\sigma - \tau \leq \alpha \leq \sigma$ the function

$$y(\sigma) + \int\limits_{\sigma}^{\sigma+\tau} \eta(\gamma,\sigma-\gamma)y(\gamma)d\gamma -$$
$$- \int\limits_{\sigma}^{\alpha+\tau} \eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma$$

is well defined and $y(\alpha)$ can be defined for values of $\alpha$ from the interval $[\sigma-\tau,\sigma]$. If $y$ is well defined on $[\sigma-\tau,\sigma]$ we can define it by the same procedure on $[\sigma-2\tau,\sigma-\tau]$, and thus through the step-by-step method ($[\sigma-2\tau,\sigma-\tau]\to[\sigma-3\tau,\sigma-2\tau]$ and so on) we can derive a theorem of the existence and uniqueness for equation (2). If the initial function defined on $[\sigma,\sigma+\tau]$ is of bounded variation then the solution defined by it is of bounded variation, and if the initial function is continuous, the solution is continuous as well.

Let us consider the matrix solution of equation (2) defined by the condition $Y(\alpha,\sigma)=0$ for $\sigma<\alpha\le\sigma+\tau, Y(\sigma,\sigma)=E$. Then

$$Y(\alpha,\sigma)+\int\limits_{\alpha}^{\alpha+\tau}\eta(\gamma,\alpha-\gamma)Y(\gamma,\sigma)d\gamma=E.$$

Let $x(t)$ be an arbitrary solution of system (1) and $y(t)$ be an arbitrary solution of system (2).

We consider

$$\int\limits_{\sigma}^{t}\dot{x}(\alpha)y(\alpha)d\alpha+\int\limits_{\sigma}^{t}x(\alpha)dy(\alpha)= \\ =x(t)y(t)-x(\sigma)y(\sigma).$$

The left hand side of the equation, after the substitution of (1) and (2), can be represented as

$$\begin{aligned}&\int\limits_{\sigma}^{t}[\int\limits_{-\tau}^{0}x(\alpha+s)d_s\eta(\alpha,s)]y(\alpha)d\alpha- \\ &-\int\limits_{\sigma}^{t}x(\alpha)d_\alpha\int\limits_{\alpha}^{\alpha+\tau}\eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma+ \\ &+\int\limits_{\sigma}^{t}f(\alpha)y(\alpha)d\alpha= \\ &=\int\limits_{\sigma}^{t}[\int\limits_{\alpha-\tau}^{\alpha}x(s)d_s\eta(\alpha,s-\alpha)]y(\alpha)d\alpha- \\ &-\int\limits_{\sigma}^{t}x(\alpha)d_\alpha\int\limits_{\alpha}^{\alpha+\tau}\eta(\gamma,\alpha-\gamma)y(\gamma)d\gamma+ \\ &+\int\limits_{\sigma}^{t}f(\alpha)y(\alpha)d\alpha.\end{aligned}$$

Considering that

$$d_s\int\limits_{p(s)}^{r(s)}\varphi(s,\alpha)d\alpha=[\varphi(s,r(s))(r(s))'- \\ -\varphi(s,p(s))(p(s))'+\int\limits_{p(s)}^{r(s)}\varphi'_s(s,\alpha)d\alpha]ds$$

we can change integral bounds according to the Fig. 1.

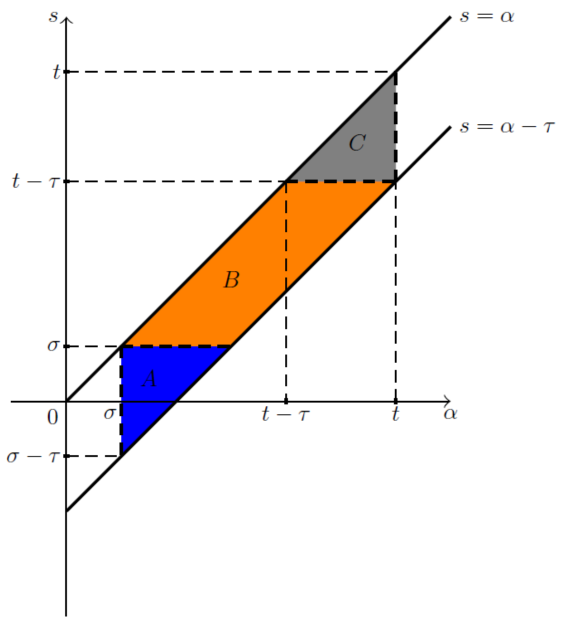


Fig. 1. Integral bounds;

