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GLOBAL SOLVABILITY OF MIXED PROBLEM FOR HYPERBOLIC SYSTEM OF THE FIRST ORDER STOCHASTIC EQUATIONS

За допомогою методу характеристик та теореми Банаха про нерухому точку доведено існування та єдиність розв'язку мішаної задачі для стохастичної гіперболічної напівлінійної системи рівнянь першого порядку.

We prove the existence and uniqueness theorem for stochastic partial differential equations of first order. The proof is based on the characteristics method, Banach fixed point theorem and a space metric with weighted functions.

1. Introduction. A lot of processes are modeled by differential hyperbolic systems of first order equations: flows on the network, the level of demand in the economy, population distribution, and others (see for example [1] - [3]). We consider the case where a process is modeled by a mixed nonlinear problem for semilinear hyperbolic system. Input parameters of a continuous model are determined from empirical data. Empirical data cannot be considered completely accurate, so it is advisable to take into account the error of the model input date. In this paper we consider the error of input data as the product of a nonlinear function by a white noise process.

The existence of a global solution for semilinear hyperbolic systems of first order equations are researched by J. Turo, namely the Cauchy problem [4] and the Cauchy problem with delay [5]. Particular cases of the Cauchy problem were considered by Ogawa in [9] and Gikhman [8].

We used the method of Maulenov–Myshkys [6] for finding a global solution of the mixed problem for semilinear hyperbolic systems. In this paper we find sufficient conditions for the existence and uniqueness of global solutions to mixed problem for a semilinear hyperbolic system of first order stochastic equations. The results are established using the method of characteristics and the Banach fixed point theorem.

2. Statement of the problem. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We assume that there exist a set of sub-σ-algebras $\mathcal{F}_t, t \in [0,T]$ of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $0 \leqslant s \leqslant t$. The process w being a p-dimensional standard Wiener process adapted to $\mathcal{F}_t$, $t \in [0,T]$ such that $w(t+h;\omega) - w(t;\omega)$, $h > 0$, is independent on $\mathcal{F}_t, t \in [0,T]$.

Let $L_p = L_p(\Omega; \mathbb{R}^n)$, $1 \leqslant p < \infty$ be the space of all random variables $\xi : \Omega \to \mathbb{R}^n$ with finite L_p-norm $\|\xi\|_p = \left(E\,|\xi|_\alpha^p\right)^{1/p}$, where E is the expectation; $|\cdot|_\alpha$ is norm on $\mathbb{R}^n$, which is defined by the rule $|y(x,t)|_\alpha = \max\limits_{i\in I} |y_i(x,t)|\alpha_i(x,t)$, $y(x,t) \in \mathbb{R}^n$, $\alpha_i \in C(\overline{\Pi}; (0,+\infty))$, $\Pi = (0,l) \times (0,T)$. Denote by $C\left(\overline{\Pi}; L_j\right), j \in \{1,2\}$ the space of all processes $y : \overline{\Pi} \to L_j$ which are continuous (y has a version with continuous sample paths almost surely) and adapted to the $\mathcal{F}_t$ for each $x \in [0,l]$. We consider on $C\left(\overline{\Pi}; L_j\right)$ the norm $\|y\| = \sup\limits_{(x,t)\in\overline{\Pi}} \|y(x,t)\|_j, j \in \{1,2\}$.

In a domain Π we consider a semilinear hyperbolic system of first order equations with a random coefficient

$$\frac{\partial y}{\partial t}(x,t;\omega) + \lambda(x,t;\omega)\frac{\partial y}{\partial x}(x,t;\omega) = \\ = f(y(x,t;\omega), x, t; \omega) + \\ + g(y(x,t;\omega), x, t;\omega)\dot{w}(t;\omega), \quad (1)$$

where $y : \Pi \times \Omega \to \mathbb{R}^n$ is a vector function solution, λ is a map from $\overline{\Pi} \times \Omega$ to the space $n \times n$ of diagonal of real-valued matrices

$$\lambda(x,t;\omega) = diag(\lambda_1(x,t;\omega), \lambda_2(x,t;\omega),\ldots,\lambda_n(x,t;\omega)),$$

$f : \mathbb{R}^n \times \Pi \times \Omega \to \mathbb{R}^n$ is a nonlinear drift vector function, g is a map from $\mathbb{R}^n \times \Pi \times \Omega$ to the space $p \times p$ of real-valued matrices $M_{p\times p}(\mathbb{R})$ (the diffusion function), $\dot{w}(t;\omega)$ is the p-dimensional white noise process.

