УДК 517.911

©2016 year A. O. Tsukanova

National Technical University of Ukraine “Kiev Polytechnic Institute”

ON EXISTENCE AND UNIQUENESS OF MILD SOLUTION TO THE CAUCHY PROBLEM FOR ONE NEUTRAL STOCHASTIC DIFFERENTIAL EQUATION OF REACTION-DIFFUSION TYPE IN HILBERT SPACE

Доведено теорему існування та єдиності м’якого роз’вязку задачі Коші для стохастичного диференціального рівняння нейтрального типу в гільбертовому просторі $L_2(\mathbb{R}^d)$.

The theorem on existence and uniqueness of mild solution to the Cauchy problem for one neutral stochastic differential equation in Hilbert space $L_2(\mathbb{R}^d)$ has been proved.

1. Introduction. Questions on existence and uniqueness of solution to stochastic differential equations (SDEs from now on) under some given initial-boundary conditions in various functional spaces, in particularly, in Hilbert spaces, have been extensively studied by a variety of authors. There exists especial interest around ***neutral SDEs***. An essential feature of such equations is the phenomena of delay within so-called “derivative”. In [1] its authors have considered an initial-value problem for an abstract SDE of such type in Hilbert space and have proved the theorem on existence and uniqueness of its ***mild solution***. But conditions of this theorem are formulated in a general form. Therefore it is rather complicated to check them directly while solving specific applied problems. Hence it is important to find conditions, convenient to check, that are expressed in terms of coefficients of the equation under investigation. If such conditions are found, it will be possible to check them immediately while solving concrete problems. But it is only possible to do in some particular cases, one of which will be studied in the paper. It consists of five sections and is organized as follows. The second section concerns with formulation of the problem. After it, in the third section, some already known results from the theory of partial differential equations and one fact from the heat semi-group theory are gathered. The fourth section contains formulation of the main result. The last, fifth, section is devoted to it’s proof.

2. Formulation of the problem. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. The following initial-value problem for nonlinear neutral stochastic integro-differential equation of reaction-diffusion type is considered

$$d\left(u(t,x) + \int\limits_{\mathbb{R}^d} b(t,x,\xi)u(t-h,\xi)d\xi\right) =$$
$$= \big(\Delta_x u(t,x) + f(t,u(t-h),x)\big)dt +$$
$$+\,\sigma(t,u(t-h),x)dW(t,x),\ 0<t\le T,\ x\in\mathbb{R}^d,$$
$$u(t,x) = \phi(t,x),\ -h\le t\le 0,\ x\in\mathbb{R}^d, \qquad (1)$$

where $T>0$ is a fixed real number, $h>0$ — an arbitrary real number, $\Delta_x \equiv \sum\limits_{i=1}^{d} \partial^2_{x_i}$ — d-measurable operator of Laplace, $\partial^2_{x_i} \equiv \frac{\partial^2}{\partial x_i^2}$, $i\in\{1,\ldots,d\}$, $W(t,x)$ — $L_2(\mathbb{R}^d)$-valued Q-Wiener process, $\{f,\sigma\} : [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $b\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are some given functions to be specified later, $\phi\colon [-h,0]\times\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is an initial-datum function. The theorem on existence and uniqueness of ***mild solution*** to the problem (1) will be proved.

3. Preliminaries. In what follows, in order to prove the main result, lemmas 1 — 4 from [4] will be needed and the following fact from the theory of heat semi-group.

Lemma [2; 3, c. 188]**.** *Operators* $S(t)\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ *generate the solution of homogeneous Cauchy problem for heat-equation* (see lemma 1 from [4] for details) *by the rule*

$$u(t,x) = (S(t)g(\,\cdot\,))(x) = \\ = \int\limits_{\mathbb{R}^d} \mathscr{K}(t, x-\xi) g(\,\cdot\,) d\xi,$$

and form $(C_0$-$)$*semi-group of operators, an infinitesimal generator of which is Laplacian* Δ_x*. Moreover, this semi-group is contractive, i.e.*

$$\|(S(t)g(\,\cdot\,))(x)\|^2_{L_2(\mathbb{R}^d)} \le \|g(x)\|^2_{L_2(\mathbb{R}^d)}, \\ g(\,\cdot\,) \in L_2(\mathbb{R}^d). \tag{2}$$

A couple of notations, given below, will be used hereinafter. Let filtration of σ-algebras $\{\mathcal{F}_t, t \ge 0\}$ is generated by $L_2(\mathbb{R}^d)$-valued Q-Wiener process $W(t,x) = \sum\limits_{n=1}^{\infty} \sqrt{\lambda_n} e_n(x) \times \times \beta_n(t)$, where $\{\beta_n(t), n \in \{1,2,\dots\}\}$ are independent standard one-dimensional real-valued Brownian motions, sequence $\{\lambda_n, n \in \{1,2,\dots\}\}$ of positive real numbers is such that

$$\sum_{n=1}^{\infty} \lambda_n < \infty, \tag{3}$$

and system of vectors $\{e_n(x), n \in \{1,2,\dots\}\}$ forms an orthonormal basis in $L_2(\mathbb{R}^d)$ such that

$$\sup_{n \in \{1,2,\dots\}} ess\sup_{x \in \mathbb{R}^d} |e_n(x)| \le 1. \tag{4}$$

