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CENTERS IN CUBIC DIFFERENTIAL SYSTEMS WITH HOMOGENEOUS INVARIANT STRAIGHT LINES

We solve the problem of the center with at least three invariant straight lines for a cubic differential system with a singular point $O(0,0)$ of a center or focus type having homogeneous invariant straight lines.

1. Introduction

Consider the cubic differential system

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + \\ &+ mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + \\ &+ qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1). It arises the problem of distinguishing between a center and a focus, i.e. of finding the coefficient conditions under which $O(0,0)$ is a center. In this paper we study the problem of the center assuming that (1) has invariant straight lines.

The derivation of necessary conditions for a singular point $O(0,0)$ to be a center for (1) often involves extensive use of computer algebra and we obtain them by calculating the Lyapunov quantities, which are polynomials in the coefficients of (1). The necessary conditions are shown to be sufficient by a variety of methods. A number of techniques, of progressively wider application, have been developed.

A theorem of Poincaré in [9] says that a singular point $O(0,0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral $F(x,y) = C$ in a neighborhood of $O(0,0)$. It is known [1] that the origin is a center for system (1) if and only if the system has an analytic integrating factor of the form

$$\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y)$$

in a neighborhood of $O(0,0)$, where μ_k are homogeneous polynomials of degree k.

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$dF/dt = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

Quantities $L_j,\ j=\overline{1,\infty}$ are polynomials with respect to the coefficients of system (1) called to be the Lyapunov quantities [8]. The origin $O(0,0)$ is a center for (1) if and only if

$$L_j = 0,\ j = \overline{1,\infty}.$$

An algebraic curve $f(x,y) = 0$ is said to be an invariant algebraic curve of system (1) if there exists a polynomial $K(x,y)$ such that

$$P \cdot \partial f/\partial x + Q \cdot \partial f/\partial y = K \cdot f.$$

The polynomial K is called the cofactor of the invariant algebraic curve $f = 0$. If the cubic system (1) has sufficiently many invariant algebraic curves $f_j(x,y) = 0,\ j = \overline{1,q}$, then in most cases the first integral (integrating factor) can be constructed in the Darboux form

$$f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_q^{\alpha_q} \tag{2}$$

with $\alpha_j \in \mathbb{C}$ not all zero. In this case we say that system (1) is Darboux integrable.

System (1) has the Darboux first integral (Darboux integrating factor) of form (2) if and only if there exist constants $\alpha_j \in \mathbb{C}$, not all identically zero such that

$$\alpha_1 K_1(x,y) + \cdots + \alpha_q K_q(x,y) \equiv 0$$

$$\Big(\sum_{j=1}^{q} \alpha_j K_j(x,y) + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0\Big),$$

where K_j is the cofactor of Φ_j for $j = \overline{1,q}$.

The problem of the center was solved for quadratic systems and for cubic symmetric systems. If cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of finding a finite number of necessary and sufficient conditions for the center is still open. It was possible to find a finite number of conditions for the center only in some particular cases (see, for example, [2-7, 10-14]).

The problem of the center was solved for some cubic systems with at least three invariant straight lines ([2, 3, 7, 14]) and for some classes of cubic systems (1) with two invariant straight lines and one invariant conic ([4, 5]).

The goal of this paper is to obtain the center conditions for cubic differential system (1) with homogeneous invariant straight lines.

The paper is organized as follows. In Section 2 we find conditions for cubic system (1) to have two homogeneous invariant straight lines. In Sections 3 and 4 we solve the problem of the center for (1) with four invariant straight lines and with three invariant straight lines of which two are homogeneous. In Section 5 for cubic system (1) with two homogeneous invariant straight lines we find sufficient conditions for $O(0,0)$ to be a center.

2. Two homogeneous invariant straight lines

In this section we find conditions under which cubic system (1) has two homogeneous invariant straight lines.