Then we obtain

$$\begin{aligned}&x(t)y(t)-x(\sigma)y(\sigma)= \\ &\left.\begin{aligned}&=\int\limits_{\sigma-\tau}^{\sigma}x(s)d_s\int\limits_{\sigma}^{s+\tau}\eta(\alpha,s-\alpha)y(\alpha)d\alpha- \\ &-\int\limits_{\sigma-\tau}^{\sigma}x(s)\eta(\alpha,-\tau)y(s+\tau)ds+\end{aligned}\right\}\text{A} \\ &\left.\begin{aligned}&+\int\limits_{\sigma}^{t-\tau}x(s)d_s\int\limits_{s}^{s+\tau}\eta(\alpha,s-\alpha)y(\alpha)d\alpha- \\ &-\int\limits_{\sigma}^{t-\tau}x(s)\eta(\alpha,-\tau)y(s+\tau)ds+\end{aligned}\right\}\text{B} \\ &\underbrace{+\int\limits_{t-\tau}^{t}x(s)d_s\int\limits_{s}^{t}\eta(\alpha,s-\alpha)y(\alpha)d\alpha-}_{\text{C}}\end{aligned}$$

$$-\int_{\sigma}^{t} x(s)d_s \int_{s}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha +$$

$$+\int_{\sigma}^{t} f(\alpha)y(\alpha)d\alpha.$$

From this we derive

$$x(t)y(t) - x(\sigma)y(\sigma) =$$
$$= \int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha -$$
$$-\int_{t-\tau}^{t} x(s)d_s \int_{s}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha -$$
$$-\int_{t-\tau}^{t} x(s)d_s \int_{t}^{s} \eta(\alpha, s-\alpha)y(\alpha)d\alpha -$$
$$-\int_{\sigma}^{t} x(s-\tau)\eta(\alpha,-\tau)y(s)ds +$$
$$+\int_{\sigma}^{t} f(\alpha)y(\alpha)d\alpha.$$

This equation also can be written as

$$x(t)y(t) - x(\sigma)y(\sigma) =$$
$$= \int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha -$$
$$-\int_{t-\tau}^{t} x(s)d_s \int_{t}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha -$$
$$-\int_{\sigma}^{t} x(s-\tau)\eta(\alpha,-\tau)y(s)ds +$$
$$+\int_{\sigma}^{t} f(\alpha)y(\alpha)d\alpha.$$

Therefore,

$$x(t)y(t) + \int_{t-\tau}^{t} x(s)d_s \int_{t}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha =$$
$$= x(\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)y(\alpha)d\alpha -$$
$$-\int_{\sigma}^{t} x(s-\tau)\eta(\alpha,-\tau)y(s)ds +$$
$$+\int_{\sigma}^{t} f(\alpha)y(\alpha)d\alpha.$$

Using the same procedure for the solution $x(t)$ of system (1) and the matrix solution $Y(\alpha, \sigma)$ of system (2) we have

$$x(t)Y(t,t) - x(\sigma)Y(\sigma,t) =$$
$$= \int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)Y(\alpha,t)d\alpha +$$
$$+\int_{t-\tau}^{t} x(s)d_s \int_{s}^{t} \eta(\alpha, s-\alpha)Y(\alpha,t)d\alpha +$$
$$+\int_{\sigma}^{t-\tau} x(s)d_s \int_{s}^{s+\tau} \eta(\alpha, s-\alpha)Y(\alpha,t)d\alpha -$$
$$-\int_{\sigma}^{t} \eta(\alpha,-\tau)x(s-\tau)Y(s,t)ds +$$
$$+\int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha.$$