Let $\mathfrak{B}_{2,T}$ denotes Banach space of all $L_2(\mathbb{R}^d)$-valued $\mathcal{F}_t$-measurable for almost all $0 \le t \le T$ random processes $\Phi\colon [0,T] \times \Omega \to \to L_2(\mathbb{R}^d)$, that are continious in t for almost all $\omega \in \Omega$, with the norm $\|\Phi\|_{\mathfrak{B}_{2,T}} = = \sqrt{\sup\limits_{0 \le t \le T} \mathbf{E}\|\Phi(t)\|^2_{L_2(\mathbb{R}^d)}}$. The further result guarantees existence and uniqueness for $0 \le \le t \le T$ of ***mild solution*** to (1) in $\mathfrak{B}_{2,T}$.

4. Main result. The following assumptions are the main, presumed to be true in the paper:

4.1) functions $\{f, \sigma\}\colon [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $b\colon [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are measurable with respect to their arguments;

4.2) an initial-datum function $\phi\colon [-h,0] \times \mathbb{R}^d \times \times \Omega \to \mathbb{R}$ is $\mathcal{F}_0$-measurable, independent from $W(t,x)$, $t \ge 0$, and such that

$$\sup_{-h \le t \le 0} \mathbf{E}\|\phi(t)\|^2_{L_2(\mathbb{R}^d)} < \infty. \tag{5}$$

Definition. *Continuous random process*

$$u\colon [-h,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$$

is called ***mild solution*** *of problem* (1)*, if it*
1) *is* $\mathcal{F}_t$*-measurable for almost all* $-h \le t \le T$*;*
2) *satisfies the following integral equation*

$$u(t,x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t, x-\xi) \Bigg(\phi(0) + \\ + \int\limits_{\mathbb{R}^d} b(0,\xi,\zeta)\phi(-h,\zeta) d\zeta \Bigg) d\xi - \\ - \int\limits_{\mathbb{R}^d} b(t,x,\xi) u(t-h,\xi) d\xi - \\ - \int\limits_0^t \Bigg(\Delta_x \int\limits_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \times \\ \times \Bigg(\int\limits_{\mathbb{R}^d} b(s,\xi,\zeta) u(s-h,\zeta) d\zeta \Bigg) d\xi \Bigg) ds + \\ + \int\limits_0^t \int\limits_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) f(s, u(s-h), \xi) d\xi ds + \\ + \int\limits_0^t \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Bigg(\int\limits_{\mathbb{R}^d} \mathscr{K}(t-s, x-\xi) \times \\ \times \sigma(s, u(s-h), \xi) e_n(\xi) d\xi \Bigg) d\beta_n(s),$$

$0 \le t \le T$, $x \in \mathbb{R}^d$,
$u(t,x) = \phi(t,x)$, $-h \le t \le 0$, $x \in \mathbb{R}^d$*;*

3) *satisfies the following condition*

$$\mathbf{E} \int\limits_0^T \|u(t)\|^2_{L_2(\mathbb{R}^d)} dt < \infty.$$

The following theorem is valid.

Theorem (existence and uniqueness of mild solution in $\mathfrak{B}_{2,T}$). *Let's suppose*

assumptions **4.1**, **4.2** *to hold true, and besides the following conditions to be valid:*
1) functions $\{f,\sigma\}$ *satisfy linear-growth and Lipschitz conditions by their second argument, i.e. there exists* $L>0$ *such that*

$$|f(t,u,x)|\le \chi(t,x)+L|u|,\quad 0\le t\le T,\ u\in\mathbb{R},\ x\in\mathbb{R}^d, \tag{6}$$
$$|f(t,u,x)-f(t,v,x)|\le L|u-v|,\quad 0\le t\le T,\ \{u,v\}\subset\mathbb{R},\ x\in\mathbb{R}^d,$$
$$|\sigma(t,u,x)|\le L\big(1+|u|\big),\quad 0\le t\le T,\ u\in\mathbb{R},\ x\in\mathbb{R}^d, \tag{7}$$
$$|\sigma(t,u,x)-\sigma(t,v,x)|\le L|u-v|,\quad 0\le t\le T,\ \{u,v\}\subset\mathbb{R},\ x\in\mathbb{R}^d,$$

where function $\chi\colon [0,T]\times\mathbb{R}^d\to[0,\infty)$ *is such that*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\chi^2(t,x)dx<\infty; \tag{8}$$

2) function b *satisfies the conditions*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\sqrt{\int_{\mathbb{R}^d}b^2(t,x,\zeta)d\zeta}\,dx<\infty, \tag{9}$$
$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}b^2(t,x,\zeta)d\zeta dx<\infty; \tag{10}$$