Definition 1. *A straight line*

$$1 + Ax + By = 0,\ A, B \in \mathbb{C} \tag{3}$$

is said to be invariant for (1), if there exists a polynomial with complex coefficients $K(x,y)$ such that the following identity holds

$$\begin{aligned} AP(x,y) + BQ(x,y) \equiv \\ (1 + Ax + By)K(x,y). \end{aligned} \tag{4}$$

If cubic system (1) has complex invariant straight lines then obviously they occur in complex conjugated pairs $1+Ax+By=0$ and $1+\overline{A}x+\overline{B}y=0$. As homogeneous invariant straight lines $Ax+By=0$ the cubic system (1) can have only the lines [3]

$$x - iy = 0,\ x + iy = 0,\ i^2 = -1. \tag{5}$$

Identifying the coefficients of the monomials $x^\nu y^j$ in (4), we reduce this identity to a system of nine equations for the unknowns A, B, $c_{\nu j}$, $\nu + j = 1, 2$. We find that $K(x,y) = -Bx + Ay + (aA - gB + AB)x^2 + (cA - dB + B^2 - A^2)xy + (fA - bB - AB)y^2$ and A, B are the solutions of the system

$$\begin{aligned} &F_1 \equiv (A+b)B^2 - (l + fA)B + rA = 0, \\ &F_2 \equiv (B+a)A^2 - (k + gB)A + sB = 0, \\ &F_3 \equiv B^3 - 2A^2B + fA^2 - dB^2 + \\ &\qquad + (c-b)AB - pA + nB = 0, \\ &F_4 \equiv A^3 - 2AB^2 - cA^2 + gB^2 + \\ &\qquad + (d-a)AB + mA - qB = 0. \end{aligned} \tag{6}$$

Theorem 1. *Cubic system (1) has two homogeneous invariant straight lines $x \pm iy = 0$ if and only if the following set of conditions holds*

$$\begin{aligned} &g = b + c,\ f = a + d, \\ &q = p + l - k,\ s = m + n - r. \end{aligned} \tag{7}$$

Proof. Let cubic system (1) have homogeneous invariant straight lines. Then by Definition 1 the straight lines $l_{1,2} \equiv x \mp iy = 0$ are invariant straight lines for (1) if and only if

$$P(x,y) \mp iQ(x,y) \equiv (x \mp iy)K(x,y), \tag{8}$$

where $K(x,y) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2$.

Identifying in (8) the coefficients of the monomials in x and y, we find that

$$\begin{aligned} &c_{10} = a \pm ig,\ c_{20} = k \pm is, \\ &c_{02} = p - k - q \pm i(m + n - s), \\ &c_{00} = \pm i,\ c_{01} = c - g \pm i(a + d), \\ &c_{11} = m - s \pm i(k + q) \end{aligned}$$

and

$$\begin{aligned} &f - a - d \pm i(b + c - g) = 0, \\ &r + s - m - n \pm i(l - k + p - q) = 0. \end{aligned}$$

Direct calculations show that $f - a - d = 0$, $b + c - g = 0$, $r + s - m - n = 0$, $l - k + p - q = 0$ and cubic system (1) has two homogeneous invariant straight lines of form (5) if and only if set of conditions (7) holds. The cofactors of the invariant straight lines are $K_2(x,y) = \overline{K_1(x,y)}$, $K_1(x,y) = i + (a + i(b + c))x + (-b + i(a + d))y + (k + is)x^2 + (m - s + i(k + q))xy + (p - k - q + i(m + n - s))y^2$.

3. Four invariant straight lines and centers

In this section we find conditions for cubic system (1) to have four distinct invariant straight lines, two of which are homogeneous, i.e. of the form $l_{1,2} \equiv x \mp iy = 0$, $i^2 = -1$. Then we obtain necessary and sufficient conditions for $O(0,0)$ to be a center.

For this purpose, we assume that set of conditions (7) holds. In what follows we will consider the problem of finding conditions for the existence of two more invariant straight lines of form (3) and divide the study into two subcases: invariant straight lines (3) are parallel and invariant straight lines (3) are nonparallel.

3.1. Let cubic system (1) have two parallel invariant straight lines l_3 and l_4 of form (3) (real or complex conjugated $l_4 = \overline{l_3}$), then by a rotation of axes we can make them to be parallel to the axis of ordinates Oy. Note that by a rotation of axes the linear part of (1) and the invariant straight lines $x \mp iy = 0$ stay their forms respectively.

For $f = a + d,\ g = b + c,\ l = k - p + q,\ r = m + n - s$ and $B = 0$, $A \neq 0$, system (6) becomes

$$\begin{array}{l} m + n - s = 0,\ aA - k = 0, \\ (a + d)A - p = 0,\ A^2 - cA + m = 0. \end{array} \tag{9}$$

Then (9) has two distinct solutions if and only if $s = m + n,\ a = k = d = p = 0$, $m(c^2 - 4m) \neq 0$. In this case we obtain the following set of conditions for the existence of four distinct invariant straight lines:

(e1) $a = d = f = k = p = r = 0,\ g = b + c,\ l = q,\ s = m + n,$

$m(c^2 - 4m) \neq 0$. The invariant straight lines are $x \pm iy = 0,\ 2 + (c \pm \sqrt{c^2 - 4m})x = 0$.

