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LIMIT CYCLE BIFURCATIONS OF A LIÉNARD SYSTEM WITH CUBIC RESTORING AND POLYNOMIAL DAMPING FUCTIONS

In this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the limit cycle problem for a Liénard system with cubic restoring and polynomial damping functions.

1. Introduction. We consider Liénard equations

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0 \tag{1}$$

and the corresponding dynamical systems in the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{2}$$

There are many examples in the natural sciences and technology in which such equations and related systems are applied [1]–[10]. They are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems, where $f(x)$ represents a coefficient of the damping force and $g(x)$ represents the restoring force or stiffness, when modeling wind rock phenomena and surge in jet engines [2], [8]. Such systems can be also used to model resistor-inductor-capacitor circuits with nonlinear circuit elements. Recently, e. g., the Liénard system (2) has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [10]. There are also some examples of using Liénard type systems in ecology and epidemiology [7].

In this paper, we suppose that system (2), where $g(x)$ is cubic and $f(x)$ is arbitrary polynomial, has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x\,(1 + \beta_1\,x + \beta_2\,x^2) + \\ &y\,(\alpha_0 + \alpha_1\,x + \ldots + \alpha_{2k}\,x^{2k}). \end{aligned} \tag{3}$$

2. Limit cycle bifurcations of a special Liénard polynomial system. By means of our bifurcationally geometric approach [11]–[13], we will consider the Liénard system (3). Its finite singularities are determined by the algebraic system

$$x\,(1 + \beta_1\,x + \beta_2\,x^2) = 0, \quad y = 0. \tag{4}$$

It always has an anti-saddle at the origin and, in general, can have at most three finite singularities which lie on the x-axis: a saddle and two anti-saddles or two saddles and an anti-saddle, or a saddle-node and an anti-saddle, or a saddle and an anti-saddle, or a unique anti-saddle at the origin. At infinity, system (3) has two singular points: a node at the “ends” of the x-axis and a saddle at the “ends” of the y-axis. For studying the infinite singularities, the methods applied in [1] for Rayleigh’s and van der Pol’s equations and also Erugin’s two-isocline method developed in [11] can be used; see [12], [13].

Following [11], we will study limit cycle bifurcations of (3) by means of a canonical system containing field rotation parameters of (3) [1], [11].

Theorem 1. *The special Liénard polynomial system* (3) *with limit cycles can be reduced to the canonical form*

$$\begin{aligned} \dot{x} &= y \equiv P(x,y), \\ \dot{y} &= -x\,(1+\beta_1\,x \pm\, x^2) \\ &+ y\,(\alpha_0 + x + \ldots + x^{2k-1} + \alpha_{2k}\,x^{2k}) \\ &\equiv Q(x,y), \end{aligned} \tag{5}$$

where β_1 is fixed and $\alpha_0,\ \alpha_2, \ldots,\ \alpha_{2k}$ are field rotation parameters of (5).

Proof. Let all the parameters $\alpha_i,\ i = 0,1,\ldots,2k$, vanish in system (5),

$$\dot{x} = y, \quad \dot{y} = -x\,(1+\beta_1\,x+\beta_2\,x^2), \tag{6}$$

and consider the corresponding equation

$$\frac{dy}{dx} = \frac{-x\,(1+\beta_1\,x+\beta_2\,x^2)}{y} \equiv F(x,y). \tag{7}$$

Since $F(x,-y) = -F(x,y)$, the direction field of (7) (and the vector field of (6) as well) is symmetric with respect to the x-axis. It follows that for arbitrary values of the parameters β_1 and β_2 system (6) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, the even parameter β_2 of system (3) can be supposed to be equal, e. g., to ± 1: $\beta_2 = \pm 1$.