Let us define sets of indices:

$$I = \{1, 2, ..., n\};$$

$$I_0 = \{i \in I | \lambda_i(x,t;\omega) > 0,\ (x,t;\omega) \in \overline{\Pi} \times \Omega\};$$

$$I_l = \{i \in I | \lambda_i(x,t;\omega) < 0,\ (x,t;\omega) \in \overline{\Pi} \times \Omega\},$$

where $m_1 = card(I_0)$, $m_2 = card(I_l)$.

For system (1) we set initial conditions

$$y(x,0;\omega) = y^0(x;\omega),\ x \in [0,l] \tag{2}$$

and boundary conditions for $t \in [0,T]$

$$y_{i\in I_0}(0,t;\omega) = \gamma_i^0(y_{i\notin I_0}(0,t;\omega),t;\omega), \tag{3}$$

$$y_{i\in I_l}(l,t;\omega) = \gamma_i^l(y_{i\notin I_l}(l,t;\omega),t;\omega), \tag{4}$$

where $y^0 : [0,l]\times\Omega \to \mathbb{R}^n$; $\gamma^0 : \mathbb{R}^{n-m_1}\times[0,T]\times\Omega \to \mathbb{R}^{m_1}$, $\gamma^l : \mathbb{R}^{n-m_2}\times[0,T]\times\Omega \to \mathbb{R}^{m_2}$ are nonlinear functions.

3. Main results.

Let us define the functions by

$$\alpha_i(x,t) = \begin{cases} e^{px(l-x)-at}, & i \in I_0,\ i \in I_l; \\ e^{px-at}, & i \in I_0,\ i \notin I_l; \\ e^{p(l-x)-at}, & i \notin I_0,\ i \in I_l; \\ e^{pl-at}, & i \notin I_0,\ i \notin I_l, \end{cases}$$

for all $(x,t) \in \overline{\Pi}$ and all $i \in I$.

By L we denote the Lipschitz constant with respect to the solution $y = y(x,t)$, Λ is the constant such that $\Lambda = \max\limits_{\substack{i\in I,\\ (x,t)\in\overline{\Pi}}} |\lambda_i(x,t)|$. The function $h = h(y,x,t)$, $h : \mathbb{R}^n \times \overline{\Pi} \to \mathbb{R}$ satisfies the Lipschitz condition if it satisfies the condition

$$|h(y,x,t) - h(\tilde{y},x,t)| \leqslant L \max_{i\in I} |y_i - \tilde{y}_i|,$$

where $y, \tilde{y} \in \mathbb{R}^n$.

Fix $a_0,\ p_0 > 0$ then the parameters a and p are defined by

$$p = \sup\{\ln(4L)/(2l), 0\} + p_0,$$

$$a = \sup\left\{p\Lambda, pl\Lambda, 2L(1+T)\times \times e^{\max\{pl,pl^2/4\}}\right\} + a_0.$$

We denote by $\xi = \varphi_i(\tau;x,t), i \in I$ the solution of the Cauchy problem

$$\frac{d\xi}{d\tau} = \lambda_i(\xi,\tau),\ \xi\big|_{\tau=t} = x, (x,t) \in \overline{\Pi}.$$

Moreover, these solutions are characteristics of system (1). Suppose that $(\varphi_i(\chi_i(x,t);x,t), \chi_i(x,t))$, $i \in I$ are the points of intersection of the characteristics $\varphi_i = \varphi_i(\tau;x,t)$ and the boundary of the domain Π in the direction of decreasing argument τ.

Let us define the domains:

$$\Pi^i = \left\{(x,t) \in \overline{\Pi} |\ \chi_i(x,t) = 0\right\},\ i \in I;$$

$$\Pi_0^i = \left\{(x,t) \in \overline{\Pi} |\ \varphi_i(\chi_i(x,t);x,t) = 0\right\}, i \in I_0;$$

$$\Pi_l^i = \left\{(x,t) \in \overline{\Pi} |\ \varphi_i(\chi_i(x,t);x,t) = l\right\}, i \in I_l.$$