3) for each point $x\in\mathbb{R}^d$ *there exist partial derivatives* $\partial_{x_i}b$, $\partial_{x_ix_j}b$, $\{i,j\}\subset\{1,\dots,d\}$, *and gradient-vector* $\nabla_x b$ *and Hesse-matrix* $D_x^2 b$ *satisfy the condition*

$$|\nabla_x b(t,x,\xi)|+\|D_x^2 b(t,x,\xi)\|\le\psi(t,x,\xi),\quad 0\le t\le T,\ \{x,\xi\}\subset\mathbb{R}^d, \tag{11}$$

where function $\psi\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ *is such that the following condition*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi^2(t,x,\zeta)d\zeta dx<\infty, \tag{12}$$

comes true, and besides for each point $x_0\in\mathbb{R}^d$ *there exists its vicinity* $B_\delta(x_0)$ *and nonnegative function* $\varphi(t,x,x_0,\delta)$ *such that*

$$\sup_{0\le t\le T}\varphi(t,\,\cdot\,,x_0,\delta)\in L_2(\mathbb{R}^d),\ \delta\in\mathbb{R}^+, \tag{13}$$

$$|\psi(t,x,\zeta)-\psi(t,x_0,\zeta)|\le\varphi(t,\zeta,x_0,\delta)|x-x_0|,\quad 0\le t\le T,\ |x-x_0|<\delta,\ \zeta\in\mathbb{R}^d. \tag{14}$$

Then the problem (1) *will have unique for* $0\le t\le T$ *mild solution* $u\in\mathfrak{B}_{2,T}$, *if*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}b^2(t,x,\xi)d\xi dx<\frac{1}{4}. \tag{15}$$

5. Proof of the theorem. The proof is based on the classical theorem from functional analysis — Banach theorem on a fixed point. According to it, let's consider an operator $\Psi\colon\mathfrak{B}_{2,T}\to\mathfrak{B}_{2,T}$ with an action

$$\begin{aligned}\big(\Psi u\big)(t)=&\int_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)\bigg(\phi(0)+\\&+\int_{\mathbb{R}^d}b(0,\xi,\zeta)\phi(-h,\zeta)d\zeta\bigg)d\xi-\\&-\int_{\mathbb{R}^d}b(t,x,\xi)u(t-h,\xi)d\xi-\\&-\int_0^t\Bigg(\Delta_x\int_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\times\\&\times\bigg(\int_{\mathbb{R}^d}b(s,\xi,\zeta)u(s-h,\zeta)d\zeta\bigg)d\xi\Bigg)ds+\\&+\int_0^t\int_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)f(s,u(s-h),\xi)d\xi ds+\\&+\int_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\bigg(\int_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\times\\&\times\sigma(s,u(s-h),\xi)e_n(\xi)d\xi\bigg)d\beta_n(s)=\sum_{j=0}^{4}I_j(t),\end{aligned}$$

$$0\le t\le T,\ x\in\mathbb{R}^d,$$
$$u(t,x)=\phi(t,x),\ -h\le t\le 0,\ x\in\mathbb{R}^d,$$

and prove that this operator is contractive. In order to do it, firstly let's show that $\Psi u\in\mathfrak{B}_{2,T}$ for each $u\in\mathfrak{B}_{2,T}$. For this purpose five norms $\|I_j(s)\|^2_{\mathfrak{B}_{2,t}}=\sup_{0\le s\le t}\mathbf{E}\|I_j(s)\|^2_{L_2(\mathbb{R}^d)}$, $j\in\{0,\dots,4\}$, must be estimated.

Taking into account property (2), Cauchy-Schwartz inequality and assumptions (5), (10), one obtains for $\|I_0(s)\|^2_{\mathfrak{B}_{2,t}}$

$$\|I_0(s)\|^2_{\mathfrak{B}_{2,t}} = \sup_{0\le s\le t} \mathbf{E}\|I_0(s)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= \sup_{0\le s\le t} \mathbf{E}\Bigg\|\int_{\mathbb{R}^d} \mathscr{K}(s, x-\xi)\Bigg(\phi(0)+$$

$$+\int_{\mathbb{R}^d} b(0,\xi,\zeta)\phi(-h,\zeta)d\zeta\Bigg)d\xi\Bigg\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le 2\sup_{0\le s\le t} \mathbf{E}\Bigg\|\int_{\mathbb{R}^d} \mathscr{K}(s,x-\xi)\phi(0)d\xi\Bigg\|^2_{L_2(\mathbb{R}^d)} +$$

$$+2\sup_{0\le s\le t} \mathbf{E}\Bigg\|\int_{\mathbb{R}^d} \mathscr{K}(s,x-\xi)\times$$

$$\times\Bigg(\int_{\mathbb{R}^d} b(0,\xi,\zeta)\phi(-h,\zeta)d\zeta\Bigg)d\xi\Bigg\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le 2\mathbf{E}\,\|\phi(0)\|^2_{L_2(\mathbb{R}^d)} + 2\mathbf{E}\Bigg\|\int_{\mathbb{R}^d} b(0,x,\zeta)\times$$