Let now all the parameters α_i with even indexes and the odd parameter β_1 vanish in system (5),

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 \pm\, x^2) \\ &+ y\,(\alpha_1\,x + \alpha_3\,x^3 + \ldots + \alpha_{2k-1}\,x^{2k-1}), \end{aligned} \tag{8}$$

and consider the corresponding equation

$$\frac{dy}{dx} = \frac{-x(1\pm x^2) + y(\alpha_1 x + \ldots + \alpha_{2k-1}x^{2k-1})}{y} \equiv G(x,y). \tag{9}$$

Since $G(-x,y) = -G(x,y)$, the direction field of (9) (and the vector field of (8) as well) is symmetric with respect to the y-axis. It follows that for arbitrary values of the parameters $\alpha_1,\ \alpha_3, \ldots,\ \alpha_{2k-1}$ system (6) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the odd parameters α_i of system (3) can be supposed to be equal, e. g., to 1: $\alpha_1 = \alpha_3 = \ldots = \alpha_{2k-1} = 1$.

Inputting the odd parameter β_1 into system (8),

$$\begin{aligned} \dot{x} &= y \equiv R(x,y), \\ \dot{y} &= -x\,(1+\beta_1\,x \pm\, x^2) \\ &+ y\,(x + x^3 + \ldots + x^{2k-1}) \\ &\equiv S(x,y), \end{aligned} \tag{10}$$

and calculating the determinant

$$\Delta_{\beta_1} = RS'_{\beta_1} - SR'_{\beta_1} = -x^2 y,$$

we can see that the vector field of (10) is rotated symmetrically (in opposite directions) with respect to the x-axis and that the finite singularities (centers and saddles) of (10) moving along the x-axis (except the center at the origin) do not change their type or join in saddle-nodes. Therefore, we can fix the odd parameter β_1 in system (5), fixing the position of its finite singularities on the x-axis.

To prove that the even parameters $\alpha_0,\ \alpha_2, \ldots,\ \alpha_{2k}$ rotate the vector field of (5), let us calculate the following determinants:

$$\Delta_{\alpha_0} = PQ'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0,$$

$$\Delta_{\alpha_2} = PQ'_{\alpha_2} - QP'_{\alpha_2} = x^2y^2 \geq 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta_{\alpha_{2k}} = PQ'_{\alpha_{2k}} - QP'_{\alpha_{2k}} = x^{2k}y^2 \geq 0.$$

By definition of a field rotation parameter [1], [11], for increasing each of the parameters $\alpha_0,\ \alpha_2, \ldots,\ \alpha_{2k}$, under the fixed others, the vector field of system (5) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (5) is rotated in the negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (3), it is sufficient to consider the canonical system (5) containing only its even parameters $\alpha_0,\ \alpha_2, \ldots,\ \alpha_{2k}$ which rotate the vector field of (5) under the fixed parameter β_1. The theorem is proved. □

By means of the canonical system (5), let us study global limit cycle bifurcations of (3) and prove the following theorem.

Theorem 2. *The special Liénard polynomial system* (3) *can have at most* $k+1$ *limit cycles in* $(k:1)$*-distribution.*

Proof. According to Theorem 1, for the study of limit cycle bifurcations of system (3), it is sufficient to consider the canonical system (5) containing the field rotation parameters α_0, $\alpha_2, \ldots, \alpha_{2k}$ of (3) under the fixed parameter β_1.

Let all these parameters vanish:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 \pm x^2) \\ &\quad + y\,(x + x^3 + \ldots + x^{2k-1}). \end{aligned} \tag{11}$$

Suppose that (11) has three finite singularities: a saddle, S, and two anti-saddles, O at the origin and A on the x-axis (all other cases are considered absolutely similarly). System (11) is symmetric with respect to the y-axis and has centers as anti-saddles. Its center domains are bounded by separatrix loops of the saddle S lying on the x-axis between O and A. If to input the parameter β_1 into (11), we will get again system (10) the vector field of which is rotated symmetrically (in opposite directions) with respect to the x-axis. The finite singularities S, O, and A of (10) do not change their type and the center domains of O and A will be bounded by separatrix loops of the saddle S of (10) [1], [11].

Let us input successively the field rotation parameters α_0, $\alpha_2, \ldots,$ α_{2k} into system (10) beginning with the parameters at the highest degrees of x and alternating with their signs; see [12], [13]. So, begin with the parameter α_{2k} and let, for definiteness, $\alpha_{2k} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1\, x \pm x^2) \\ &\quad + y\,(x + x^3 + \ldots + x^{2k-1} + \alpha_{2k}\, x^{2k}). \end{aligned} \tag{12}$$

In this case, the vector field of (12) is rotated in the positive direction (counterclockwise) turning the center O at the origin into a nonrough (weak) unstable focus. The other center A becomes a rough unstable focus [1], [11].