3.2. Let now cubic system (1) have two nonparallel invariant straight lines l_3 and l_4 of form (3) (real or complex conjugated) intersecting at a point (x_0, y_0). The intersection point (x_0, y_0) is a singular point for (1) with real coordinates. By rotating the system of coordinates and rescaling the axes of coordinates, we obtain that $x_0 = 0$, $y_0 = 1$. As a point $(0,1)$ is a singular point for (1), then $P(0,1) = Q(0,1) = 0$. These equalities yield

$$s = a + d + m + n + 1,\ q = -b - k + p.$$

In this case the equation of each invariant straight line can be written into the form $1 + Ax - y = 0$. For $f = a + d,\ g = b + c,\ l = k - p + q,\ r = m + n - s$ and $B = -1$, system (6) becomes

$$\begin{array}{l} F_2 \equiv (a-1)A^2 + (b + c - k)A - \\ \quad - a - d - m - n - 1 = 0, \\ F_3 \equiv (a + d + 2)A^2 + (b - c - p)A - \\ \quad - d - n - 1 = 0, \\ F_4 \equiv A^3 - cA^2 + (a - d + m - 2)A + \\ \quad + c - k + p = 0. \end{array} \tag{10}$$

Reduce the equation $F_3 = 0$ of (10) by n from $F_2 = 0$ and by p from $F_4 = 0$, then $F_3 \equiv f_1 f_2 = 0$, where $f_1 = A^2 - cA + a + m,\ \ f_2 = A^2 + 1$.

Suppose $f_1 = 0$. We reduce the equations $F_2(A) = 0, F_4(A) = 0$ by A^2 from $f_1 = 0$, then system (10) becomes

$$\begin{array}{l} F_2 \equiv (ac + b - k)A - a^2 - \\ \quad - am - d - n - 1 = 0, \\ F_4 \equiv (d + 2)A + k - c - p = 0, \\ f_1 \equiv A^2 - cA + a + m = 0. \end{array} \tag{11}$$

System (11) has two distinct solutions if $F_2 \equiv 0,\ F_4 \equiv 0$ and $c^2 - 4(a + m) \neq 0$. Under the above assumptions we get the following set of conditions for the existence of four distinct invariant straight lines

(e2) $d = -2,\ f = a - 2,\ g = b + c,\ k = ac + b,\ l = -b,\ q = -(b + c),\ n = 1 - a^2 - am,\ p = ac + b - c,\ r = 1 - a,\ s = (a + m)(1 - a),$

$c^2 - 4(a + m) \neq 0$. The invariant straight lines are $x \mp iy = 0,\ \ 1 + A_1 x - y = 0,\ \ 1 + A_2 x - y = 0$, where A_1, A_2 are distinct roots of the equation $A^2 - cA + a + m = 0$.

Assume $f_1 \neq 0$ and let $f_2 = 0$. In this case $f_2 = 0$ yields $A = \pm i$. Substituting this into (11) we obtain

$$\begin{array}{l} (b + c - k)i - (d + 2a + m + n) = 0, \\ (a - d + m - 3)i + (2c - k + p) = 0. \end{array} \tag{12}$$

The equations of (12) imply $k = b + c,\ m = 3 - a + d,\ n = -a - 2d - 3,\ p = b - c$. From this we get the following set of conditions for the existence of four distinct invariant straight lines

(e3) $f = a + d,\ k = g = b + c,\ l = -b,\ m = 3-a+d,\ s = 1-a,\ n = -a-2d-3,\ p = b-c,\ q = -b-2c,\ r = -a-d-1,$

$((a-1)^2+b^2)((a-2)^2+b^2) \neq 0$. The invariant straight lines are $x \pm iy = 0,\ 1 \pm ix - y = 0,\ 1 + (-b \pm i(1-a))x + (1-a \pm bi)y = 0$.