From this we obtain

$$x(t) = x(\sigma)Y(\sigma,t) +$$
$$+\int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)Y(\alpha,t)d\alpha +$$
$$+\int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha -$$
$$-\int_{\sigma}^{t} \eta(\alpha,-\tau)x(s-\tau)Y(s,t)ds +$$
$$+\int_{\sigma}^{t} x(\alpha)d_\alpha[Y(\alpha,t) +$$
$$+\int_{\alpha}^{\alpha+\tau} \eta(\beta, \alpha-\beta)Y(\beta,t)d\beta].$$

Considering that

$$Y(\alpha,t) + \int_{\alpha}^{\alpha+\tau} \eta(\beta, \alpha-\beta)Y(\beta,t)d\beta = E,$$

where $E$ is the unit matrix, we get

$$x(t) = x(\sigma)Y(\sigma,t) +$$
$$+\int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)Y(\alpha,t)d\alpha -$$
$$-\int_{\sigma}^{t} \eta(\alpha,-\tau)x(s-\tau)Y(s,t)ds +$$
$$+\int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha.$$

Let $X(t,\sigma)$ be the matrix whose rows for $t > \sigma$ are solutions of a homogeneous system, $X(t,\sigma) = 0$ for $t < \sigma$, $X(\sigma,\sigma) = E$. It will then follow that $X(t,\sigma) = Y(\sigma,t)$.

Thus we finally obtain

$$x(t) = x(\sigma)X(t,\sigma) +$$
$$+ \int\limits_{\sigma-\tau}^{\sigma} x(s)d_s \int\limits_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha -$$
$$- \int\limits_{\sigma}^{t} \eta(\alpha,-\tau)x(s-\tau)X(t,s)ds +$$
$$+ \int\limits_{\sigma}^{t} f(\alpha)X(t,\alpha)d\alpha.$$

In the following remark points $(a),(b),(c)$ look almost like points $(a),(b),(c)$ at the beginning of Section 2. The only differences are $\eta_{ij}(t,s) = \eta_{ij}(t,0)$ at point $(a)$ and $\eta_{ij}(t,s) \equiv 0$ at point $(b)$.

**Remark 1.** *For the system of differential-difference equations of delay-type*

$$\dot{x}(t) = \int\limits_{-\infty}^{0} x(t+s)d_s\eta(t,s) \quad + \quad f(t),$$

*where*

*(a)* $\eta_{ij}(t,s)$ *are defined for* $t \geq 0$, $-\infty \leq s \leq \infty$, $\eta_{ij}(t,s) = \eta_{ij}(t,0)$ *for* $s \geq 0$;

*(b) there exist functions* $\tau_{ij}(t) > 0, V_{ij}(t) > 0$, *bounded for* $t \geq 0$ *such that*

$$\eta_{ij}(t,s) \equiv 0 \text{ for } s \leq -\tau_{ij}(t),$$

$$\vee_{s=-\tau_{ij}(t)}^{s=0}\eta_{ij}(t,s) \leq V_{ij}(t),$$

*where, as usual,* $\vee_{s=\alpha}^{s=\beta} f(s)$ *means the total variation of the function* $f$ *on* $[\alpha,\beta]$. *We set* $\tau = sup_{i,j,t}\tau_{ij}(t)$;

*(c)* $\eta_{ij}(t,s)$ *are continuous in* $t$, *uniformly with respect to* $s$.