$$\times\phi(-h,\zeta)d\zeta\Bigg\|^2_{L_2(\mathbb{R}^d)} = 2\mathbf{E}\,\|\phi(0)\|^2_{L_2(\mathbb{R}^d)} +$$

$$+2\mathbf{E}\int_{\mathbb{R}^d}\Bigg(\int_{\mathbb{R}^d} b(0,x,\zeta)\phi(-h,\zeta)d\zeta\Bigg)^2 dx \le$$

$$\le 2\mathbf{E}\,\|\phi(0)\|^2_{L_2(\mathbb{R}^d)} + 2\Bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(0,x,\zeta)d\zeta dx\Bigg)\times$$

$$\times\mathbf{E}\int_{\mathbb{R}^d}\phi^2(-h,\zeta)d\zeta = 2\mathbf{E}\,\|\phi(0)\|^2_{L_2(\mathbb{R}^d)} +$$

$$+2\Bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(0,x,\zeta)d\zeta dx\Bigg)\mathbf{E}\,\|\phi(-h)\|^2_{L_2(\mathbb{R}^d)} <$$

$$< \infty.$$

By using Cauchy-Schwartz inequality and assumptions (5), (10), one obtains for $\|I_1(s)\|^2_{\mathfrak{B}_{2,t}}$

$$\|I_1(s)\|^2_{\mathfrak{B}_{2,t}} = \sup_{0\le s\le t} \mathbf{E}\|I_1(s)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= \sup_{0\le s\le t} \mathbf{E}\Bigg\|\int_{\mathbb{R}^d} b(s,x,\xi)u(s-h)d\xi\Bigg\|^2_{L_2(\mathbb{R}^d)} =$$

$$= \sup_{0\le s\le t} \mathbf{E}\int_{\mathbb{R}^d}\Bigg(\int_{\mathbb{R}^d} b(s,x,\xi)u(s-h,\xi)d\xi\Bigg)^2 dx \le$$

$$\le \sup_{0\le s\le t}\Bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\mathbf{E}\int_{\mathbb{R}^d} u^2(s-h,\xi)d\xi =$$

$$= \sup_{0\le s\le t}\Bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\mathbf{E}\|u(s-h)\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le \Bigg(\sup_{0\le s\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\sup_{0\le s\le t}\mathbf{E}\|u(s-h)\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le \Bigg(\sup_{0\le s\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\Bigg(\sup_{0\le s\le h}\mathbf{E}\|u(s-h)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\sup_{h\le s\le t}\mathbf{E}\|u(s-h)\|^2_{L_2(\mathbb{R}^d)}\Bigg) =$$

$$= \Bigg(\sup_{0\le s\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\Bigg(\sup_{-h\le s-h\le 0}\mathbf{E}\|u(s-h)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\sup_{0\le s-h\le t-h}\mathbf{E}\|u(s-h)\|^2_{L_2(\mathbb{R}^d)}\Bigg) =$$

$$= \Bigg(\sup_{0\le s\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\Bigg(\sup_{-h\le s\le 0}\mathbf{E}\|u(s)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\sup_{0\le s\le t-h}\mathbf{E}\|u(s)\|^2_{L_2(\mathbb{R}^d)}\Bigg) =$$

$$= \Bigg(\sup_{0\le s\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\Bigg)\times$$

$$\times\Bigg(\sup_{-h\le s\le 0}\mathbf{E}\|\phi(s)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\sup_{0\le s\le t-h}\mathbf{E}\|u(s)\|^2_{L_2(\mathbb{R}^d)}\Bigg) \le$$

$$\leq \left(\sup_{0\leq s\leq t}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\right)\times$$
$$\times\left(\sup_{-h\leq s\leq 0}\mathbf{E}\|\phi(s)\|^2_{L_2(\mathbb{R}^d)}+\right.$$
$$\left.+\sup_{0\leq s\leq t}\mathbf{E}\|u(s)\|^2_{L_2(\mathbb{R}^d)}\right)<\infty.$$

While estimating $\|I_2(s)\|^2_{\mathfrak{B}_{2,t}}$, by using Cauchy-Schwartz inequality and Fubini theorem, one concludes