Lemma 1. *The following six sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $a = d = f = k = l = p = q = r = 0,\ g = b+c,\ l = q,\ s = m+n$;

(ii) $d = -2,\ f = a-2,\ g = b+c,\ k = ac+b,\ l = -b,\ q = -(b+c),\ n = 1-a^2-am,\ p = ac+b-c,\ s = (a+m)(1-a),\ r = 1-a$;

(iii) $b = c = g = k = l = p = q = 0,\ f = a+d,\ s = 1-a, m = d-a+3, n = -a-2d-3,\ r = -a-d-1$;

(iv) $a = 1,\ b = l = s = 0,\ f = d+1,\ k = g = c,\ p = -c,\ m = d+2,\ r = -d-2,\ n = 2(-d-2),\ q = -2c$;

(v) $d = -2,\ f = a-2,\ k = g = b,\ l = -b,\ m = 1-a,\ c = 0,\ n = 1-a,\ p = b,\ q = -b,\ r = s = 1-a$;

(vi) $c = -2b,\ d = -2a,\ f = -a,\ g = k = l = -b,\ m = -3(a-1),\ n = 3(a-1),\ p = q = 3b,\ r = a-1,\ s = -(a-1)$.

Proof. If one of conditions (i)–(v) holds the cubic system (1) has four invariant straight lines two of which are homogeneous invariant straight lines. In cases (i), (ii), (iv) and (v) we find the first integral of the Darboux form

$$l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} l_4^{\alpha_4} = C,$$

which consists of invariant straight lines:

In case (i): $l_{1,2} = x \pm iy,\ l_{3,4} = 2 + (c \pm \sqrt{c^2-4m})x$ and $\alpha_1 = \alpha_2 = m\sqrt{c^2-4m},\ \alpha_3 = n\sqrt{c^2-4m} + 2bm - cn,\ \alpha_4 = n\sqrt{c^2-4m} - 2bm + cn$.

In case (ii): $l_{1,2} = x \pm iy,\ l_{3,4} = 2 + (c \pm \sqrt{c^2-4a-4m})x - 2y$ and $\alpha_1 = \alpha_2 = -\sqrt{c^2-4a-4m},\ \alpha_3 = a\sqrt{c^2-4a-4m} - ac - 2b,\ \alpha_4 = a\sqrt{c^2-4a-4m} + ac + 2b$.

In case (iv): $l_{1,2} = x \pm iy,\ l_{3,4} = 1 \pm ix - y$ and $\alpha_1 = \alpha_2 = -1,\ \alpha_3 = \alpha_4 = 1$.

In case (v): $l_{1,2} = x \pm iy,\ l_{3,4} = 1 \pm ix - y$ and $\alpha_1 = \alpha_2 = 1,\ \alpha_3 = -a - ib,\ \alpha_4 = -a + ib$.

In case (iii) we find an integrating factor of the Darboux form

$$\mu = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} l_4^{\alpha_4},$$

where $l_{1,2} = x \pm iy,\ l_{3,4} = 1 \pm ix - y$ and $\alpha_1 = \alpha_2 = (d-2a+6)/(2a-2),\ \alpha_3 = \alpha_4 = (4a+d)/(2-2a)$.

If condition (vi) holds then cubic system (1) has six invariant straight lines two of which are homogeneous invariant straight lines. We find the first Darboux integral of the form

$$l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} l_4^{\alpha_4} l_5^{\alpha_5} l_6^{\alpha_6} = C,$$

where $l_{1,2} = x \pm iy,\ l_{3,4} = 1 \pm ix - y,\ l_5 = 1 + (i - ia - b)x + (1 - a + ib)y,\ l_6 = 1 + (-i + ia - b)x + (1 - a - ib)y$ and $\alpha_1 = \alpha_2 = (a-2)^2 - b^2,\ \alpha_{3,4} = 1 - 2 \mp ib,\ \alpha_{5,6} = 3a - a^2 - 2 - b^2 \mp ib$.

Theorem 2. *Suppose cubic system (1) has at least four invariant straight lines two of which are homogeneous. Then the origin $O(0,0)$ is a center for (1) if and only if $L_1 = L_2 = 0$.*

Proof. We compute the first two Lyapunov quantities L_1 and L_2 for (1) by algorithm proposed in [13] assuming that one set of conditions (e1)–(e3) holds. In the expressions of L_j, we will neglect the denominators and non-zero factors.

In case (e1) the vanishing of L_1 gives $q = 0$, then use Lemma 1, (i).

In case (e2) we find $L_1 = 0$, then use Lemma 1, (ii).