*the solution can be represented in the form [ Halanay (1966) (p. 366)]*

$$x(t) = x(\sigma)X(t,\sigma) +$$
$$+ \int\limits_{\sigma-\tau}^{\sigma} x(s)d_s \int\limits_{\sigma}^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha +$$
$$+ \int\limits_{\sigma}^{t} f(\alpha)X(t,\alpha)d\alpha,$$

*where* $X(t,s)$ *is the matrix whose rows for* $t > s$ *are solutions of the homogeneous system and has the following properties:*

1. $X(t,s) \equiv 0$, *for* $t < s$;
2. $X(t,t) = E$, $E$ *- unit matrix;*

**Example 1.** *We consider the system of differential-difference equations*

$$\dot{x}(t) = \int\limits_{-\infty}^{0} x(t+s)d_s\eta(t,s) \quad + \quad \varphi(t), \quad x \in \mathbb{R}^n, \tag{3}$$

*where*

$$\eta(t,s) = \begin{cases} 0, & s \in (-\infty,-\tau], \\ B, & s \in (-\tau,0), \\ A+B, & s \in [0,+\infty), \end{cases}$$
$$\tau(t) = \tau = const.$$

*Then system* (3) *can be represented in the form*

$$\dot{x}(t) = \int\limits_{-\infty}^{0} x(t+s)d_s\eta(t,s) + \varphi(t) =$$
$$= Ax(t) + Bx(t-\tau) + \varphi(t).$$

*The solution of system* (3) *can be written in the form (Remark* 1*)*

$$x(t) = x^0(0)X(t,0) +$$
$$+ \int\limits_{-\tau}^{0} x(s)d_s \int\limits_{0}^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha +$$
$$+ \int\limits_{0}^{t} \varphi(s)X(t,s)ds,$$

*For* $0 \leq \alpha \leq s+\tau$ *we have* $-\tau \leq s-\alpha \leq s \leq 0$. *I.e.*

$$\eta(\alpha, s-\alpha) = B \quad for \quad -\tau \le s-\alpha \le 0.$$

*Thus we have finally obtained*

$$x(t) = x^0(0)X(t,0) + +B\int_{-\tau}^{0} x(s)X(t,s+\tau)ds + +\int_0^t \varphi(s)X(t,s)ds,$$

*which is Cauchy's formula for the system of differential-difference equations of delay type with constant delay [ Bellman and Cooke (1963) (p. 201) ].*

## 3 Auxiliary Mathematical Results

We will need the following nine lemmas for the proof of Theorem 1 in Section 4.

**Lemma 1.** *Let the map $F(x,y), F : X\times Y \to K(\mathbb{R}^n), X, Y \in K(\mathbb{R}^n)$ be continuous. Then the map $G(x) = \bigcap_{y\in Y} F(x,y)$ is upper semicontinuous on the set $X \cap domG$. [Chikrii (1997) (p.14)]*

**Lemma 2.** *Let $X \in K(\mathbb{R}^n), F(x), F : X \to K(\mathbb{R}^n)$ be a measurable (Borel) map. Then the selection*

$$f(x) = lex\ min\ F(x), \quad x \in X,$$

*is measurable (Borel). [Chikrii (1997) (p.15)]*

**Lemma 3.** *Let $X \in K(\mathbb{R}^n), T > 0, f(t,x) : [0,T]\times X \to \mathbb{R}$ be a bounded, measurable in $t$ and upper semicontinuous in $x$ function. Then the function $f(t) = \inf_{x\in X} f(t,x)$ is measurable on $[0,T]$. [Ioffe and Tihomirov (1979) (p. 345)]*

**Lemma 4.** *Let $X, Y, Z \in K(\mathbb{R}^n)$, a function $f(x)$, $f : X \to Y$ be Lebesgue measurable (Borel), function $g(y)$, $g : Y \to Z$, - Borel (upper semicontinuous). Then the function $h(x) = g(f(x))$, $h : X \to Z$ is Lebesgue measurable (Borel) [Ioffe and Tihomirov (1979) (p. 345)].*

**Lemma 5.** *Let $X, Y, M \in K(\mathbb{R}^n)$ and $F(x,y)$, $F : X\times Y \to K(\mathbb{R}^n)$ be upper semicontinuous set-valued map, $f(x)$, $f : X \to \mathbb{R}^n$ be continuous, $f(X)\cap M = \emptyset$ and $con(M - f(x)) \cap F(x,y) \ne \emptyset$ for all $x \in X$, $y \in Y$. Then the function $\alpha : X\times Y \to \mathbb{R}$, defined by the formula*

$$\alpha(x,y) = \max\{\alpha \ge 0 : \alpha(M - f(x)) \cap F(x,y) \ne \emptyset\}$$