We define by $f(y[\varphi_i(\tau;x,t)]) = f(y(\varphi_i(\tau;x,t),\tau),\varphi_i(\tau;x,t),\tau)$ the value of the function $f = f(y(x,t),x,t)$ on the characteristic curve $\varphi_i = \varphi_i(\tau;x,t)$. Along the characteristic curves, the component $y_i, i \in I$ of the solution of system (1) satisfies the following system of stochastic operator equations

$$y_i(x,t) = \mathfrak{R}_i[y](x,t) + \int\limits_{\chi_i(x,t)}^{t} f_i(y[\varphi_i(\tau;x,t)])d\tau + \sum_{j=1}^{n} \int\limits_{\chi_i(x,t)}^{t} g_{ij}(y[\varphi_i(\tau;x,t)])dw_j(\tau), \tag{5}$$

where the operator is defined by $\mathfrak{R}_i[y](x,t) = \begin{cases} y_i^0(\varphi_i(0;x,t)), & (x,t) \in \Pi^i, \\ \gamma_i^0(y_{i\notin I_0}(0,\chi_i(x,t)),\chi_i(x,t)), & (x,t) \in \Pi_0^i, \\ \gamma_i^l(y_{i\notin I_l}(l,\chi_i(x,t)),\chi_i(x,t)), & (x,t) \in \Pi_l^i. \end{cases}$

Definition. *A vector function $y \in C\left(\overline{\Pi}; L_2\right)$ that satisfies system of operator equations (5) on the set $\overline{\Pi}$ and conditions (2)–(4) is called a solution of problem (1)–(4).*

Theorem. *Suppose the following conditions hold:*

1. $\lambda \in C(\overline{\Pi};\mathbb{R}^n) \cap Lip_x(\overline{\Pi};\mathbb{R}^n)$;

2. $f \in C(\mathbb{R}^n \times \overline{\Pi}; L_1(\Omega;\mathbb{R}^n)) \cap Lip_y(\mathbb{R}^n \times \overline{\Pi}; L_1(\Omega;\mathbb{R}^n))$,
$g \in C(\mathbb{R}^n \times \overline{\Pi}; L_2(\Omega; M_{n\times p}(\mathbb{R}))) \cap Lip_y(\mathbb{R}^n \times \overline{\Pi}; L_2(\Omega; M_{n\times p}(\mathbb{R})))$;

3. $y^0 \in C([0,l];\mathbb{R}^n)$;

4. $\gamma^0 \in C(\mathbb{R}^{n-m_1} \times [0,T]; L_2(\Omega;\mathbb{R}^{m_1})) \cap Lip_y(\mathbb{R}^{n-m_1} \times [0,T]; L_2(\Omega;\mathbb{R}^{m_1}))$,
$\gamma^l \in C(\mathbb{R}^{n-m_2} \times [0,T]; L_2(\Omega;\mathbb{R}^{n-m_2})) \cap Lip_y(\mathbb{R}^{n-m_2} \times [0,T]; L_2(\Omega;\mathbb{R}^{n-m_2}))$;

5. $y_i^0(0) = \gamma^0(y^0_{j\notin I_0}(0), 0)$, $i \in I_0$,
$y^0_{i\in I_l}(l) = \gamma^l(y^0_{j\notin I_l}(l), 0)$, $j \in I_l$ *(zero-order compatibility conditions).*

Then there exists a unique solution of problem (1)–(4).

Доведення. We introduce an operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n), \mathcal{A} : C\left(\overline{\Pi}; L_2\right) \to C\left(\overline{\Pi}; L_2\right)$, where $\mathcal{A}_i$ are defined by the right hand sides of operator system (5), i.e.

$$\mathcal{A}_i[y](x,t) = \mathfrak{R}_i[y](x,t) + \mathcal{F}_i[y](x,t) + \\ + \mathcal{G}_i[y](x,t),\ i \in I,$$

where

$$\mathcal{F}_i[y](x,t) = \int\limits_{\chi_i(x,t)}^{t} f_i(y[\varphi_i(\tau;x,t)])d\tau,$$

$$\mathcal{G}_i[y](x,t) = \sum_{j=1}^{n} \int\limits_{\chi_i(x,t)}^{t} g_{ij}(y[\varphi_i(\tau;x,t)])dw_j(\tau)$$

for all $i \in I$.

Note that the space $C\left(\overline{\Pi}; L_2\right)$ is complete with respect to the imposed norm. This fact follows from the inequalities

$$\sup_{(x,t)\in\overline{\Pi}} \left(E\left(\max_{i\in I} |y_i(x,t)|\alpha_i(x,t)\right)^2\right)^{1/2} \leqslant$$
$$\leqslant e^{\max\{pl, pl^2/4\}} \sup_{(x,t)\in\overline{\Pi}} \left(E\left(\max_{i\in I} |y_i(x,t)|\right)^2\right)^{1/2},$$

$$\sup_{(x,t)\in\overline{\Pi}} \left(E\left(\max_{i\in I} |y_i(x,t)|\alpha_i(x,t)\right)^2\right)^{1/2} \geqslant$$
$$\geqslant e^{-aT} \sup_{(x,t)\in\overline{\Pi}} \left(E\left(\max_{i\in I} |y_i(x,t)|\right)^2\right)^{1/2}.$$

Thus, finding the solution of problem (1)–(4) is reduced to the finding of a fixed point of the operator $\mathcal{A}$ in the space $C\left(\overline{\Pi}; L_2\right)$.

