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RATIONALLY REVERSIBLE CUBIC SYSTEMS

For cubic differential system with a singular point $O(0,0)$ a weak focus it was found coefficient conditions for $O(0,0)$ to be a center. The presence of a center was proved by using the method of rational reversibility.

1. Introduction

By using a nondegenerate transformation of variables and a time rescaling, a cubic system with a singular point with pure imaginary eigenvalues ($\lambda_{1,2} = \pm i$, $i^2 = -1$) can be brought to the form

$$\begin{array}{l} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + \\ \quad +mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + \\ \quad +qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{array} \qquad (1)$$

where the variables x, y and coefficients $a, b, \ldots, s$ in (1) are assumed to be real. Then the origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. it is a weak focus. It arises the problem of distinguishing between a center and a focus, i.e. of finding the coefficient conditions under which $O(0,0)$ is, for example, a center. This problem is the so called the "problem of the center"and the corresponding conditions are called the "center conditions".

The derivation of necessary conditions for a center existence often involves extensive use of computer algebra (see, for example, [10], [12]), in many cases making very heavy demands on the available algorithms and hardware. The necessary conditions are shown to be sufficient by a variety of methods. A number of techniques, of progressively wider application, have been developed.

A theorem of Poincaré in [13] says that the singular point $O(0,0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral $F(x,y) = C$ in a neighborhood of $O(0,0)$. It is known [1] that the origin is a center for system (1) if and only if the system has in the neighborhood of $O(0,0)$ an analytic integrating factor of the form

$$\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y),$$

where μ_k are homogeneous polynomials of degree k.

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$dF/dt = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

Quantities L_j, $j = \overline{1,\infty}$ are polynomials with respect to the coefficients of system (1) called to be the Lyapunov quantities [11]. The origin $O(0,0)$ is a center for (1) if and only if

$$L_j = 0, \ j = \overline{1,\infty}.$$

A singular point $O(0,0)$ is a center for (1) if the equations of (1) are invariant under reflection in a line through the origin and reversion of time, called time-reversible systems. The classical condition is that the system is invariant under one or other of the transformations $(x,y,t) \to (-x,y,-t)$ or $(x,y,t) \to (x,-y,-t)$. The first corresponds to reflection the y-axis and the second to reflection in the x-axis.

The time-reversibility in two-dimensional autonomous systems was studied in [14] and the relation between time-reversibility and the center-focus problem was discussed in [18].

In [9] by using the method of rational reversibility it was found center conditions for cubic differential system (1) with one invariant straight line.

The paper is organized as follows. In Section 2 we present the results concerning the problem of the center for cubic systems (1)

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and formulate the main result. In Section 3 we describe the algorithm to transform a cubic system to one which is symmetric in a line by means of a rational transformation and finally in Section 4 we find six series of conditions for (1) to be rationally reversible and therefore a singular point $O(0,0)$ to be a center.

2. Statement of the main result

An algebraic curve $f(x,y)=0$ is said to be an invariant curve of system (1) if there exists a polynomial $K(x,y)$ such that

$$P\cdot\partial f/\partial x+Q\cdot\partial f/\partial y=K\cdot f.$$

The polynomial K is called the cofactor of the invariant algebraic curve $f=0$. If the cubic system (1) has sufficiently many invariant algebraic curves $f_j(x,y)=0,\ j=\overline{1,q}$, then in most cases a first integral (an integrating factor) can be constructed in the Darboux form

$$f_1^{\alpha_1}f_2^{\alpha_2}\cdots f_q^{\alpha_q}$$

with $\alpha_j\in\mathbb{C}$ not all zero. In this case we say that the system (1) is Darboux integrable.

In [20] Żołądek mentioned three general mechanisms for producing centers: searching for 1) a Darboux first integral or 2) a Darboux–Schwarz–Christoffel first integral or by 3) generating centers by rational reversibility, and he claimed that these are sufficient for producing all cases of real polynomial differential systems with centers. This conjecture is still open, even for cubic systems (1).

The problem of the center was solved for quadratic systems and for cubic symmetric systems. If the cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of finding a finite number of necessary and sufficient conditions for the center is still open. It was possible to find a finite number of conditions for the center only in some particular cases (see, for example, [2-10, 15-17, 19]).