*is upper semicontinuous on the set $X\times Y$. [Chikrii (1997) (p.17)]*

**Lemma 6.** *Let $X \in K(\mathbb{R}^n)$, $f(s,x)$, $f : [0,T]\times X \to \mathbb{R}$ be a measurable in $s$ and continuous in $x$ function, uniformly bounded on $[0,T]\times X$. Then*

$$\inf_{x(\cdot)\in\Omega_x} \int_0^T f(s,x(s))ds = \int_0^T \inf_{x\in X} f(s,x)ds,$$

*where $\Omega_x = \{x(\cdot) : x(t) \in X, t \ge 0, x(t)$ is measurable$\}$. [Ioffe and Tihomirov (1979) (p. 356)]*

**Lemma 7.** *Let $X \in K(\mathbb{R}^n), \lambda(x), \lambda : \mathbb{R}^n \to \mathbb{R}$ be an upper semicontinuous (measurable) function. Then the map $\lambda(x)\cdot X$ is Borel (measurable). [Chikrii (1997) (p.15)]*

**Lemma 8.** *Let $X \in K(\mathbb{R}^n)$, set-valued maps $F(x), F : X \to K(\mathbb{R}^n)$ and $G(x), G : X \to K(\mathbb{R}^n)$ be meadurable (Borel), function $f(x,y), x \in X, y \in Y, f(x,y) \in \mathbb{R}^n$ be measurable (Borel) in $x \in X$ and continuous in $y \in Y$. Then the set-valued map*

$$H(x) = \{y \in G(x) : f(x,y) \in F(x)\}$$

*is measurable (Borel). [Chikrii (1992) (p.26)]*

**Lemma 9.** *Let $X \in K(\mathbb{R}^n), \alpha(s), \alpha : [0,\infty) \to \mathbb{R}$ be a nonnegative bounded measurable function. Then*

$$\int_0^T \alpha(s)Xds = \int_0^T \alpha(s)ds\cdot coX, \quad T > 0.$$

*[Ioffe and Tihomirov (1979) (p. 349)]*

## 4 Differential-Difference Games of Delay-Type with Variable Delay

We consider a pursuit problem given by the system of the differential-difference equations of delay-type studied by Halanay (1966) (p. 362)

$$\dot{z}(t) = \int_{-\infty}^{0} z(t+s)d_s\eta(t,s) \; + \; \varphi(u,v),$$
$$z \in \mathbb{R}^n, u \in U, v \in V, \qquad (4)$$

where

(a) $\eta_{ij}(t,s)$ are defined for $t \geq 0$, $-\infty \leq s \leq \infty$, $\eta_{ij}(t,s) = \eta_{ij}(t,0)$ for $s \geq 0$;

(b) there exist functions $\tau_{ij}(t) > 0, V_{ij}(t) > 0$, bounded for $t \geq 0$ such that

$$\eta_{ij}(t,s) \equiv 0 \text{ for } s \leq -\tau_{ij}(t),$$
$$\vee_{s=-\tau_{ij}(t)}^{s=0} \eta_{ij}(t,s) \leq V_{ij}(t),$$

where, as usual, $\vee_{s=\alpha}^{s=\beta} f(s)$ means the total variation of a function $f$ on $[\alpha, \beta]$. We set $\tau = sup_{i,j,t}\tau_{ij}(t)$;

(c) $\eta_{ij}(t,s)$ are continuous in t, uniformly with respect to s.

$\varphi(u,v), \varphi : U \times V \to \mathbb{R}^n$ is a jointly continuous function, $U \in K(\mathbb{R}^n), V \in K(\mathbb{R}^n)$.

