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ON A GENERALIZATION OF LEIBNITZ'S RULE FOR THE SECOND-ORDER DERIVATIVE ON THE CASE OF FUNCTIONALS

Одержано опис усіх лінійних функціоналів на просторі функцій, аналітичних у довільній області, які задовольняють співвідношення, що є узагальненням правила Лейбніца знаходження похідної другого порядку від добутку функцій.

We obtain the description of all linear functionals satisfying a generalization of Leibnitz's rule for the second-order derivative for two functions on the space of analytic functions in an arbitrary domain.

Let G be an arbitrary domain of the complex plane. Let $\mathcal{H}(G)$ denote the space of all analytic functions in G equipped with the topology of compact convergence. By $\mathcal{H}^*(G)$ we denote the set of all linear functionals on $\mathcal{H}(G)$.

Generalizing the formula for the differentiation of the product of two functions Rubel [1] posed and solved the problem of finding all pairs of linear continuous functionals L and M on the space $\mathcal{H}(G)$ that satisfy the relation

$$L(fg) = L(f)M(g) + L(g)M(f) \qquad (1)$$

for arbitrary functions f and g of $\mathcal{H}(G)$. Such pairs of functionals Rubel called derivation pairs of functionals. Later Nandakumar, [2] and Zalcman, [3] solved Rubel's problem in the class of linear functionals on the space $\mathcal{H}(G)$ by different ways.

In [4] Nandakumar and Kannappan (see also [5]) solved the problem of finding all pairs of linear functionals L and M on $\mathcal{H}(G)$ that satisfy the following functional equation

$$L(fg) = L(f)L(g) - M(g)M(f) \qquad (2)$$

for any $f, g \in \mathcal{H}(G)$. Note that all continuous linear functionals L and M on $\mathcal{H}(G)$ satisfying (2) was completely characterized in [6]. An generalization of Rubel's equation was solved in [7]. In [8] all Rubel's derivation triples was completely characterized. In [9] was solved generalized Rubel's equation.

In the light of the above-mentioned results there naturally arises the problem of finding of all linear functionals L, M, N on $\mathcal{H}(G)$ that satisfy the relation

$$L(fg) = L(f)M(g) + L(g)M(f) + N(f)N(g) \qquad (3)$$

for any f and g of $\mathcal{H}(G)$. The purpose of this paper is to solve this problem. Note that in the case when $N = 0$ equation (3) coincides with Rubel's equation (1).

Lemma 1. *Let G be an arbitrary domain of the complex plane. Let L, M, N be arbitrary functionals on $\mathcal{H}(G)$. Then there exists a nonzero polynomial of the degree at most 3 which is a zero of L, M, N.*

Proof. The system

$$\begin{cases} aL(z^3) + bL(z^2) + cL(z) + dL(1) = 0; \\ aN(z^3) + bN(z^2) + cN(z) + dM(1) = 0; \\ aM(z^3) + bM(z^2) + cM(z) + dN(1) = 0 \end{cases} \qquad (4)$$

is homogeneous and the number of equations is less than the number of unknown quantities. Therefore, (4) has a non-trivial solution with respect to a, b, c, d. Let (a_1, b_1, c_1, d_1) be a non-trivial solution of (4). Then the polynomial $h(z) = a_1z^3 + b_1z^2 + c_1z + d_1$ is the desired one. Lemma is proved.

Lemma 2. *Let functionals $L, M, N \in \mathcal{H}^*(G)$ satisfy (3) and let $L \neq 0$. Let there exists $h \in \mathcal{H}(G)$ such that $L(h) = M(h) = N(h) = 0$. Then $L(fh) = M(fh) = N(fh) = 0$ hold for any $f \in \mathcal{H}(G)$.*

Proof. The equality (3) and properties of h imply that $L(fh) = 0$ for any $f \in \mathcal{H}(G)$. Let f be an arbitrary function of $\mathcal{H}(G)$. Replacing in (3) pf instead of f and setting $g = pf$ we get $N(fh) = 0$ for any $f \in \mathcal{H}(G)$. Since $L \neq 0$, there exists $g_0 \in \mathcal{H}(G)$ such that $L(g_0) \neq 0$. Replacing in (3) fh instead of f and setting $g = g_0$ we get $M(fh) = 0$ for any $f \in \mathcal{H}(G)$. Lemma is proved.

Lemma 3. *Let $L, M, N \in \mathcal{H}^*(G)$ be arbitrary functionals satisfying* (3) *and* $L \neq 0$. *Then there exists a polynomial p of the degree* 3 *such that* $L(p) = M(p) = N(p) = 0$.