By the definitions of $\chi_i(x,t)$

$$\begin{array}{ll} \chi_i(x,t) \leqslant t - x/\Lambda, & i \in I_0, \\ \chi_i(x,t) \leqslant t - (l-x)/\Lambda, & i \in I_l. \end{array}$$

Denote by $\Delta_k\mathcal{H}(y^k) = \Delta_k\mathcal{H}(y^k) = \mathcal{H}(y^1) - \mathcal{H}(y^2)$. The Cauchy-Schwarz inequality implies that

$$E\left(|\Delta_k\mathcal{F}_i[y^k](x,t)\alpha_i(x,t)|^2\right) \leqslant$$
$$\leqslant E\left(\left|\int\limits_{\chi_i(x,t)}^{t} f_i(y[\varphi_i(\tau;x,t)])\alpha_i(x,t)d\tau\right|^2\right) \leqslant$$
$$\leqslant tE\left(\int\limits_{0}^{t} |f_i(y[\varphi_i(\tau;x,t)])|^2 \alpha_i^2(x,t)d\tau\right) \leqslant$$
$$\leqslant TL^2 \int_0^t E\Big(\max_{i\in I} |\Delta_k y^k(\varphi_i(\tau;x,t),\tau)\times$$
$$\times \alpha_i(x,t)|^2\Big)d\tau \leqslant TL^2 \|\Delta_k y^k\|^2 \times$$
$$\times \int_0^t \max_{i,j\in I} \left(\frac{\alpha_i(x,t)}{\alpha_j(\varphi_i(\tau;x,t),\tau)}\right)^2 d\tau, i \in I. \quad (6)$$

We get a similar estimate for the components $\mathcal{G}_i, i \in I$ of the operator $\mathcal{G}$

$$E\left(|\Delta_k\mathcal{G}_i[y^k](x,t)\alpha_i(x,t)|^2\right) \leqslant$$
$$\leqslant E\left(\left|\sum_{j=1}^{n} \int\limits_{\chi_i(x,t;\omega)}^{t} g_{ij}(y[\varphi_i(\tau;x,t)])\alpha_i(x,t)dw_j(\tau)\right|^2\right) \leqslant$$
$$\leqslant 2\sum_{j=1}^{n} E\left(\int\limits_{\chi_i(x,t)}^{t} g_{ij}(y[\varphi_i(\tau;x,t)])\alpha_i(x,t)d\tau\right)^2 \leqslant$$
$$\leqslant 2\sum_{j=1}^{n} E\left(\int\limits_{\chi_i(x,t)}^{t} |g_{ij}(y[\varphi_i(\tau;x,t)])\alpha_i(x,t)|^2 d\tau\right) \leqslant$$

$$\leqslant 2L^2 \sum_{j=1}^{n} \int_0^t E\Big(\max_{i\in I} \big|\Delta_k y^k(\varphi_i(\tau;x,t),\tau)\times$$

$$\times \alpha_i(x,t)\big|^2 \Big) d\tau \leqslant 2L^2 \|\Delta_k y^k\|^2 \times$$

$$\times \int_0^t \max_{i,j\in I} \left(\frac{\alpha_i(x,t)}{\alpha_i(\varphi_i(\tau;x,t),\tau)} \right)^2 d\tau.$$

By the choice of functions $\alpha_i = \alpha_i(x,t)$, we get

$$E\left(\max_{i\in I} \left| \Delta_k \mathfrak{R}_i[y^k](x,t)\alpha_i(x,t) \right|^2 \right) \leqslant$$

$$\leqslant L^2 \max \left\{ \max_{i\in I_0, j\notin I_0} \left(\frac{\alpha_i(x,t)}{\alpha_j(0,\chi_i(x,t))} \right)^2, \right.$$

$$\left. \max_{i\in I_l, j\notin I_l} \left(\frac{\alpha_i(x,t)}{\alpha_j(l,\chi_i(x,t))} \right)^2 \right\} \|\Delta_k y^k\|^2.$$