The initial state of system (4) is an absolutely continuous function

$$z(t) = z^0(t), \quad t \in [-\tau, 0].$$

At a moment $t$ we have

$$z^t(\cdot) = \{z(t+s), -\tau \leq s \leq 0\}.$$

The terminal set is cylindrical and will be denoted by

$$M^* = M_0 + M \qquad (5)$$

where $M_0$ is a linear subspace of $\mathbb{R}^n$, $M$ is a nonempty compact in the orthogonal complement $L$ to $M_0$ in $\mathbb{R}^n$. We consider a local convergence problem with a fixed time. We will use the symbol $\pi$ to denote the orthogonal projection from $\mathbb{R}^n$ to $L$. Let us denote the set-valued maps

$$W(t,s,v) = \pi X(t,s)\varphi(U,v),$$
$$W(t,s) = \bigcap_{v \in V} W(t,s,v), \quad t \geq 0$$

where $X(t,\sigma)$ is the matrix whose rows for $t > \sigma$ are solutions of homogeneous system $X(t,\sigma) = 0$, for $t < \sigma$ $X(\sigma,\sigma) = E$.

**Condition 1.** *(Pontryagin's condition)* $W(t,0) \neq \emptyset$ *for all* $t \geq 0$.

As the matrix $X(t,s)$ is continuous, the set-valued map $W(t,s,v)$ is also continuous on the set $[0,+\infty) \times V$. Consequently, when condition 1 is performed, by Lemma 1 $W(t,0)$ is upper semicontinuous and therefore a Borel map. Hence, by Lemma 2, there exists at least one Borel selection $\gamma(t,s)$, $\gamma(t,s) \in W(t,0)$, $t \geq 0$.

From now on, $\Gamma = \{\gamma(\cdot,\cdot) \in \Gamma, \quad t \geq 0\}$ denotes the set of all Borel selections of the set-valued map $W(t,0)$. Fix some $\gamma(\cdot,\cdot) \in \Gamma$, put

$$\xi(t, z^0(\cdot), \gamma(\cdot,\cdot)) = \pi X(t,0)z^0(0) +$$
$$+ \int_{-\tau}^{0} \pi z^0(s) d_s \int_{0}^{s+\tau} \eta(\alpha, s-\alpha) X(t,\alpha) d\alpha +$$
$$+ \int_{0}^{t} \gamma(t,s) ds \qquad (6)$$

and look at the following resolving function

$$\alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot)) =$$
$$\sup\{\alpha \geq 0 : [W(t,s,v) - \gamma(t,s)] \cap$$
$$\cap \alpha[M - \xi(t,z^0(\cdot),\gamma(\cdot,\cdot))] \neq \emptyset\} \qquad (7)$$

where $0 \leq s \leq t, \quad z \in \mathbb{R}^n, \quad v \in V$.

Since $0 \in W(t,s,v) - \gamma(t,s)$ for all $v \in V$, $0 \leq s \leq t$, for $\xi(t,z^0(\cdot),\gamma(\cdot,\cdot)) \in M$ we have that the function $\alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot)) = +\infty$ for all $s \in [0,t]$, $v \in V$. If $\xi(t,z^0(\cdot),\gamma(\cdot,\cdot)) \notin M$ then resolving function (7) takes finite values which are uniformly bounded jointly in $s \in [0,t]$, $v \in V$.

Let us define the function

$$T(z^0(\cdot),\gamma(\cdot,\cdot)) = \inf\{t \geq 0 :$$
$$\int_0^t \inf_{v \in V} \alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot))ds \geq 1\},$$
$$\gamma \in \Gamma. \qquad (8)$$

If the inequality in the braces does not hold for all $t \geq 0$, then we set $T(z^0(\cdot),\gamma(\cdot,\cdot)) = +\infty$. If $\xi(t,z^0(\cdot),\gamma(\cdot,\cdot)) \notin M$ then by Lemma 3 the function $\inf_{v \in V} \alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot))$ is measurable in $s$. The function

$\alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot))$ is a jointly Borel function by Lemma 4 and is uniformly bounded in $s,v$ by Lemma 5. Consequently, by Lemma 6 the function $\inf_{v\in V}\alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot))$ is integrable on $[0,t]$. If $\xi(t,z^0(\cdot),\gamma(\cdot,\cdot)) \in M$ then $\inf_{v\in V}\alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot)) = +\infty,\ s\in[0,t]$, and it is natural to set the integral equal to $+\infty$, therefore the inequality in the definition of function $T(z^0(\cdot),\gamma(\cdot,\cdot))$ holds automatically.