In case (e3) the first Lyapunov quantity is $L_1 = c(a-1) - b(d+2)$. Assume $b = 0$. If $c = 0$, then $L_1 = 0$ and use Lemma 1, (iii). If $c \neq 0, a = 1$, then $L_1 = 0$ and use Lemma 1, (iv).

Assume $b \neq 0$, then $L_1 = 0$ yields $d = (ac - 2b - c)/b$. The second Lyapunov quantity is $L_2 = c(2b + c)$. If $c = 0$, then $L_2 = 0$ and use Lemma 1, (v). If $c \neq 0$ and $c = -2b$, then $L_2 = 0$ and use Lemma 1, (vi).

4. Three invariant straight lines and centers

In this section we find conditions for the existence of three invariant straight lines two of which are homogeneous and solve the problem of the center.

Theorem 3. *Cubic system (1) has three invariant straight lines of the form*

$$l_{1,2} \equiv x \pm iy = 0,\ l_3 \equiv 1 - x = 0 \tag{13}$$

if and only if the following set of conditions is satisfied

$$\begin{array}{l} f = a + d,\ g = b + c,\ k = -a, \\ m = -c - 1,\ l = d + q,\ r = 0, \\ p = -a - d,\ n = c + s + 1. \end{array} \tag{14}$$

Proof. Assume condition (7) holds and let cubic system (1) have one nonhomogeneous invariant line of the form $1 + Ax + By = 0$. This line is real, otherwise, we must have also the invariant straight line $1+\bar{A}x+\bar{B}y = 0$. The problem of the center for cubic system (1) with four invariant straight lines two of which are homogeneous was considered in Section 3. Via a rotation of axes about the origin and under the transformation $x \to \gamma x,\ y \to \gamma y,\ \gamma \in R \setminus 0$, the invariant line $1 + Ax + By = 0$ becomes $1 - x = 0$. For $1 - x = 0$ identity (4) gives $k = -a,\ m = -c - 1,\ p = -f,\ n = s - m$ and we obtain set of conditions (14). The cofactor of $l_3 \equiv 1 - x = 0$ is $K_3(x, y) = -y - ax^2 - (c + 1)xy - (a + d)y^2$.

Lemma 2. *The following three sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $d = l = m = q = r = 0,\ c = -1,\ f = a,\ k = p = -a,\ g = b - 1,\ n = s$;

(ii) $c = -2,\ d = r = 0,\ q = f = l = a,\ g = b - 2,\ m = 1,\ n = s - 1,\ k = p = -a$;

(iii) $b = m = 1/2,\ c = (-3)/2,\ f = a+d,\ g = -1,\ k = -a,\ r = 0,\ l = (a + d)/2,\ p = -(a + d),\ q = (a - d)/2,\ s = (2n + 1)/2$.

Proof. If either condition (i) or (ii) holds, then cubic system (1) has three invariant straight lines of form (13). We find a Darboux integrating factor of the form

$$\mu = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3}$$

with $\alpha_1 = \alpha_2 = \alpha_3 = -1$ in case (i) and $\alpha_1 = \alpha_2 = -1,\ \alpha_3 = -2$ in case (ii).

If (iii) holds then cubic system (1) with invariant straight lines (13) is rationally reversible. Indeed, in this case there exists a transformation [6] $X = 2x/(2 - x),\ Y = 2y/(2-x)$ that brings system (1) to the system

$$\begin{array}{l} \dot{X} = (4 - X^2)(Y + aX^2 + (a + d)Y^2), \\ \dot{Y} = -X(4 + 4dY + (4n + 1)X^2 + \\ \quad + (4n + 2)Y^2 + aX^2Y + (a + d)Y^3) \end{array}$$

for which $X = 0$ is an axes of symmetry. The obtained system has a center at $X = Y = 0$ and hence the origin is a center for (1).

Lemma 3. *The following two sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $a = d = f = k = l = p = q = r = 0,\ g = b + c,\ m = -c - 1,\ n = c + s + 1$;

(ii) $f = a + d,\ g = b + c,\ k = -a,\ m = -c - 1,\ l = bd - a(c+1),\ n = [(c+1)(bd - a(c+2))]/d,\ q = d(b-1) - a(c+1),\ p = -a-d,\ s = [(c+1)(d(b-1) - a(c+2))]/d,\ r = 0$.