The problem of the center was completely solved for cubic systems with at least three invariant straight lines ([3, 4, 5, 17]) and for some classes of cubic systems (1) with two invariant straight lines and one invariant conic ([6, 7, 8]). The main results of these works are summarized in the following theorem.

Theorem 1. Every center in the cubic differential system (1) with:

1) three invariant straight lines comes from a Darboux integrating factor or a rational reversibility.

2) two invariant straight lines and one invariant conic comes from a Darboux first integral or a Darboux integrating factor.

The goal of this paper is to obtain the center conditions for cubic differential system (1) by using the method of rational reversibility. Our main result is the following one.

Theorem 2. The cubic differential system (1) is rationally reversible if and only if one of the conditions 1)– 6) is satisfied.

We shall prove Theorem 2 in Section 4. There we shall provide explicit expressions for center conditions.

3. Bilinear transformation in cubic systems

It is well known from Poincaré [13] that if a differential system with a singular point $O(0,0)$ a weak focus is invariant by the reflection with respect, for example, to the axis $X=0$ and reversion of time then $O(0,0)$ is a center for (1) ($X=0$ is called the axis of symmetry). It is clear that (1) has a center at $O(0,0)$ if there exists a diffeomorphism $\Phi: U\to V$, $\Phi=\{X=\varphi(x,y),\ Y=\psi(x,y)\}$, $\Phi(0,0)=(0,0)$, which brings system (1) to a system with the axis of symmetry.

In [12] is described an algorithm based on application of Gröebner bases in the search for a bilinear transformation, which is invertible in a neighbourhood of the origin and transform a given system to one which is symmetric in a line. This algorithm is applied to find center conditions for some cubic systems.

In this section we shall consider a general mechanism to produce center by rational reversibility. We seek a transformation of the form

$$x=\frac{a_1X+b_1Y}{a_3X+b_3Y-1},\ y=\frac{a_2X+b_2Y}{a_3X+b_3Y-1}\quad(2)$$

with $a_1b_2-b_1a_2\neq0$ and $a_j,\ b_j\in\mathbb{R},\ j=1,2,3.$ The condition $a_1b_2-b_1a_2\neq0$ guarantees that (2) is invertible in a neighborhood of $O(0,0)$ and the singular point is mapped to $X=Y=$

0. Applying the transformation (2) to (1) we obtain a system of the form

$$\dot{X} = \frac{P(X,Y)}{R(X,Y)}, \quad \dot{Y} = \frac{Q(X,Y)}{R(X,Y)},$$

whose orbits in some neighborhood of $O(0,0)$ are the same as those of the system

$$\begin{aligned} \dot{X} &= \sum_{i+j=0}^{4} U_{ij}X^iY^j \equiv P(X,Y), \\ \dot{Y} &= \sum_{i+j=0}^{4} V_{ij}X^iY^j \equiv Q(X,Y), \end{aligned} \qquad (3)$$

where U_{ij}, V_{ij} are polynomials in the coefficients of the original system and the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ of the transformation.

The requirement is to show that $a_1, a_2, a_3, b_1, b_2, b_3$ can be chosen so that the system (3) is symmetric in the Y-axis, i.e. the transformation (2) brings in some neighborhood of $O(0,0)$ the system (1) to one equivalent with a polynomial system

$$\begin{aligned} \dot{X} &= Y + M(X^2, Y), \\ \dot{Y} &= -X(1 + N(X^2, Y)). \end{aligned} \qquad (4)$$