**Theorem 1.** *Let Condition* 1 *hold for the conflict-controlled process* (4), (5), *$M$ be a convex set, $T(z^0(\cdot),\gamma^0(\cdot,\cdot)) < +\infty$ for initial position $z^0(\cdot)$, and some selection $\gamma^0(\cdot,\cdot)\in\Gamma$. Then the trajectory of process* (4) *can be brought from the initial position $z^0(\cdot)$ to the terminal set at moment $T(z^0(\cdot),\gamma^0(\cdot,\cdot))$.*

Outline of the proof.

We consider two cases: $\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) \notin M$ and $\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) \in M$. For the case $\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) \notin M$ we set the pursuer's control on two intervals: active $[0,t_*]$ and passive $[t_*,T]$. For the case $\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) \in M$ we choose the pursuer's control on a whole interval $[0,T)$. According to Cauchy's formula and taking into account the pursuer's control law we obtain the desired results.

## 5 Connection with Pontryagin's First Direct Method

**Theorem 2.** *(Pontryagin's theorem) Let Condition* 1 *hold for the conflict-controlled process* (4), (5) *and for some initial state $z^0(\cdot)$ : $P(z^0(\cdot)) < +\infty$, where*

$$P(z^0(\cdot)) = min\{t \geq 0 : \pi X(t,0)z^0(0) + \int_{-\tau}^{0}\pi z^0(s)d_s\int_{0}^{s+\tau}\eta(\alpha,s-\alpha)X(t,\alpha)d\alpha \in M - \int_0^t W(t,s)ds\}. \quad (9)$$

*Then the trajectory of process* (4) *can be brought from the initial position $z^0(\cdot)$ to the terminal set at moment $P(z^0(\cdot))$.*

Proof.

Denote $P_0 = P(z^0(\cdot))$. Then

$$\pi X(t,0)z^0(0) + \int_{-\tau}^{0}\pi z^0(s)d_s\int_{0}^{s+\tau}\eta(\alpha,s-\alpha)X(t,\alpha)d\alpha \in M - \int_0^t W(t,s)ds.$$

This means there exist $m\in M$ and a selection $\gamma(\cdot,\cdot)\in\Gamma$ such that

$$\pi X(t,0)z^0(0) + \int_{-\tau}^{0}\pi z^0(s)d_s\int_{0}^{s+\tau}\eta(\alpha,s-\alpha)X(t,\alpha)d\alpha = m - \int_0^t \gamma(t,s)ds.$$

Let us consider the set-valued map

$$U(s,v) = \{u\in U : \pi X(P_0,s)\varphi(u,v) - \gamma^0(P_0,s) = 0\},\ s\in[0,P_0],\ v\in V. \quad (10)$$

It is Borel jointly in $s,v$. The selection

$$u(s,v) = lexminU(s,v)$$

is a Borel function jointly in $s,v$ by Lemma 2. We set the pursuer's control on the interval $[0,P_0]$ equal to

$$u(s) = u(s,v(s)).$$

According to (10), in view of (9), we get

$$\pi z(P_0) = \pi X(P_0,0)z^0(0) + \int_{-\tau}^{0}\pi z^0(s)d_s\int_{0}^{s+\tau}\eta(\alpha,s-\alpha)X(P_0,\alpha)d\alpha + \int_0^{P_0}\pi X(P_0,s)\varphi(u(s),v(s))ds = m\in M. \quad (11)$$

This completes the proof of the theorem.

**Theorem 3.** *Suppose Condition* 1 *hold for conflict-controlled process* (4), (5). *Then for any initial state $z^0(\cdot)$ there exists a selection $\gamma^0(\cdot,\cdot)\in\Gamma$ such that*

$$T(z^0(\cdot),\gamma^0(\cdot,\cdot)) \leq P(z^0(\cdot)).$$

Proof.