Proof. In these two cases the system (1) has four invariant straight lines and the Darboux first integral

$$(x^2 + y^2) l_3^{\alpha_3} l_4^{\alpha_4} = C.$$

In case (i): $l_3 = 1 - x,\ l_4 = 1 + (c + 1)x$ and $\alpha_3 = -2(b + c + s + 1)/(c + 2),\ \alpha_4 = 2(b + bc - c - s - 1)/((c + 1)(c + 2))$, where $(c + 1)(c + 2) \neq 0$.

In (ii): $l_3 = 1 - x,\ l_4 = 1 + (c + 1)x + dy$ and $\alpha_3 = 2(a + ac - bd)/d,\ \alpha_4 = 2a/d$, where $d \neq 0$.

Theorem 4. *Suppose cubic system (1) has three invariant straight lines of form (13). Then the origin O(0, 0) is a center for (1) if and only if $L_j = 0,\ j = \overline{1,\ 7}$.*

Proof. We compute the first seven Lyapunov quantities for (1) by algorithm proposed in [13] assuming that set of conditions (13) holds. In the expressions of $L_j,\ j =$

$\overline{1,7}$, we will neglect the denominators and nonzero factors.

The vanishing of the first Lyapunov quantity gives $q = bd - ac - a - d$. The second Lyapunov quantity looks like $L_2 = f_1 f_2$, where

$$f_1 = a(c+1)(c+2) - d(c+1)(b-1) + ds,$$
$$f_2 = 4b + 2c + 1.$$

If $f_1 = 0$ and $d = 0$, we find $a(c+1)(c+2) = 0$. If $a = 0$, then use Lemma 3, (i); if $a \neq 0,\ c = -1$, then use Lemma 2, (i); if $a \neq 0,\ c = -2$, then use Lemma 2, (ii).

If $f_1 = 0$ and $d \neq 0$, we have $s = [(c+1)(d(b-1) - a(c+2))]/d$. In this case the origin is a center by Lemma 3, (ii).

Assume $f_1 \neq 0$ and let $f_2 = 0$, then $b = -(1+2c)/4$. The third Lyapunov quantity looks like $L_3 = g_1 g_2$, where $g_1 = 2c + 3$ and

$$g_2 = 48a^2 + 40ad + 2c + 8d^2 + 8s + 3.$$

If $g_1 = 0$, then use Lemma 2, (iii) and if $g_1 \neq 0$, $g_2 = 0$, then $s = -(48a^2 + 40ad + 2c + 8d^2 + 3)/8$. In this case $L_4 = h_1 h_2$, where

$$h_1 = 4(2a+d)^2 + 1,$$
$$h_2 = 80a^2 + 88ad - 2c^2 - 6c + 22d^2 - 5.$$

It is evident that $h_1 = 0$ has no real solutions. In the next three Lyapunov quantities the factor h_1 will be omitted. Next we reduce the Lyapunov quantities L_5, L_6 by h_2, and L_7 by h_2 and L_5. We have

$$L_5 = 1280a^4 + 1536a^3d + 384a^2d^2 - 416a^2 - 128ad^3 - 480ad - 48d^4 - 136d^2 + 1,$$

$$L_6 = L_5(1488a^2 + 1480ad - 66c + 368d^2 - 79),$$

$$L_7 = 20971520a^8 - 45132ad + 31457280a^7d + 15728640a^6d^2 - 12684d^2 - 11390976a^6 + 2621440a^5d^3 + 69264ad^3 - 15857664a^5d - 6986752a^4d^2 + 1641728a^4 - 952832a^3d^3 + 2085312a^3d + 773312a^2d^2 - 42320a^2 + 93.$$

The system $h_2 = L_5 = L_6 = L_7 = 0$ has no real solutions. Note that $h_2 = L_5 = 0$ (i.e. $h_2 = L_5 = L_6 = 0$) has real solutions.

Indeed, if we assume $a = 0$, then it is evident that the system $h_2 = 22d^2 - 2c^2 - 6c - 5 = 0$, $L_5 = 1 - 136d^2 - 48d^4 = 0$ has real solutions. Hence, the vanishing of the Lyapunov quantities $L_j,\ j = \overline{1,6}$ does not imply the origin to be a center for (1). Theorem is proved.

We summarize necessary and sufficient conditions for the origin to be a center in the following theorem.