The obtained system has an axis of symmetry $X = 0$ and therefore $O(0,0)$ is a center for (1). The system (4) is equivalent to the system (3) if the following conditions are satisfied:

$$V_{40} = 0, \; U_{31} \equiv V_{22} = 0, \; U_{13} \equiv V_{04} = 0,$$

$$U_{10} \equiv V_{01} = 0, \; U_{00} = 0, \; V_{00} = 0$$

and

$$\begin{aligned}
V_{04} &\equiv a_3[sb_1^4 + rb_2^4 + b_1b_2((k+q)b_1^2 \\
&\quad +(m+n)b_1b_2 + (l+p)b_2^2)] = 0, \\
V_{22} &\equiv a_3[(2p - 3k - q)a_1a_2b_2^2 \\
&\quad +da_2b_1^2a_3 + na_2^2b_1^2 + ca_1b_2^2a_3 \\
&\quad +(3l + p - 2q)a_2^2b_1b_2 \\
&\quad +(3(r+s) - 2(m+n))a_2^2b_2^2 \\
&\quad +ma_1^2b_2^2 + (2b + c - 2g)a_2b_1b_2a_3 \\
&\quad +(2f - 2a - d)a_2b_2^2a_3 + a_3^2] = 0, \\
U_{30} &\equiv 2aa_1^2b_2a_3 + [(m-s)a_1 \\
&\quad +(p-q)a_2 + 2(c-g)a_3]a_1a_2b_2 \\
&\quad +ka_1^3b_2 + a_2^3(lb_1 - nb_2 + rb_2) \\
&\quad +2a_2^2a_3(bb_1 - db_2 + fb_2) = 0, \\
U_{12} &\equiv (qa_2 + 2ga_3)b_1^3 + [2(a+d)a_3 \\
&\quad +(m + 2n - 3s)a_2]b_1^2b_2 \\
&\quad +[(3l - 3k + 2p - 2q)a_2 \\
&\quad +2(b+c)a_3]b_1b_2^2 + [2fa_3 + pa_1 \\
&\quad -(2m + n - 3r)a_2]b_2^3 = 0, \\
V_{03} &\equiv (ka_2 - ga_3)b_1^3 + [(m-s)a_2 \\
&\quad -(a+d)a_3]b_1^2b_2 + [(p-q)a_2 \\
&\quad -(b+c)a_3]b_1b_2^2 + [la_1- \\
&\quad -(n-r)a_2 - fa_3]b_2^3 = 0, \\
V_{21} &\equiv qa_1^3b_2 + (m + 2n - 3s)a_1^2a_2b_2 \\
&\quad +(d-a)a_1^2a_3b_2 + (3l - 3k \\
&\quad +2p - 2q)a_1a_2^2b_2 + [pb_1 + (3r- \\
&\quad -2m - n)b_2]a_2^3 + (2b - g)a_1a_2a_3b_2 \\
&\quad +[(f - 2a)b_2 - (b-c)b_1]a_2^2a_3 = 0, \\
V_{02} &\equiv aa_2b_1^2 + (c-g)b_1b_2a_2 \\
&\quad +(ba_1 - da_2 + fa_2)b_2^2 - a_3 = 0, \\
V_{20} &\equiv ga_1^3 + (a+d)a_1^2a_2 \\
&\quad +(b+c)a_1a_2^2 + fa_2^3 + 2a_3 = 0, \\
U_{11} &\equiv [db_1 + (2b + c - 2g)b_2]a_2b_1 \\
&\quad +[ca_1 + (2f - 2a - d)a_2]b_2^2 \\
&\quad +3a_3 = 0, \\
U_{01} &\equiv b_1^2 + b_2^2 - 1 = 0, \\
U_{10} &\equiv a_1b_1 + a_2b_2 = 0, \\
V_{10} &\equiv a_1^2 + a_2^2 - 1 = 0.
\end{aligned} \qquad (5)$$

Next we shall study the compatibility of (5). If (5) is compatible, then the cubic system (1) with a weak focus at $O(0,0)$ is rationally reversible and a singular point $O(0,0)$ is a center.

4. Rationally reversible cubic systems

In this section we prove the Theorem 2 by studying the compatibility of (5). It is easy to verify that the equations $U_{01} = 0$, $V_{10} = 0$ of (5) admit the following parametrization

$$\begin{aligned} a_1 &= (2u)/(u^2+1), \;\; a_2 = (u^2-1)/(u^2+1), \\ b_1 &= (2v)/(v^2+1), \;\; b_2 = (v^2-1)/(v^2+1), \end{aligned}$$

where u and v are some real parameters. In this case $U_{10} \equiv j_1j_2 = 0$, where $j_1 = uv + u - v + 1$, $j_2 = uv - u + v + 1$.