Since $(a+b+c)^2 \leqslant 2(a^2+b^2+c^2)$, it follows that

$$E\left(\left| \Delta_k \mathcal{A}_i[y^k](x,t)\alpha_i(x,t) \right|^2 \right) \leqslant$$

$$\leqslant 2E\left(\left| \Delta_k \mathfrak{R}_i[y^k](x,t)\alpha_i(x,t) \right|^2 + \right.$$

$$+ \left| \Delta_k \mathcal{F}_i[y^k](x,t)\alpha_i(x,t)|^2 \right| +$$

$$\left. + \left| \mathcal{G}_i[y^k](x,t)\alpha_i(x,t) \right|^2 \right).$$

By the choice a and p, we get

$$\sup_{(x,t)\in\overline{\Pi}} \max_{i\in I_0, j\notin I_0} \left(\frac{\alpha_i(x,t)}{\alpha_j(0,\chi_i(x,t))} \right)^2 =$$

$$= \sup_{(x,t)\in\overline{\Pi}} \max_{i\in I_l, j\notin I_l} \left(\frac{\alpha_i(x,t)}{\alpha_j(l,\chi_i(x,t))} \right)^2 = e^{-2pl}$$

and

$$\int_0^t \max_{i,j\in I} \left(\frac{\alpha_i(x,t)}{\alpha_j(s,\sigma)} \right)^2 d\sigma \leqslant \frac{e^{\max\left\{2pl, pl^2/2\right\}}}{2a},$$

since $p\Lambda \max\{1,l\} < a$.

As a result, we obtain

$$\|\Delta_k \mathcal{A}[y^k]\| \leqslant 2L \left(e^{-2pl} + \right.$$

$$\left. + \frac{(1+T)e^{\max\left\{2pl, pl^2/2\right\}}}{2a} \right)^{1/2} \|\Delta_k y^k\|,$$

where $2L\left(e^{-2pl} + \dfrac{(1+T)e^{\max\left\{2pl, pl^2/2\right\}}}{2a} \right)^{1/2} < 1.$

Then $\mathcal{A}$ is a contractive operator on $C\left(\overline{\Pi}; L_2\right)$ with the chosen functions $\alpha_i = \alpha_i(x,t)$ and parameters a, p. Therefore, by the Banach fixed point theorem, there exists a unique fixed point of the operator $\mathcal{A}$ in $C\left(\overline{\Pi}; L_2\right)$. This point is a solution of problem (1)–(4).

REFERENCES

1. *Lax P.* Partial differential equations. / P. Lax. Courant Institute of Mathematical Sciences. – 2006. – N 4. – 285 p.
2. *Arnold V. I.* Close-to-convex schlicht functions. / V. I. Arnold // Michigan Math. J. – 1952. – V.1. – N 2. – P. 169–185.
3. *Hillen T., Hadeler K. P.* Hyperbolic systems and transport equations in mathematical biology. / T. Hillen , K. P. Hadeler // Analysic and numeries for conservation laws. Berlin: Springer. – 2005. – P. 257–279.
4. *Turo J.* Study of first order stochastic partial differential equations using integral contractors. / J. Turo // Applicable Analisys. – 1999. – Vol. 70(3-4). – P. 281–291.
5. *Turo J.* Existens and uniques of solutions to a class of stochastic functional partial differentional equations via integral contractors. / J. Turo // Journal for Analisys and its Applications.– 1999. – Vol. 18. – No. 1. – P. 131–141.
6. *Maulenov O., Myshkys A. D.* The solvability of mixed problems for a degenerate semi-linear hyperbolic system on the interval. / O. Maulenov, A. D. Myshkys // Proceedings of the Academy of Sciences of the Kazakh SSR. Mathematical and Physics Series – 1981. – N5. – P. 25—29. (in Russian)
7. *Rozhdestvenskii B. L., Yanenko N. N.* Systems of quasilinear equations and their applications to gas dynamics. / B. L. Rozhdestvenskii , N. N. Yanenko. M.: Nauka, 1978. – 592 p. (in Russian)
8. *Gikhman I. I., Miestechkina T. M.* Cauchy problem for first order stochastic partial differential equation. / I. I. Gikhman , T. M. Miestechkina // Teor. Sluchainych Protsessov, 1983. – P. 25–28 (in Russian).
9. *Ogawa S.* A partial differential equation with white noise as a coefficient. / S. Ogawa // Z. Wahr. verw. –1973 – N 28. – P. 53–71.