Theorem 5. *The origin is a center for (1), with at least three invariant straight lines two of which are homogeneous, if and only if one of the conditions of Lemmas 1–3 holds.*

5. Two homogeneous invariant straight lines and centers

In this section assuming that cubic system (1) has two homogeneous invariant straight lines we find sufficient conditions for the origin to be a center for (1).

Lemma 4. *The following three sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $c = -2b,\ d = -2a,\ f = -a,\ n = 2r - m,\ g = -b,\ p = -l,\ q = -k,\ s = r;$

(ii) $a = d = f = 0,\ g = b + c,\ k = l,\ m = (2br + cn - cr)/(2b),\ p = q = [l(b+c)]/b,\ s = (2bn + cn - cr)/(2b);$

(iii) $c = (bd)/a,\ f = a + d,\ g = [b(a+d)]/a,\ p = q = [l(a+d)]/a,\ k = l,\ m = (2ar + dn - dr)/(2a),\ s = (2an + dn - dr)/(2a).$

Proof. Assume condition (7) is satisfied, then cubic system (1) has two homogeneous invariant straight lines of the form $x \pm iy = 0$. We find the integrating factor of the form

$$\mu = (x + iy)^{\alpha_1}(x - iy)^{\alpha_2}.$$

In case (i): $\alpha_2 = \alpha_1 = -2$; in case (ii): $\alpha_2 = \alpha_1 = (c - 2b)/(2b)$; in case (iii): $\alpha_2 = \alpha_1 = (d - 2a)/(2a)$.

Lemma 5. *The following four sets of conditions are sufficient conditions for the origin to be a center for system (1):*

(i) $b = c = g = k = l = p = q = 0,\ f = a + d,\ r = m + n - s;$

(ii) $b = [a(1-u^2)]/(2u),\ c = [d(1-u^2)]/(2u),\ g = [(a+d)(1-u^2)]/(2u),\ n = [(q - 3k)(u^4 - 6u^2 + 1) + 4m(u^3 - u)]/[4u(u^2 - 1)],\ l = k,\ f = a + d,\ r = [(q-k)(u^4 - 6u^2 + 1) + 4m(u^3 - u)]/[4u(u^2 - 1)],\ p = q,\ s = [k(6u^2 - u^4 - 1) + 2mu(u^2 - 1)]/[2u(u^2 - 1)];$

(iii) $c = -3b$, $f = a + d$, $g = -2b$, $k = -2ab$, $l = b(a + d)$, $m = 2b^2$, $p = -2b(a + d)$, $q = b(a - d)$, $r = 0$, $s = 2b^2 + n$;

(iv) $c = [(3a + d)(1 - u^2) - 6bu]/(2u)$, $g = [(3a + d)(1 - u^2) - 4bu]/(2u)$, $l = [a(3a + d)(u^2 - 1) + 2(3ab + bd + k)u]/(2u)$, $m = [r(u^2+1)^4+2(au^2-a+2bu)((5a+2d)(u^6-1) + (11a-2d)(u^2-u^4) + b(10u^5 - 12u^3 + 10u))]/(u^2+1)^4$, $s = [n(u^2+1)^4+2(au^2-a+2bu)((5a+2d)(u^6-1)+(11a-2d)(u^2-u^4) + b(10u^5 - 12u^3 + 10u))]/(u^2 + 1)^4$, $f = a + d$, $q = [2pu + (3a + d)(au^2 - a + 2bu)]/(2u)$, $r = [2(5ab+bd+k)(u^{11}-u)+2(4b^2-9a^2-3ad)(u^{10}+u^2)+a(3a+d)(u^{12}+1) + 2(3k-5bd-33ab)(u^9-u^3) + (61a^2-ad-64b^2)(u^8+u^4)+4(45ab-3bd+k)(u^7-u^5) + 4(28b^2 - 23a^2 + 3ad)u^6]/[4u^2(u^2 + 1)^4]$, $n = [2(k - 10ab - 2bd)(u^9 + u) + 8(10ab + k)(u^7 + u^3) + 2(14a^2 + ad - 12b^2)(u^8 - u^2) + 4(10b^2 + ad - 8a^2)(u^6 - u^4)+4(3k-14ab+2bd)u^5+2a(2a+d)(1-u^{10})]/[(u^2+1)^4(u^2-1)]$, $p = [(12ab+2bd+k)(u^9+u)+(12b^2-5ad-19a^2)(u^8-u^2)+4(k-16ab-2bd)(u^7+u^3)+2(21a^2-3ad-26b^2)(u^6-u^4)+a(3a+d)(u^{10}-1)+2(52ab-10bd+3k)u^5]/[u(u^2+1)^4]$.