The correctness of the assertion of Lemma 3 follows from Lemmas 1–2.

Assume that the functionals $L, M, N \in \mathcal{H}^*(G)$ satisfy (3). If $L = 0$, then $N = 0$, M is an arbitrary functional on $\mathcal{H}(G)$ and these functionals satisfy (3).

Now we find all solutions of (3) such that $L \neq 0$. Let z_i, $i = \overline{1,3}$ be arbitrary different points of the domain G. For the further we define the following sets of linear functionals on $\mathcal{H}(G)$:

$$S_3 = \{K \in \mathcal{H}^*(G) : Kf = k_1 f(z_1) + k_2 f(z_2) + $$
$$+ k_3 f(z_3), z_i \in G, k_i \in \mathbb{C}, i = \overline{1,3}\},$$
$$S_2 = \{K \in \mathcal{H}^*(G) : Kf = k_1 f(z_1) + k_2 f'(z_1) + $$
$$+ k_3 f(z_2), z_1, z_2 \in G, k_i \in \mathbb{C}, i = \overline{1,3}\},$$
$$S_1 = \{K \in \mathcal{H}^*(G) : Kf = k_1 f(z_1) + k_2 f'(z_1) + $$
$$+ k_3 f''(z_1), z_1 \in G, k_i \in \mathbb{C}, i = \overline{1,3}\}.$$

Theorem 1. *Let functionals $L, M, N \in \mathcal{H}^*(G)$ satisfy* (3) *and let $L \neq 0$. Then L, M, N belong to the one of the classes S_1, S_2, S_3.*

Proof. Assume that the functionals L, M, N of $\mathcal{H}^*(G)$ satisfy (3), and let $L \neq 0$. Then, according to Lemma 3 there exists a polynomial p of the degree 3 such that $L(p) = M(p) = N(p) = 0$. Without loss of generality we can assume that the coefficient of z^3 of p is equal to 1. We now consider the possible cases of the presence of roots of the polynomial p.

(i) Let p has three different roots z_1, z_2, z_3. We first consider the case when $z_1, z_2, z_3 \in G$. Take an arbitrary function $f \in \mathcal{H}(G)$ and consider the function $f_1(z) = f(z) - s(z)$, where

$$s(z) = f(z_1)\frac{(z - z_2)(z - z_3)}{(z_1 - z_2)(z_1 - z_3)} + f(z_2)\cdot$$
$$\cdot\frac{(z - z_1)(z - z_3)}{(z_2 - z_1)(z_2 - z_3)} + f(z_3)\frac{(z - z_1)(z - z_2)}{(z_3 - z_1)(z_3 - z_2)}.$$

Since $f_1 \in \mathcal{H}(G)$ and z_j $j = \overline{1,3}$ are zeros of $f_1(z)$, $f_1(z) = p(z) f_2(z)$ if $z \in G$, where f_2 is a function of $\mathcal{H}(G)$. Therefore,

$$f(z) = p(z) f_2(z) + s(z).$$

Using Lemma 2 and the property of the polynomial p we get

$$L(f) = l_1 f(z_1) + l_2 f(z_2) + l_3 f(z_3), \quad (5)$$

$$M(f) = m_1 f(z_1) + m_2 f(z_2) + m_3 f(z_3), \quad (6)$$

$$N(f) = n_1 f(z_1) + n_2 f(z_2) + n_3 f(z_3), \quad (7)$$

where $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,3}$. Thus, in this case the functionals L, M, N belong to the class S_3.

Now consider the case when two roots of p lie in G but the third root lies outside of the domain G. For the definiteness we assume that $z_1, z_2 \in G$, $z_3 \notin G$. Take an arbitrary function $f \in \mathcal{H}(G)$ and represent f in the form

$$f(z) = (z - z_1)(z - z_2) g(z) + f(z_1)\frac{z - z_2}{z_1 - z_2} + $$
$$+ f(z_2)\frac{z - z_1}{z_2 - z_1},$$

where g is some function of $\mathcal{H}(G)$. Since $z_3 \notin G$, it follows that for all $z \in G$

$$f(z) = p(z)\frac{g(z)}{z - z_3} + f(z_1)\frac{z - z_2}{z_1 - z_2} + $$
$$+ f(z_2)\frac{z - z_1}{z_2 - z_1}.$$

Since $\frac{g(z)}{z - z_3}$ belongs to $\mathcal{H}(G)$, as in the previous case we have

$$L(f) = l_1 f(z_1) + l_2 f(z_2),$$

$$M(f) = m_1 f(z_1) + m_2 f(z_2),$$

$$N(f) = n_1 f(z_1) + n_2 f(z_2),$$

where $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,2}$. Thus, in this case the functionals L, M, N belong to the class S_3, where $l_3 = m_3 = n_3 = 0$ and an arbitrary point $z_3 \in G$ is different from z_1 and z_2.