$$\|I_2(s)\|^2_{\mathfrak{B}_{2,t}}=\sup_{0\leq s\leq t}\mathbf{E}\|I_2(s)\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\leq s\leq t}\mathbf{E}\left\|\int\limits_0^s\left(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.\right.$$
$$\left.\left.\times\left(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\tau-h,\zeta)d\zeta\right)d\xi\right)d\tau\right\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\leq s\leq t}\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_0^s\left(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.\right.$$
$$\left.\left.\times\left(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\tau-h,\zeta)d\zeta\right)d\xi\right)d\tau\right)^2dx\leq$$
$$\leq\sup_{0\leq s\leq t}s\mathbf{E}\int\limits_{\mathbb{R}^d}\int\limits_0^s\left(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.$$
$$\left.\times\left(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\tau-h,\zeta)d\zeta\right)d\xi\right)^2d\tau dx\leq$$
$$\leq t\sup_{0\leq s\leq t}\mathbf{E}\int\limits_{\mathbb{R}^d}\int\limits_0^s\left(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.$$
$$\left.\times\left(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\tau-h,\zeta)d\zeta\right)d\xi\right)^2d\tau dx=$$
$$=t\sup_{0\leq s\leq t}\mathbf{E}\int\limits_0^s\int\limits_{\mathbb{R}^d}\left(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.$$
$$\left.\times\left(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)u(\tau-h,\zeta)d\zeta\right)d\xi\right)^2dxd\tau\leq$$
$$\leq Ct\sup_{0\leq s\leq t}\mathbf{E}\int\limits_0^s\int\limits_{\mathbb{R}^d}\left\|D_x^2\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)\times\right.$$
$$\left.\times u(\tau-h,\zeta)d\zeta\right\|^2dxd\tau=Ct\times$$
$$\times\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\left\|D_x^2\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)\times\right.$$
$$\left.\times u(\tau-h,\zeta)d\zeta\right\|^2dxd\tau,\qquad(16)$$

if conditions of lemma 4 from [4] are valid, where

$$u(\tau,x)=\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\left(\int\limits_{\mathbb{R}^d}b(\tau,\xi,\zeta)\times\right.$$
$$\left.\times u(\tau-h,\zeta)d\zeta\right)d\xi,$$
$$g(\tau,x)=\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta.\qquad(17)$$

Here $\nabla_x\equiv(\partial_{x_1}\ldots\partial_{x_d})^{\mathrm{T}}$, $D_x^2\equiv$ $\equiv\begin{pmatrix}\partial^2_{x_1} & \ldots & \partial_{x_1x_d}\\ \vdots & \ddots & \vdots\\ \partial_{x_dx_1} & \ldots & \partial^2_{x_d}\end{pmatrix}$, $\|\cdot\|$ is the corresponding matrix norm. Thus the aim is to verify that conditions of lemma 4 from [4] are executed for our function g, defined by (17). In order to do it, it is necessary to prove that

1) with probability one for each $0\leq\tau\leq t$

$$\int\limits_{\mathbb{R}^d}b(\tau,\cdot,\zeta)u(\tau-h,\zeta)d\zeta\in L_1(\mathbb{R}^d);\qquad(18)$$

2) $|\nabla_x g|\in L_2(\mathbb{R}^d)$, $\|D_x^2 g\|\in L_2(\mathbb{R}^d)$. (19)

1. While proving (18), one obtains via using Cauchy-Schwartz inequality and conditions (9), (5)

$$\mathbf{E}\int_{\mathbb{R}^d}\left|\int\limits_{\mathbb{R}^d}b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta\right|dx\leq$$
$$\leq\left(\int\limits_{\mathbb{R}^d}\sqrt{\int\limits_{\mathbb{R}^d}b^2(\tau,x,\zeta)d\zeta dx}\right)\times$$

$$\times \sqrt{\mathbf{E}\int\limits_{\mathbb{R}^d} u^2(\tau-h,\zeta)d\zeta} \le$$

$$\le \left(\sup_{0\le\tau\le t}\int\limits_{\mathbb{R}^d}\sqrt{\int\limits_{\mathbb{R}^d} b^2(\tau,x,\zeta)d\zeta dx}\right)\times$$

$$\times\sqrt{\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}} \le$$

$$\le \left(\sup_{0\le\tau\le t}\int\limits_{\mathbb{R}^d}\sqrt{\int\limits_{\mathbb{R}^d} b^2(\tau,x,\zeta)d\zeta dx}\right)\times$$

$$\times\Big(\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}\Big)^{\frac{1}{2}} < \infty,$$

therefore with probability one

$$\int\limits_{\mathbb{R}^d}\left|\int\limits_{\mathbb{R}^d} b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta\right|dx < \infty.$$

2. Condition (19) will be proved for $|\nabla_x g|$, since for $\|D_x^2 g\|$ it is similar.

Firstly it is necessary to show differentiability of (17) at the point $x = x_0$ — an arbitrary point from $\mathbb{R}^d$.

Let $B_\delta(x_0)$ be the vicinity from p. 3 of the theorem. One obtains through the use of conditions (11) and (14)

$$|\nabla_x b(\tau,x,\zeta)u(\tau-h,\zeta)| \le \psi(\tau,x,\zeta)\times$$
$$\times|u(\tau-h,\zeta)| = \big(\psi(\tau,x,\zeta)-\psi(\tau,x_0,\zeta)+$$
$$+\psi(\tau,x_0,\zeta)\big)|u(\tau-h,\zeta)| \le \big(|\psi(\tau,x,\zeta)-$$
$$-\psi(\tau,x_0,\zeta)|+\psi(\tau,x_0,\zeta)\big)|u(\tau-h,\zeta)| \le$$
$$\le \big(\varphi(\tau,\zeta,x_0,\delta)|x-x_0|+\psi(\tau,x_0,\zeta)\big)\times$$
$$\times|u(\tau-h,\zeta)| \le \big(\delta\varphi(\tau,\zeta,x_0,\delta)+$$
$$+\psi(\tau,x_0,\zeta)\big)|u(\tau-h,\zeta)|.$$