Proof. If one of conditions (i)–(iv) holds, the cubic system is rationally reversible. We find a transformation of the form [6]

$$x = \frac{a_1X + b_1Y}{a_3X + b_3Y - 1},\ y = \frac{a_2X + b_2Y}{a_3X + b_3Y - 1}$$

with $a_1b_2 - b_1a_2 \neq 0$ and $a_j,\ b_j \in \mathbb{R}$, $j = 1, 2, 3$ which brings system (1) to one equivalent with a polynomial system

$$\dot{X} = Y + M(X^2, Y),$$
$$\dot{Y} = -X(1 + N(X^2, Y)).$$

The obtained system has an axis of symmetry $X = 0$ and therefore $O(0, 0)$ is a center for (1).

In case (i): $x = X/(Y + 1)$, $y = Y/(Y + 1)$; in case (ii): $x = (2uX-u^2Y+Y)/[(u^2+1)(Y-1)]$, $y = (2uY+u^2X-X)/[(u^2+1)(Y-1)]$; in case (iii): $x = X/(1+bX)$, $y = Y/(1+bX)$; in case (iv): $x = (2uX - u^2Y + Y)/[(au^2 + 2bu - a)X - u^2 - 1]$, $y = (2uY + u^2X - X)/[(au^2 + 2bu - a)X - u^2 - 1]$.

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REFERENCES

1. *Amel'kin, V.V., Lukashevich, N.A., Sadovsky, A.P.*, Non-linear oscillations in the systems of second order// Minsk. –1982 (in Russian).

2. *Cozma, D., Şubă, A.*, Partial integrals and the first focal value in the problem of centre// Nonlinear Differ. Equ. and Appl. –1995. – **2**,–P. 21–34.

3. *Cozma, D., Şubă, A.*, The solution of the problem of center for cubic differential systems with four invariant straight lines// Scientific Annals of the "Al.I.Cuza"University, Mathematics. –1998. – **XLIV**, s.I.a. – P. 517–530.

4. *Cozma, D.*, The problem of the center for cubic systems with two parallel invariant straight lines and one invariant conic// Nonlinear Differ. Equ. and Appl. –2009. –**16**. – P. 213–234.

5. *Cozma D.*, Center problem for cubic systems with a bundle of two invariant straight lines and one invariant conic// Bull. of Acad. of Sci. of Moldova. Mathematics. –2012. –**68**, no.1. –P.32–49.

6. *Cozma D.*, Rationally reversible cubic systems// Scientific Bulletin of Chernivtsi University. Series Mathematics. –2012. –**2**. –P. 114–119.

7. *Cozma, D., Şubă, A.*, Solution of the problem of the centre for a cubic differential system with three invariant straight lines// Qualitative Theory of Dynamical Systems. –2001. –**2**, no.1. –P. 129–143.

8. *Lyapunov, A. M.*, Probléme général de la stabilité du mouvement// Ann. of Math. Stud. –1947. –**17**. Princeton University Press.

9. *Poincaré, H.*, Mémoire sur les courbes définies par une équation différentielle// Oeuvres de Henri Poincaré. –1951. –**1**, Gauthiers–Villars, Paris.

10. *Sadovskii, A.P.* Solution of the center and focus problem for a cubic system of nonlinear oscillations// Differ. Equa. –1997. –**33**, no.2. – P. 236–244.

11. *Sadovskii, A.P.* Solution of the center-focus problem for a cubic system reducible to a Lienard system// Differen. Uravn. –2006. –**42**, no.1. – P. 11–22.

12. *Sadovskii, A.P., Shcheglova T.V.* Solution of the center-focus problem for a nine-parameter cubic system// Differ. Equa. –2011. –**47**, no.2. – P. 208–223.

13. *Şubă, A.*, On the Liapunov quantities of two-dimensional autonomous system of differential equations with a critical point of centre or focus type// Bulletin of Baia Mare University, Math. and Infor. – 1998. –**13**. –P. 153—170.

14. *Şubă, A., Cozma, D.*, Solution of the problem of center for cubic differential systems with three invariant straight lines in generic position// Qualitative Theory of Dynamical Systems. –2005. –**6**. – P. 45–58.