Fix α_{2k} and input the parameter $\alpha_{2k-2} < 0$ into (12):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1\, x \pm x^2) \\ &\quad + y\,(x + x^3 + \ldots + \alpha_{2k-2} x^{2k-2} \\ &\quad + x^{2k-1} + \alpha_{2k}\, x^{2k}). \end{aligned} \tag{13}$$

Then the vector field of (13) is rotated in the opposite direction (clockwise) and the focus O immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of x changes) generating a stable limit cycle. The focus A will also generate a stable limit cycle for some value of α_{2k-2} after changing the character of its stability. Under further decreasing α_{2k-2}, both limit cycles will expand disappearing on separatrix loops of (13) [1], [11].

Denote the limit cycle surrounding the origin by Γ_1, the domain outside the cycle by D_1, the domain inside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding this singular point. It is clear that, under decreasing the parameter α_{2k-2}, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [12], [13].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\alpha^*_{2k} > 0$ and $\alpha^*_{2k-2} < 0$. Return to system (10) and change the inputting order for the field rotation parameters. Input first the parameter $\alpha_{2k-2} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1\, x \pm x^2) \\ &\quad + y\,(x + \ldots + \alpha_{2k-2} x^{2k-2} + x^{2k-1}). \end{aligned} \tag{14}$$

Fix it under $\alpha_{2k-2} = \alpha^*_{2k-2}$. The vector field of (14) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter $\alpha_{2k} > 0$ into (14), we get again system (13) the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle Γ_1 will appear from a separatrix loop for some value of α_{2k}. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing α_{2k} to the value α^*_{2k}. It follows that there are no values of $\alpha^*_{2k-2} < 0$ and $\alpha^*_{2k} > 0$ for which a semi-stable limit cycle could appear in the domain D_1.

This contradiction proves the uniqueness of a limit cycle surrounding the origin O in system (13) for any values of the parameters α_{2k-2} and α_{2k} of different signs. Obviously, if these parameters have the same sign, system (13) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than one limit cycle surrounding the other its singular point A.

Let system (13) have the unique limit cycle Γ_1 surrounding the origin O and a unique limit cycle surrounding A. Fix the parameters $\alpha_{2k} > 0$, $\alpha_{2k-2} < 0$ and input the third parameter, $\alpha_{2k-4} > 0$, into this system:

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x\,(1 + \beta_1\, x \pm\, x^2) + \\ &y\,(x + x^3 + \ldots +\, \alpha_{2k-4}x^{2k-4} + \\ &\alpha_{2k-2}\, x^{2k-2} \,+\, x^{2k-1} \,+\, \alpha_{2k}\, x^{2k}). \end{aligned} \tag{15}$$

The vector field of (15) is rotated counterclockwise, the focus at the origin O changes the character of its stability and the second (unstable) limit cycle, Γ_2, immediately appears from this point. The limit cycle surrounding A can only disappear in this point (because of its roughness) under increasing the parameter α_{2k-4}. Under further increasing α_{2k-4}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter α_{2k-4} [12], [13].

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i.e., that for some set of values of the parameters, $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, and $\alpha^*_{2k-4} > 0$, such a semi-stable cycle exists. Return to system (10) again and input first the parameters $\alpha_{2k-4} > 0$ and $\alpha_{2k} > 0$:

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x\,(1 + \beta_1\, x \pm\, x^2) \\ &+ y\,(x + x^3 + \ldots + \alpha_{2k-4}x^{2k-4} + \\ &x^{2k-3} + \alpha_{2k}\, x^{2k}). \end{aligned} \tag{16}$$

Both parameters act in a similar way: they rotate the vector field of (16) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under $\alpha_{2k-4} = \alpha^*_{2k-4}$, $\alpha_{2k} = \alpha^*_{2k}$ and input the parameter $\alpha_{2k-2} < 0$ into (16) getting again system (15). Since, by our assumption, this system has two limit cycles surrounding the origin for $\alpha_{2k-2} > \alpha^*_{2k-2}$, there exists some value of the parameter, α^{12}_{2k-2} ($\alpha^*_{2k-2} < \alpha^{12}_{2k-2} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (15) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing α_{2k-2}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing α_{2k-2}. The distance between the spirals of the domain D_2 will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k-2} < \alpha^{12}_{2k-2}$ [12], [13].