Let's verify that

$$\big(\delta\varphi(\tau,\cdot,x_0,\delta)+\psi(\tau,x_0,\cdot)\big)\times$$
$$\times|u(\tau-h,\cdot)| \in L_1(\mathbb{R}^d). \qquad (20)$$

Using Cauchy-Schwartz inequality and assumptions (13), (12), (5) yields

$$\mathbf{E}\int\limits_{\mathbb{R}^d}\big(\delta\varphi(\tau,\zeta,x_0,\delta)+\psi(\tau,x_0,\zeta)\big)\times$$
$$\times|u(\tau-h,\zeta)|d\zeta = \delta\mathbf{E}\int\limits_{\mathbb{R}^d}\varphi(\tau,\zeta,x_0,\delta)\times$$
$$\times|u(\tau-h,\zeta)|d\zeta + \mathbf{E}\int\limits_{\mathbb{R}^d}\psi(\tau,x_0,\zeta)\times$$
$$\times|u(\tau-h,\zeta)|d\zeta \le$$
$$\le\Bigg(\delta\sqrt{\int\limits_{\mathbb{R}^d}\varphi^2(\tau,\zeta,x_0,\delta)d\zeta}+$$
$$+\sqrt{\int\limits_{\mathbb{R}^d}\psi^2(\tau,x_0,\zeta)d\zeta}\Bigg)\times$$
$$\times\sqrt{\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}} \le$$
$$\le\Bigg(\delta\sqrt{\int\limits_{\mathbb{R}^d}\varphi^2(\tau,\zeta,x_0,\delta)d\zeta}+$$
$$+\sqrt{\int\limits_{\mathbb{R}^d}\psi^2(\tau,x_0,\zeta)d\zeta}\Bigg)\times$$
$$\times\Big(\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+$$
$$+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}\Big)^{\frac{1}{2}} < \infty,$$

hence with probability one

$$\int\limits_{\mathbb{R}^d}\big(\delta\varphi(\tau,\zeta,x_0,\delta)+\psi(\tau,x_0,\zeta)\big)|u(\tau-h,\zeta)|d\zeta < \infty.$$

Thus, according to local theorem on differentiability of an integral by parameter, for function (17) there exists its gradient $\nabla_x g$ and

$$\nabla_x\int\limits_{\mathbb{R}^d} b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta =$$
$$=\int\limits_{\mathbb{R}^d}\nabla_x b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta. \qquad (21)$$

It remains to prove that

$$\nabla_x \int\limits_{\mathbb{R}^d} b(\tau, \cdot, \zeta) u(\tau-h,\zeta)d\zeta \in L_2(\mathbb{R}^d).$$

Since, according to (21), (11), Cauchy-Schwartz inequality, conditions (12) and (5),

$$\mathbf{E}\int\limits_{\mathbb{R}^d}\left|\nabla_x \int\limits_{\mathbb{R}^d} b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta\right|^2 dx =$$
$$= \mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\nabla_x b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta\right)^2 dx \le$$
$$\le \mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|\nabla_x b(\tau,x,\zeta)u(\tau-h,\zeta)|d\zeta\right)^2 dx \le$$
$$\le \mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\psi(\tau,x,\zeta)|u(\tau-h,\zeta)|\big)d\zeta\right)^2 dx \le$$
$$\le \left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\mathbf{E}\int\limits_{\mathbb{R}^d}u^2(\tau-h)d\zeta =$$
$$= \left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\mathbf{E}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)} \le$$
$$\le \sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times \mathbf{E}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)} \le$$
$$\le \sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times \sup_{0\le\tau\le t}\mathbf{E}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)} \le$$
$$\le \sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\left(\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+\right.$$
$$\left.+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}\right) < \infty,$$

one concludes that

$$\int\limits_{\mathbb{R}^d}\left|\nabla_x \int\limits_{\mathbb{R}^d} b(\tau,x,\zeta)u(\tau-h,\zeta)d\zeta\right|^2 dx < \infty.$$

Thus conditions of lemma 4 from [4] are valid, hence in (16)

$$\|I_2(s)\|^2_{\mathfrak{B}_{2,t}} \le Ct\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\left\|D_x^2\int\limits_{\mathbb{R}^d} b(\tau,x,\zeta)\times\right.$$
$$\left.\times u(\tau-h,\zeta)d\zeta\right\|^2 dxd\tau \le$$
$$\le Ct\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\|D_x^2 b(\tau,x,\zeta)\|\times\right.$$
$$\left.\times|u(\tau-h,\zeta)|d\zeta\right)^2 dxd\tau \le$$
$$\le Ct\int\limits_0^t\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\mathbf{E}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d\tau \le$$
$$\le Ct^2\sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\left(\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+\right.$$
$$\left.+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}\right) < \infty.$$