Next assume $j_1 = 0$, then $v = (1+u)/(1-u)$ and $U_{10} \equiv 0$. The case $j_2 = 0$ is equivalent with $j_1 = 0$ if we take into consideration that $j_2(u,v) = j_1(-u,-v)$.

3.1. $a_3=0$. In this case $V_{04}\equiv 0$ and $V_{22}\equiv 0$.

If $u=0$ or $u=-1$, then from the equations of (5) we obtain respectively the following two series of conditions for the existence of a center:

1)
$$a=d=f=k=l=p=q=0;$$
2)
$$b=c=g=k=l=p=q=0.$$

Assume $u(u+1)\neq 0$, then the equations of (5) yields the following series of conditions for the existence of a center:

3)
$$a=[(d-f)(u^6-7u^4+7u^2-1)+b(20u^3-6u^5-6u)]/[2(u^2-1)^3],$$
$$c=[f(1-u^6)+b(12u^3-2u^5-2u)+(4d-7f)(u^2-u^4)]/[2u(u^2-1)^2],$$
$$g=[(d+f)(1-u^6)+(7d-f)(u^4-u^2)+b(2u^5-12u^3+2u)]/[4u(u^2-1)^2],$$
$$l=[(m-s)(2u^7-14u^5+14u^3-2u)+k(20u^6+20u^2-u^8-54u^4-1)]/[2u^2(u^2-1)^2],$$
$$q=[3k(u^4-6u^2+1)^2+8pu^2(u^2-1)^2-6u(m-s)(u^4-6u^2+1)(u^2-1)]/[8u^2(u^2-1)^2],$$
$$n=[3k(u^4-6u^2+1)(u^8-20u^6+54u^4-20u^2+1)+2u(4pu^3-4pu+3su^4-18su^2+3u)(u^4-6u^2+1)(u^2-1)-mu(3u^2-1)(u^2-3)(u^4-14u^2+1)(u^2-1)]/[32u^3(u^2-1)^3],$$
$$r=[k(u^4-6u^2+1)(u^8-4u^6+22u^4-4u^2+1)+2m(u^4-6u^2+1)^2(1-u^2)+8pu^2(u^8-8u^6+14u^4-8u^2+1)+2su(u^2+1)^4(u^2-1)]/[32u^3(u^2-1)^3].$$

3.2. $a_3\neq 0$. In this case from the equation $V_{02}=0$ of (5) we get
$$a_3=[a(u^2-1)^3+u^2(8bu-4(d-f)(u^2-1))-2(c-g)u(u^2-1)^2]/(u^2+1)^3.$$

If $u=0$ or $u=-1$, then from the equations of (5) we obtain respectively the following two series of conditions for the existence of a center:

4)
$$f=-2a,\ \ d=-3a,\ \ p=a(b-c),\ \ n=2a^2,\ \ k=ag,\ \ s=0,\ \ l=-2ab,\ \ q=-2ag;$$

5)
$$g=-2b,\ \ c=-3b,\ \ q=b(a-d),\ \ m=2b^2,\ \ l=bf,\ \ r=0,\ \ k=-2ab,\ \ p=-2bf.$$

Assume $u(u+1)\neq 0$, then the equations of (5) yields the following series of conditions for the existence of a center:

6)
$$a=[(6b+2c)u(10u^2-3u^4-3)+2d(u^6-11u^4+11u^2-1)+5f(7u^4-u^6-7u^2+1)]/[4(u^4+1)(u^2-1)],$$
$$g=[14bu(u^4-6u^2+1)+2cu(5u^4-14u^2+5)+(3f-2d)(u^6-15u^4+15u^2-1)]/[8u(u^4+1)],$$
$$k=[(2df-3f^2+4s)(u^{20}+1)+4(bd-5bf+cd-4cf)(u^{19}-u)+2(4s-14b^2-24bc-10c^2+4d^2-24df+31f^2)(u^{18}+u^2)+(256b^2+336bc+112c^2-96d^2+362df-391f^2+4s)(u^{16}+u^4)+8(157f^2-106b^2-140bc-42c^2+20d^2-124df)(u^{14}+u^6)+2(1536b^2+1752bc+456c^2-464d^2+1866df-1851f^2-4s)(u^{12}+u^8)+4(1389f^2-1994b^2-1592bc-334c^2+428d^2-1528df-4s)u^{10}+4(65bf-31bd-23cd+50cf)(u^{17}-u^3)+8(67bd-152bf+39cd-93cf)\cdot(u^{15}-u^5)+8(287cf-181bd+476bf-129cd)(u^{13}-u^7)+8(760bd-1385bf+332cd-617cf)(u^{11}-u^9)]/[8u(u^4+1)^2(u^2+1)^4(u^2-1)],$$
$$l=[(2df-3f^2-8s)(u^{16}+1)+4(bd-5bf+cd-4cf+4q)(u^{15}-u)+4(32s-7b^2-12bc-5c^2+2d^2-17df+21f^2)(u^{14}+u^2)+4(123bf-41bd-33cd+74cf-28q)(u^{13}-u^3)+4(158b^2+204bc+54c^2-44d^2+196df-195f^2-64s)(u^{12}+u^4)+4(479bd-913bf+247cd-442cf+36q)(u^{11}-u^5)+4(96s-969b^2-868bc-187c^2+254d^2-879df+747f^2)(u^{10}+u^6)+4(2543bf-1527bd-615cd+1016cf-60q)(u^9-u^7)+2(3656b^2+2832bc+552c^2-848d^2+2798df-2289f^2-248s)u^8]/[128u^3(u^4+1)^2(u^2-1)],$$

$$
\begin{aligned}
p = {} & [3(2df - 3f^2 - 8s)(u^{24} + 1) \\
& + 12(bd - 5bf + cd - 4cf + 4q)(u^{23} - u) \\
& + 2(444b^2 + 744bc + 348c^2 - 248d^2 \\
& + 978df - 1101f^2 + 120s)(u^{20} + u^4) \\
& + 4(24s - 21b^2 - 36bc - 15c^2 + 6d^2 \\
& - 49df + 60f^2)(u^{22} + u^2) + 4(24s \\
& - 1521b^2 - 2836bc - 1123c^2 + 462d^2 \\
& - 2589df + 3612f^2)(u^{18} + u^6) \\
& + 4(237bf - 119bd - 95cd + 222cf \\
& - 4q)(u^{21} - u^3) + 4(763bd - 1857bf \\
& + 611cd - 1598cf - 60q)(u^{19} - u^5) \\
& + 4(12747bf - 4535bd - 3735cd \\
& + 9252cf - 76q)(u^{17} - u^7) \\
& + (45984b^2 + 62912bc + 18592c^2 \\
& -12608d^2 + 62394df - 71367f^2 \\
& + 24s)(u^{16} + u^8) + 8(13331bd - 29307bf \\
& + 7427cd - 15456cf - 44q)(u^{15} - u^9) \\
& + 8(5910d^2 - 23421b^2 - 23012bc - 5575c^2 \\
& - 23257df + 22740f^2 - 24s)(u^{14} + u^{10} \\
& + 8(61569bf - 33111bd - 14927cd \\
& + 28154cf - 28q)(u^{13} - u^{11}) + 4(76404b^2 \\
& + 66616bc + 14932c^2 - 18024d^2 \\
& + 66126df - 61515f^2 - 120s)u^{12}]/ \\
& [128u^3(u^4 + 1)^4(u^2 + 1)^4(u^2 - 1)],
\end{aligned}
$$