The case when one of the points z_1, z_2, z_3 lies in G but other two points lie outside of the domain G is similar to the previous case. Here we obtain that L, M, N belong to the class S_3.

If $z_1, z_2, z_3 \notin G$, then $L = M = N = 0$, but it's not possible.

(ii) Now consider the case when among the roots z_1, z_2, z_3 of p are exactly two equal roots. For the definiteness we assume that $z_3 = z_1$ and $z_1 \neq z_2$. Suppose that $z_1, z_2 \in G$. Take an arbitrary $f \in \mathcal{H}(G)$ and consider the function

$$f_1(z) = f(z) - f(z_1) - f'(z_1)(z - z_1) -$$
$$- \frac{f(z_2) - f(z_1) - f'(z_1)(z_2 - z_1)}{(z_2 - z_1)^2}(z - z_1)^2.$$

Since $f_1 \in \mathcal{H}(G)$ and $f_1(z_1) = f_1'(z_1) = 0$ and $f_1(z_2) = 0$, $f_1(z) = p(z)f_2(z)$ if $z \in G$. Herewith $f_2 \in \mathcal{H}(G)$. Therefore

$$f(z) = p(z)f_2(z) + f(z_1) + f'(z_1)(z - z_1) +$$
$$+ \frac{f(z_2) - f(z_1) - f'(z_1)(z_2 - z_1)}{(z_2 - z_1)^2}(z - z_1)^2.$$

Similar to the case (i) we obtain that

$$L(f) = l_1 f(z_1) + l_2 f'(z_1) + l_3 f(z_2), \qquad (8)$$

$$M(f) = m_1 f(z_1) + m_2 f'(z_1) + m_3 f(z_2), \qquad (9)$$

$$N(f) = n_1 f(z_1) + n_2 f'(z_1) + n_3 f(z_2), \qquad (10)$$

where $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,3}$. Thus, the functionals L, M, N belong to S_2. In the case $z_1 \in G$, $z_2 \notin G$ we can represent an arbitrary function $f \in \mathcal{H}(G)$ in the form

$$f(z) = p(z)\frac{f_2(z)}{z - z_2} + f(z_1) + f'(z_1)(z - z_1).$$

If $z_2 \in G$, $z_1 \notin G$ we use the following representation:

$$f(z) = p(z)\frac{f_2(z)}{(z - z_1)^2} + f(z_2),$$

and in both these cases $f_2 \in \mathcal{H}(G)$. Using these representations we get that in each of these cases the functionals L, M, N belong to the class S_2. The case $z_1, z_2 \notin G$ is not possible, since $L \neq 0$.

(iii) Let $p(z) = (z - z_1)^3$, $z_1 \in G$. Take an arbitrary function $f \in \mathcal{H}(G)$. Using Taylor's formula we get

$$f(z) = p(z)f_2(z) + f(z_1) + f'(z_1)(z - z_1) +$$
$$+ \frac{f''(z_1)}{2}(z - z_1)^2,$$

where $f_2 \in \mathcal{H}(G)$, $z \in G$. Then, similar to previous cases we obtain that L, M, N can be represent in the following form

$$L(f) = l_1 f(z_1) + l_2 f'(z_1) + l_3 f''(z_1), \qquad (11)$$

$$M(f) = m_1 f(z_1) + m_2 f'(z_1) + m_3 f''(z_1), \qquad (12)$$

$$N(f) = n_1 f(z_1) + n_2 f'(z_1) + n_3 f''(z_1), \qquad (13)$$

where $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,3}$. Thus, L, M, N belong to the class S_1. The case $z_1 \notin G$ is not possible, since $L \neq 0$. Theorem is proved.

Let us find further the conditions under which the functionals L, M, N from the fixed class S_i, $i = \overline{1,3}$ satisfy (3).

Lemma 4. *Let the functionals L, M, N are represented by the formulas* (5) – (7), *where z_1, z_2, z_3 are different points of the domain G, $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,3}$. In order that functionals L, M, N satisfy* (3) *it is necessary and sufficient that the following conditions*

$$l_1m_2 + l_2m_1 + n_1n_2 = 0, l_1m_3 + l_3m_1 + n_1n_3 = 0,$$
$$l_2m_3 + l_3m_2 + n_2n_3 = 0, \qquad (14)$$
$$2l_3m_3 + n_3^2 = l_3,\ 2l_2m_2 + n_2^2 = l_2,$$
$$2l_1m_1 + n_1^2 = l_1$$

hold.

