

MODULO-ELLIPTIC AND MODULO-LOXODROMIC FUNCTIONS

Встановлено взаємозв'язки між p -локсодромними та модуль-локсодромними функціями, а також квазі-еліптичними та модуль-еліптичними функціями.

Connections between p -loxodromic functions and modulo-loxodromic functions and p -elliptic functions and modulo-elliptic functions, respectively, are established.

Introduction. Denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Let ω_1, ω_2 be complex numbers such that $\text{Im} \frac{\omega_2}{\omega_1} > 0$. A meromorphic in \mathbb{C} function g is called **elliptic** [1] if for every $u \in \mathbb{C}$

$$g(u + \omega_1) = g(u), \quad g(u + \omega_2) = g(u).$$

Elliptic functions were first discovered by N. Abel as inverse functions of elliptic integrals, and their theory was developed by C. Jacobi. A more complete investigation of elliptic functions was later undertaken by K. Weierstrass, who found a simple elliptic function \wp , in terms of which all the others could be expressed.

Let $q, p \in \mathbb{C}^*$, $|q| < 1$.

Definition. [3] A meromorphic in \mathbb{C}^* function f is said to be **p -loxodromic of multiplier q** if for every $z \in \mathbb{C}^*$

$$f(qz) = pf(z).$$

In particular, if $p(z) \equiv 1$, we obtain classic loxodromic function. It was studied in the works of O. Rausenberger [9], G. Valiron [10] and Y. Hellegouarch [1]. A. Kondratyuk and his colleagues also made a great contribution to the development of this theory (see for example [2], [5], [6], [8]).

Definition. [7] A meromorphic in \mathbb{C}^* function f is said to be **modulo-loxodromic of multiplier q** if for every $z \in \mathbb{C}^*$

$$|f(qz)| = |f(z)|. \quad (1)$$

By $|\mathcal{L}|_q$ we denote the set of all modulo-loxodromic functions of multiplier q .

1. Modulo-loxodromic functions. Now let us consider p -loxodromic function of multiplier q with $p = e^{i\alpha}$. This means that the function f satisfies the condition

$$f(qz) = e^{i\alpha} f(z) \quad (2)$$

for every $z \in \mathbb{C}^*$ and fixed $\alpha \in \mathbb{R}$. Denote the class of such functions by \mathcal{L}_q^α .

Theorem 1. $\bigcup_{\alpha \in \mathbb{R}} \mathcal{L}_q^\alpha = |\mathcal{L}|_q$. **Доведення.**

Let $f \in \mathcal{L}_q^\alpha$ for some $\alpha \in \mathbb{R}$. Then the relation (10) implies

$$|f(qz)| = |e^{i\alpha} f(z)| = |f(z)|.$$

Hence, $f \in |\mathcal{L}|_q$.

Let now $f \in |\mathcal{L}|_q$. Consider the meromorphic function

$$g(z) = \frac{f(qz)}{f(z)}. \quad (3)$$

Since $f(z)$ is meromorphic in \mathbb{C}^* , using (11) we get that zero and pole sets of functions $f(z)$ and $f(qz)$ coincide. So, we conclude g is holomorphic in \mathbb{C}^* and (11) also implies

$$|g(z)| = 1$$

outside of $z = 0$, zeros and poles of f . Hence, all these points are removable. Applying the Liouville theorem we obtain that $g(z) \equiv c$, where c is a constant. Taking into account that $|g| = 1$, we deduce that there exists $\alpha \in \mathbb{R}$, such that $c = e^{i\alpha}$ and (12) implies

$$f(qz) = e^{i\alpha} f(z),$$

that is $f \in \mathcal{L}_q^\alpha$ as well as $f \in \bigcup_{\alpha \in \mathbb{R}} \mathcal{L}_q^\alpha$. This completes the proof.

2. Modulo-elliptic functions

Definition 1. A meromorphic in \mathbb{C} function g is called **quasi-elliptic**, if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$, $Im \frac{\omega_2}{\omega_1} > 0$, and $p \in \mathbb{C}^*$, $q \in \mathbb{C}^*$, such that for every $u \in \mathbb{C}$

$$g(u + \omega_1) = pg(u), \quad g(u + \omega_2) = qg(u).$$

Consider the case $p = e^{i\alpha}$, $q = e^{i\beta}$. It means that function g satisfies two conditions for every $u \in \mathbb{C}$

$$g(u + \omega_1) = e^{i\alpha}g(u), \quad g(u + \omega_2) = e^{i\beta}g(u). \quad (4)$$

Denote the class of such functions by $\mathcal{QE}_{\alpha\beta}$.

The following definition was introduced by A. Kondratyuk [4].

Definition. A meromorphic in \mathbb{C} function f is said to be **modulo-elliptic**, if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$, such that $Im \frac{\omega_2}{\omega_1} > 0$ and for every $u \in \mathbb{C}$

$$|f(u + \omega_1)| = |f(u)|, \quad |f(u + \omega_2)| = |f(u)|.$$

The class of modulo-elliptic functions is denoted by $|\mathcal{E}|$.

Theorem 2. $\bigcup_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}} \mathcal{QE}_{\alpha\beta} = |\mathcal{E}|$. **До-**

днення.

Let $f \in \mathcal{QE}_{\alpha\beta}$. Then, using (4) we have

$$|f(u + \omega_1)| = |e^{i\alpha}f(u)| = |f(u)|,$$

$$|f(u + \omega_2)| = |e^{i\beta}f(u)| = |f(u)|.$$

Hence, $f \in |\mathcal{E}|$.

Let now $f \in |\mathcal{E}|$. In other words, for every $u \in \mathbb{C}$ the following equalities are valid

$$|f(u + \omega_1)| = |f(u)|, \quad |f(u + \omega_2)| = |f(u)|.$$

Consider the first of these equalities

$$|f(u + \omega_1)| = |f(u)|, \quad u \in \mathbb{C}. \quad (5)$$

If $f(u) \neq 0$ and $f(u) \neq \infty$ we can divide (5) by $|f(u)|$ and we obtain

$$\left| \frac{f(u + \omega_1)}{f(u)} \right| = 1. \quad (6)$$

The function $g(u) = \frac{f(u + \omega_1)}{f(u)}$ is meromorphic in \mathbb{C} . From (6) it follows

that function g is holomorphic and bounded in \mathbb{C} without a set of zeros and poles of f . Since g is bounded, these points are removable, and relation (6) implies

$$\forall u \in \mathbb{C} : |g(u)| = 1.$$

By the Liouville theorem g is constant and the last equality implies there exists $\alpha \in \mathbb{R}$, such that $g(u) = e^{i\alpha}$. It means

$$\forall u \in \mathbb{C} : f(u + \omega_1) = e^{i\alpha}f(u).$$

Similarly as above, we obtain that there exists $\beta \in \mathbb{R}$ and

$$\forall u \in \mathbb{C} : f(u + \omega_2) = e^{i\beta}f(u).$$

So, we can conclude that $f \in \mathcal{QE}_{\alpha\beta}$ as well as $\bigcup_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}} \mathcal{QE}_{\alpha\beta}$.

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