$$
\begin{aligned}
r = {} & [(2df - 3f^2 - 8s)(u^{26} - 1) \\
& + 4(bd - 5bf + cd - 4cf + 4q)(u^{25} + u) \\
& + (104s - 28b^2 - 48bc - 20c^2 + 8d^2 \\
& - 62df + 75f^2)(u^{24} - u^2) \\
& + 2(146b^2 + 208bc + 78c^2 - 76d^2 \\
& + 292df - 267f^2 + 56s)(u^{22} - u^4) \\
& + 2(1691f^2 - 634b^2 - 1200bc - 582c^2 \\
& +252d^2 - 1164df - 56s)(u^{20} - u^6) \\
& + 8(38bf - 19bd - 15cd + 31cf - 8q) \\
& \cdot (u^{23} + u^3) + 8(115bd - 213bf \\
& + 75cd - 178cf - 20q)(u^{21} + u^5) \\
& + 8(1486bf - 471bd - 483cd + 1331cf \\
& - 8q)(u^{19} + u^7) + (10828b^2 + 18992bc \\
& + 6500c^2 - 3112d^2 + 18250df - 23445f^2 \\
& - 88s)(u^{18} - u^8) + 4(8223bd - 20523bf \\
& + 5471cd - 12300cf - 4q)(u^{17} + u^9) \\
& + (18768d^2 - 71256b^2 - 79392bc \\
& - 20808c^2 - 80118df + 83069f^2 - 200s) \\
& \cdot (u^{16} - u^{10}) + 16(8q - 7947bd + 15622bf \\
& - 3847cd + 7511cf)(u^{15} + u^{11}) \\
& + 4(44618b^2 + 40224bc + 9222c^2 \\
& - 10748d^2 + 40200df - 37909f^2 - 56s) \\
& \cdot (u^{14} - u^{12}) + 16(12157bd - 22291bf \\
& + 5381cd - 10054cf + 20q)u^{13}]/ \\
& [256u^4(u^4 + 1)^2(u^2 + 1)^4],
\end{aligned}
$$

$$
\begin{aligned}
m = {} & [f(2d - 3f)(u^{20} + 1) + 4(bd - 5bf \\
& + cd - 4cf + 2q)(u^{19} - u) + 2(5088b^2 \\
& + 5496bc + 1416c^2 - 1456d^2 + 5850df \\
& - 5787f^2)(u^{12} + u^8) + 2(4d^2 - 14b^2 \\
& - 24bc - 10c^2 - 32df + 39f^2)(u^{18} + u^2) \\
& + (320b^2 + 528bc + 240c^2 - 160d^2 \\
& + 586df - 711f^2)(u^{16} + u^4) + 8(453f^2 \\
& - 250b^2 - 396bc - 122c^2 + 52d^2 \\
& - 348df)(u^{14} + u^6) + 4(4293f^2 - 6026b^2 \\
& - 4920bc - 1038c^2 + 1324d^2 - 4720df)u^{10} \\
& + 8(2356bd - 4289bf + 1032cd - 1917cf \\
& + 2q)(u^{11} - u^9) + 4(6q - 39bd + 81bf \\
& - 31cd + 74cf)(u^{17} - u^3) + 8(119bd \\
& - 304bf + 91cd - 233cf + 4q)(u^{15} - u^5) \\
& + 8(4q - 577bd + 1508bf - 397cd + 883cf) \\
& \cdot (u^{13} - u^7)]/[16u^2(u^4 + 1)^2(u^2 + 1)^4],
\end{aligned}
$$

$$
\begin{aligned}
n = {} & [3(2df - 3f^2 - 8s)(u^{16} + 1) \\
& + 12(bd - 5bf + cd - 4cf + 4q)(u^{15} - u) \\
& + 4(306b^2 + 388bc + 106c^2 - 84d^2 \\
& + 380df - 397f^2 - 96s)(u^{12} + u^4) \\
& + 4(48s - 21b^2 - 36bc - 15c^2 + 6d^2 \\
& - 39df + 53f^2)(u^{14} + u^2) + 4(144s \\
& - 1467b^2 - 1356bc - 305c^2 + 378d^2 \\
& - 1369df + 1227f^2)(u^{10} + u^6) \\
& + 2(5112b^2 + 4144bc + 856c^2 - 1200d^2 \\
& + 4106df - 3523f^2 - 360s)u^8 \\
& + 4(265bf - 99bd - 75cd + 174cf - 52q) \\
& \cdot (u^{13} - u^3) + 4(789bd - 1579bf \\
& + 413cd - 782cf + 76q)(u^{11} - u^5) \\
& + 4(3773bf - 2181bd - 917cd \\
& + 1592cf - 116q)(u^9 - u^7)]/ \\
& [64u^2(u^4 + 1)^2(u^2 - 1)^2].
\end{aligned}
$$

In this way we have finished the proof of Theorem 2 and the following theorem is valid

Theorem 3. *If at least one of the following six series of conditions 1) – 6) is satisfied, then the cubic system (1) has a center at the origin.*

It is easy to see that the center conditions 1) is symmetric with 2) and 4) is symmetric with 5).