Proof. Necessity. Let $p_i(z) = \prod\limits_{j=1, j\neq i}^{3} \frac{z - z_j}{z_i - z_j}$, $i = \overline{1,3}$. Setting $f = p_k$, $g = p_l$, $k, l = \overline{1,3}$, $k \neq l$ in (3) we obtain first three conditions of (14). Setting $f = g = p_i$, $i = \overline{1,3}$ in (3) we obtain other three conditions of (14). The necessity part is proved. By the direct calculation we can obtain the sufficiency part of Lemma 4. Lemma is proved.

Lemma 5. *Let the functionals L, M, N are represented by the formulas* (8) – (10), *where z_1, z_2 are different points of the domain G, $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,3}$. In order that the functionals L, M, N satisfy* (3) *it is necessary and sufficient that the following conditions*

$$l_1m_2 + l_2m_1 + n_1n_2 = l_2\ l_1m_3 + l_3m_1 + n_1n_3 = 0,$$
$$l_2m_3 + l_3m_2 + n_2n_3 = 0, \qquad (15)$$

$2l_3m_3+n_3^2 = l_3, 2l_2m_2+n_2^2 = 0,\ 2l_1m_1+n_1^2 = l_1$

hold.

Proof. Necessity. Let $p_1(z) = \frac{(z_2-z)(z+z_2-2z_1)}{(z_1-z_2)^2}$, $p_2(z) = \frac{(z-z_1)(z-z_2)}{z_1-z_2}$, $p_3(z) = \frac{(z-z_1)^2}{(z_2-z_1)^2}$. Setting $f = p_k$, $g = p_l$, $k,l = \overline{1,3}$ in (3) we get (15). The necessity part is proved. By the direct calculation we can obtain the sufficiency part of Lemma 5. Lemma is proved.

Lemma 6. *Let the functionals* L, M, N *are represented by the formulas* (11) – (13)*, where* $z_1 \in G$, $l_i, m_i, n_i \in \mathbb{C}$, $i = \overline{1,3}$*. In order that the functionals* L, M, N *satisfy* (3) *it is necessary and sufficient that the following conditions*

$$l_1m_2+l_2m_1+n_1n_2 = l_2, l_1m_3+l_3m_1+n_1n_3 = l_3,$$

$$l_2m_3 + l_3m_2 + n_2n_3 = 0, \qquad (16)$$

$$2l_3m_3 + n_3^2 = 0,\ 2l_2m_2 + n_2^2 = 2l_3,$$

$$2l_1m_1 + n_1^2 = l_1.$$

hold.

Proof. Necessity. Let $p_1(z) = 1$, $p_2(z) = z - z_1$, $p_3(z) = \frac{(z-z_1)^2}{2}$. Setting $f = p_k$, $g = p_l$, $k,l = \overline{1,3}$, in (3) we get (16). The necessity part is proved. By a direct calculation we can obtain the sufficiency part of Lemma 6. Lemma is proved.

Theorem 1 and Lemmas 4–6 imply the main result of this paper.

Theorem 2. *In order that* L, M, N *of* $\mathcal{H}^*(G)$ *satisfy* (3) *it is necessary and sufficient that either these functionals are defined as in Lemmas 4–6, or* $L = N = 0$ *and* M *is an arbitrary functional of* $\mathcal{H}^*(G)$*.*

In the light of the above-proved theorem, there naturally arises an interesting problem of the description of all linear operators A, B, C on the space $\mathcal{H}(G)$ such that

$$(A(fg))(z) = (Af)(z)(Bg)(z)+$$

$$+(Ag)(z)(Bf)(z) + (Cf)(z)(Cg)(z) \qquad (17)$$

for any $f, g \in \mathcal{H}(G)$, $z \in G$. Notice that in case $C = 0$ all solutions of corresponding equation (17) in the class of linear continuous operators that act in spaces of analytic functions in arbitrary simply connected domains were described in [10]. In [11] Rubel's operator equation was solved in the class of linear operators that act in spaces of analytic functions in domains. In [12] all pairs of linear operators that act in the spaces of functions analytic in domains and satisfying the operator analog of the cosine addition theorem were described. In [13] a generalized Rubel's operator equation was solved.

Note that in this paper implemented generalized method for solving of the Rubel-type equations in the class of linear functionals on the space $\mathcal{H}(G)$, which can be found in [9]. In [14] some operator equation generalizing the Leibniz rule for the second derivative in other space was solved.

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