Thus, there are no such values of the parameters, $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, $\alpha^*_{2k-4} > 0$, for which system (15) would have an additional semi-stable limit cycle surrounding the origin O. Obviously, there are no other values of the parameters α_{2k}, α_{2k-2}, and α_{2k-4} for which system (15) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singular point A of (15). Therefore, three in $(2:1)$-distribution is the maximum number of limit cycles in system (15).

Suppose that system (15) has two limit cycles, Γ_1 and Γ_2, surrounding the origin O and a unique limit cycle surrounding A (this is always possible if $\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} > 0$). Fix the parameters α_{2k}, α_{2k-2}, α_{2k-4} and consider a more general system inputting the fourth parameter, $\alpha_{2k-6} < 0$, into (15):

$$
\begin{aligned}
&\dot{x} = y, \\
&\dot{y} = -x\,(1 + \beta_1\, x \pm x^2) \\
&+ y\,(x + x^3 + \ldots + \alpha_{2k-6}x^{2k-6} + \\
&x^{2k-5} + \ldots + \alpha_{2k}\, x^{2k}).
\end{aligned} \tag{17}
$$

For decreasing α_{2k-6}, the vector field of (17) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, Γ_3. With further decreasing α_{2k-6}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand disappearing on a separatrix loop of (17).

Let system (17) have three limit cycles surrounding the origin O : Γ_1, Γ_2, Γ_3. Could an additional semi-stable limit cycle appear with decreasing α_{2k-6} after splitting of which system (17) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains after decreasing α_{2k-6}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, $\alpha^*_{2k-4} > 0$, and $\alpha^*_{2k-6} < 0$ such a semi-stable cycle exists. Return to system (10) again, input first the parameters $\alpha_{2k-6} < 0$, $\alpha_{2k-2} < 0$ and then the parameter $\alpha_{2k} > 0$:

$$
\begin{aligned}
&\dot{x} = y, \\
&\dot{y} = -x\,(1 + \beta_1\, x \pm x^2) + \\
&y\,(x + x^3 + \ldots + \alpha_{2k-6}x^{2k-6} + \ldots + \\
&\alpha_{2k-2}x^{2k-2} + x^{2k-3} + \alpha_{2k}\, x^{2k}).
\end{aligned} \tag{18}
$$

Fix the parameters α_{2k-6}, α_{2k-2} under the values α^*_{2k-6}, α^*_{2k-2}, respectively. With increasing α_{2k}, a separatrix loop formed around the origin will generate a stable limit cycle, Γ_1. Fix α_{2k} under the value α^*_{2k} and input the parameter $\alpha_{2k-4} > 0$ into (18) getting system (17).

Since, by our assumption, (17) has three limit cycles for $\alpha_{2k-4} < \alpha^*_{2k-4}$, there exists some value of the parameter α^{23}_{2k-4} $(0 < \alpha^{23}_{2k-4} < \alpha^*_{2k-4})$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, with further increasing α_{2k-4}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there [12], [13].

All other combinations of the parameters α_{2k}, α_{2k-2}, α_{2k-4}, and α_{2k-6} are considered in a similar way. It follows that system (17) can have at most four limit cycles in (3 : 1)-distribution.

If we continue the procedure of successive inputting the even parameters, $\alpha_{2k}, \ldots, \alpha_2, \alpha_0$, into system (10), it is possible first to obtain k limit cycles surrounding the origin $(\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} \gg -\alpha_{2k-6} \gg \alpha_{2k-8} \gg \ldots)$ and then to conclude that the canonical system (5) (i. e., the special Liénard polynomial system (3) as well) can have at most $k+1$ limit cycles in $(k : 1)$-distribution. The theorem is proved. □

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