REFERENCES

1. *Amel'kin, V.V., Lukashevich, N.A., Sadovsky, A.P.*, Non-linear oscillations in the systems of second order// Minsk. –1982 (in Russian).

2. *Bondar, Y.L., Sadovskii, A.P.*, Variety of the center and limit cycles of a cubic system, which is

reduced to Lienard form// Bull. Acad. Sci. of Moldova. Mathematics. –2004. – **46**, no. 3,– P. 71–90.

3. *Cozma, D., Şubă, A.*, Partial integrals and the first focal value in the problem of centre// Nonlinear Differ. Equ. and Appl. –1995. – **2**,–P. 21–34.

4. *Cozma, D., Şubă, A.*, The solution of the problem of center for cubic differential systems with four invariant straight lines// Scientific Annals of the "Al.I.Cuza"University, Mathematics. –1998. – **XLIV**, s.I.a. – P. 517–530.

5. *Cozma, D., Şubă, A.*, Solution of the problem of the centre for a cubic differential system with three invariant straight lines// Qualitative Theory of Dynamical Systems. –2001. –**2**, no.1. –P. 129–143.

6. *Cozma, D.*, The problem of the center for cubic systems with two parallel invariant straight lines and one invariant conic// Nonlinear Differ. Equ. and Appl. –2009. –**16**. – P. 213–234.

7. *Cozma D.*, The problem of the center for cubic systems with two homogeneous invariant straight lines and one invariant conic// Annals of Differential Equations. –2010. –**26**, no. 4. – P. 385–399.

8. *Cozma D.*, Center problem for cubic systems with a bundle of two invariant straight lines and one invariant conic// Bulletin of Academy of Sciences of the Republic of Moldova. Mathematics. –2012. –**68**, no.1. –P.32–49.

9. *Cozma D.*, Centers in a cubic differential system with one invariant stright line// Romai Journal. – 2011. –**7**, no. 2. – P. 53–62.

10. *Levandovskyy, V., Logar, A., Romanovski, V.G.* The cyclicity of a cubic system// Open Systems & Information Dynamics. –2009. –**16**, no.4. –P.429–439.

11. *Liapunov, A. M.*, Probléme général de la stabilité du mouvement// Ann. of Math. Stud. –1947. –**17**. Princeton University Press.

12. *Lloyd, N. G., Pearson, J. M.*, Symmetry in Planar Dynamical Systems// J. Symbolic Computation. –2002. –**33**. – P. 357—366.

13. *Poincaré, H.*, Mémoire sur les courbes définies par une équation différentielle// Oeuvres de Henri Poincaré. –1951. –**1**, Gauthiers–Villars, Paris.

14. *Romanovski V. G.*, Time-Reversibility in 2-Dimensional Systems// Open Systems & Information Dynamics. –2008. –**15**, no.4. – P. 359–370.

15. *Sadovskii, A.P.* Solution of the center and focus problem for a cubic system of nonlinear oscillations// Differential Equations. –1997. –**33**, no.2. – P. 236–244.

16. *Sadovskii, A.P.* Solution of the center-focus problem for a cubic system reducible to a Lienard system// Differentsial'nye Uravneniya. –2006. –**42**, no.1. – P. 11–22.

17. *Şubă, A., Cozma, D.*, Solution of the problem of center for cubic differential systems with three invariant straight lines in generic position// Qualitative Theory of Dynamical Systems. –2005. –**6**. – P. 45–58.

18. *Teixeira, M. A., Yang Jiazhong*, The center-focus problem and reversibility// Journal of Diff. Equations. –2001. – **174**. – P. 237–251.

19. *Żołądek, H.*, The classification of reversible cubic systems with center// Topol. Meth. in Nonlin. Analysis. –1994. –**4**. – P. 79–136.

20. *Żołądek, H.*, The solution of the center-focus problem// Preprint. –1992.