Let $T = T(z^0(\cdot), \gamma^0(\cdot,\cdot))$ be the moment of the end of the game. Considering

$$T(z^0(\cdot), \gamma^0(\cdot,\cdot)) = \inf\{t \geq 0 : \int_0^t \inf_{v\in V} \alpha(t,s,z^0(\cdot),v,\gamma^0(\cdot,\cdot))ds \geq 1\}$$

we suppose that at moment $t_*$ the integral of the resolving function is equal 1, that means $\int_0^{t_*} \alpha(t,s,z^0(\cdot),v,\gamma^0(\cdot,\cdot))ds = 1$, and on the interval $[t_*, T]$ it is natural to set $\alpha = 0$. That means that $\int_0^t \alpha(t,s,z^0(\cdot),v,\gamma^0(\cdot,\cdot))ds \leq 1$ for $t \in [0,T]$. Thus $\alpha(t,s,z^0(\cdot),v,\gamma^0(\cdot,\cdot)) \neq \infty$ for all $t \in [0,T]$. Since for $\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) \in M$ we have $\alpha(t,s,z^0(\cdot),v,\gamma^0(\cdot,\cdot)) = +\infty$, one gets $\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) \notin M \quad \forall t \in [0,T]$. Therefore $\forall t \in [0,T] \quad \nexists m \in M$ such that

$$\pi X(t,0)z^0(0) + \\ + \int_{-\tau}^0 \pi z^0(s)d_s \int_0^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha + \\ + \int_0^t \gamma(t,s)ds = m.$$

I.e. for all $t \in [0,T]$

$$\pi X(t,0)z^0(0) + \\ + \int_{-\tau}^0 \pi z^0(s)d_s \int_0^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha \notin \\ \notin M - \int_0^t W(t,s)ds.$$

This means that on the interval $[0,T]$ Pontryagin's theorem does not hold and there is no $P(z^0(\cdot))$ such that the trajectory of process (4) can be brought from the initial position $z^0(\cdot)$ to the terminal set.

If $P(z^0(\cdot))$ is the moment of the end of the game then, by the virtue of Pontryagin's theorem,

$$\pi X(t,0)z^0(0) + \\ + \int_{-\tau}^0 \pi z^0(s)d_s \int_0^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha \in \\ \in M - \int_0^t W(t,s)ds.$$

Therefore there are $m \in M$ and a selection $\gamma^0(\cdot,\cdot) \in \Gamma$ such that

$$\pi X(t,0)z^0(0) + \\ + \int_{-\tau}^0 \pi z^0(s)d_s \int_0^{s+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha = \\ = m - \int_0^t \gamma^0(t,s)ds$$

or

$$\xi(t,z^0(\cdot),\gamma^0(\cdot,\cdot)) = m \in M.$$

It follows that if the resolving function satisfies $\alpha(t,s,z^0(\cdot),v,\gamma^0(\cdot,\cdot)) = +\infty$, then $\int_0^t \alpha(t,s,z^0(\cdot),v,\gamma(\cdot,\cdot))ds = +\infty > 1$. Thus by the Method of Resolving Functions we can finish the game at the moment $T = T(z^0(\cdot),\gamma^0(\cdot,\cdot))$. It follows that if the resolving function satisfies

$$T(z^0(\cdot),\gamma^0(\cdot,\cdot)) \leq P(z^0(\cdot)).$$

This completes the proof of the theorem.

## 6 Conclusions

In this article we present a general scheme of the Method of Resolving Functions for the local convergence problem with fixed time. The conflict-controlled process is described by a system of differential-difference equations of delay-type with variable delay. The performance of these sufficient conditions of Theorem 1 is enough for capturing the evader at a fixed moment of time. If the evader makes mistakes then in the course of pursuit there is a moment of switching from the Method of Resolving Functions to Pontryagin's First Direct Method. As the result, the process hits the terminal set at the predetermined time. Analyzing Theorem 2 one can also get sufficient conditions for capturing the evader at a fixed moment of time. Results obtained by the Method of Resolving Functions for such conflict-controlled process we compare to Pontryagin's First Direct Method.

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