Cauchy-Schwartz inequality, Fubini theorem and conditions (6), (2), (8), (5) yield for $\|I_3(s)\|^2_{\mathfrak{B}_{2,t}}$

$$\|I_3(s)\|^2_{\mathfrak{B}_{2,t}} = \sup_{0\le s\le t}\mathbf{E}\|I_3(s)\|^2_{L_2(\mathbb{R}^d)} =$$
$$= \sup_{0\le s\le t}\mathbf{E}\left\|\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.$$
$$\left.\times f(\tau,u(\tau-h),\xi)d\xi d\tau\right\|^2_{L_2(\mathbb{R}^d)} =$$
$$= \sup_{0\le s\le t}\mathbf{E}\int\limits_{\mathbb{R}^d}\left(\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.$$
$$\left.\times f(\tau,u(\tau-h,\xi),\xi)d\xi d\tau\right)^2 dx \le$$
$$\le \sup_{0\le s\le t}s\mathbf{E}\int\limits_{\mathbb{R}^d}\int\limits_0^s\left(\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times\right.$$

$$\times f(\tau,u(\tau-h,\xi),\xi)d\xi\Big)^2 d\tau dx \le$$

$$\le t\sup_{0\le s\le t}\mathbf{E}\int_0^s\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times|f(\tau,u(\tau-h,\xi),\xi)|d\xi\Big)^2 dxd\tau \le$$

$$\le t\sup_{0\le s\le t}\mathbf{E}\int_0^s\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\big(\chi(\tau,\xi)+L|u(\tau-h,\xi)|\big)d\xi\Big)^2 dxd\tau \le$$

$$\le 2t\sup_{0\le s\le t}\int_0^s\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\chi(\tau,\xi)d\xi\Big)^2 dxd\tau+$$

$$+2L^2t\sup_{0\le s\le t}\mathbf{E}\int_0^s\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times|u(\tau-h,\xi)|d\xi\Big)^2 dxd\tau = 2t\times$$

$$\times\sup_{0\le s\le t}\mathbf{E}\int_0^s\Big\|\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\chi(\tau)d\xi\Big\|^2_{L_2(\mathbb{R}^d)}d\tau+$$

$$+2L^2t\sup_{0\le s\le t}\mathbf{E}\int_0^s\Big\|\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times|u(\tau-h)|d\xi\Big\|^2_{L_2(\mathbb{R}^d)}d\tau \le$$

$$\le 2t\sup_{0\le s\le t}\int_0^s\|\chi(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau+$$

$$+2L^2t\sup_{0\le s\le t}\mathbf{E}\int_0^s\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d\tau \le$$

$$\le 2t\int_0^t\|\chi(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau+$$

$$+2L^2t\mathbf{E}\int_0^t\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d\tau =$$

$$=2t\int_0^t\int_{\mathbb{R}^d}\chi^2(\tau,x)dxd\tau+$$

$$+2L^2t\mathbf{E}\int_{-h}^{t-h}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d(\tau-h)=$$

$$=2t\int_0^t\int_{\mathbb{R}^d}\chi^2(\tau,x)dxd\tau+$$

$$+2L^2t\mathbf{E}\int_{-h}^0\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau+$$

$$+2L^2t\mathbf{E}\int_0^{t-h}\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau \le$$

$$\le 2t^2\sup_{0\le\tau\le t}\int_{\mathbb{R}^d}\chi^2(\tau,x)dx+$$

$$+2hL^2t\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+2L^2t\mathbf{E}\int_0^t\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau<\infty.$$

Using Cauchy-Schwartz inequality, Fubini theorem and conditions (7), (2), (3), (4), (5), one obtains

$$\|I_4(s)\|^2_{\mathfrak{B}_{2,t}}=\sup_{0\le s\le t}\mathbf{E}\|I_4(s)\|^2_{L_2(\mathbb{R}^d)}=$$

$$=\sup_{0\le s\le t}\mathbf{E}\Big\|\int_0^t\sum_{n=1}^\infty\sqrt{\lambda_n}\Big(\int_{\mathbb{R}^d}\mathscr{K}(t-s,x-\xi)\times$$

$$\times\sigma(s,u(s-h),\xi)e_n(\xi)d\xi\Big)d\beta_n(s)\Big\|^2_{L_2(\mathbb{R}^d)}=$$

$$=\sup_{0\le s\le t}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_0^s\sum_{n=1}^\infty\sqrt{\lambda_n}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\sigma(\tau,u(\tau-h,\xi),\xi)e_n(\xi)d\xi\Big)d\beta_n(\tau)\Big)^2dx=$$

$$= \sup_{0\le s\le t} \mathbf{E}\int_{\mathbb{R}^d}\int_0^s \sum_{n=1}^{\infty}\lambda_n\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\,\sigma(\tau,u(\tau-h,\xi),\xi)e_n(\xi)d\xi\Big)^2 d\tau dx =$$

$$=\sum_{n=1}^{\infty}\lambda_n \sup_{0\le s\le t}\mathbf{E}\int_0^s\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\,|\sigma(\tau,u(\tau-h,\xi),\xi)|e_n(\xi)d\xi\Big)^2 dx d\tau \le L^2\times$$

$$\times\sum_{n=1}^{\infty}\lambda_n \sup_{0\le s\le t}\mathbf{E}\int_0^s\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\,\big(1+|u(\tau-h,\xi)|\big)e_n(\xi)d\xi\Big)^2 dx d\tau = L^2\times$$

$$\times\sum_{n=1}^{\infty}\lambda_n \sup_{0\le s\le t}\mathbf{E}\int_0^s\Big\|\int_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$

$$\times\,\big(1+|u(\tau-h)|\big)e_n d\xi\Big\|^2_{L_2(\mathbb{R}^d)} d\tau \le L^2\times$$

$$\times\sum_{n=1}^{\infty}\lambda_n \sup_{0\le s\le t}\mathbf{E}\int_0^s\big\|\big(1+|u(\tau-h)|\big)\times$$

$$\times\, e_n\big\|^2_{L_2(\mathbb{R}^d)}d\tau \le 2L^2\sum_{n=1}^{\infty}\lambda_n\mathbf{E}\int_0^t\big(\|e_n\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\,\|u(\tau-h)e_n\|^2_{L_2(\mathbb{R}^d)}\big)d\tau = 2L^2\sum_{n=1}^{\infty}\lambda_n\times$$

$$\times\,\mathbf{E}\int_0^t\Big(1+\int_{\mathbb{R}^d}u^2(\tau-h,x)e_n^2(x)dx\Big)d\tau \le$$

$$\le 2L^2\sum_{n=1}^{\infty}\lambda_n\Big(t+\mathbf{E}\int_0^t\int_{\mathbb{R}^d}u^2(\tau-h,x)dxd\tau\Big) =$$

$$= 2L^2\Big(\sum_{n=1}^{\infty}\lambda_n\Big)\Big(t+$$

$$+\,\mathbf{E}\int_0^t\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d\tau\Big) =$$

$$= 2L^2\Big(\sum_{n=1}^{\infty}\lambda_n\Big)\Big(t+$$

$$+\,\mathbf{E}\int_{-h}^{t-h}\|u(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d(\tau-h)\Big) =$$

$$= 2L^2\Big(\sum_{n=1}^{\infty}\lambda_n\Big)\Big(t+\mathbf{E}\int_{-h}^{0}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau+$$

$$+\,\mathbf{E}\int_0^{t-h}\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau\Big) \le 2L^2\Big(\sum_{n=1}^{\infty}\lambda_n\Big)\times$$

$$\times\Big(t+h\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+$$

$$+\,\mathbf{E}\int_0^t\|u(\tau)\|^2_{L_2(\mathbb{R}^d)}d\tau\Big) < \infty.$$

Thus the above five estimates together imply that for $u\in\mathfrak{B}_{2,T}$

$$\|\Psi u\|^2_{\mathfrak{B}_{2,T}} = \sup_{0\le t\le T}\mathbf{E}\Big\|\sum_{j=0}^{4}I_j(t)\Big\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le 5\sup_{0\le t\le T}\mathbf{E}\sum_{j=0}^{4}\|I_j(t)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= 5\sup_{0\le t\le T}\sum_{j=0}^{4}\mathbf{E}\|I_j(t)\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le 5\sum_{j=0}^{4}\sup_{0\le t\le T}\mathbf{E}\|I_j(t)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= 5\sum_{j=0}^{4}\|I_j(t)\|^2_{\mathfrak{B}_{2,T}} < \infty.$$

Since $\mathcal{F}_t$-measurability of $(\Psi u)(t)$ is easily verified, one concludes that Ψ is well defined.

Next, it is necessary to prove that operator Ψ has a unique fixed point. Indeed, taking into account the above five inequalities and the property of linearity of integral, one obtains

$$\|I_1(s)(u)-I_1(s)(v)\|^2_{\mathfrak{B}_{2,t}} =$$

$$= \sup_{0\le s\le t}\mathbf{E}\|I_1(s)(u)-I_1(s)(v)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= \sup_{0\le s\le t}\mathbf{E}\Big\|\int_{\mathbb{R}^d}b(s,x,\xi)\times$$

$$\times\,\big(u(s-h)-v(s-h)\big)d\xi\Big\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le \left(\sup_{0\le s\le t}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx\right)\times$$
$$\times \|u-v\|^2_{\mathfrak{B}_{2,t}}, \qquad (22)$$
$$\|I_2(s)(u)-I_2(s)(v)\|^2_{\mathfrak{B}_{2,t}} =$$
$$=\sup_{0\le s\le t}\mathbf{E}\|I_2(s)(u)-I_2(s)(v)\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\le s\le t}\mathbf{E}\Bigg\|\int\limits_0^s\Bigg(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times\left(\int\limits_{\mathbb{R}^d} b(\tau,\xi,\zeta)u(\tau-h)d\zeta\right)d\xi\Bigg)d\tau-$$
$$-\int\limits_0^s\Bigg(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times\left(\int\limits_{\mathbb{R}^d} b(\tau,\xi,\zeta)v(\tau-h)d\zeta\right)d\xi\Bigg)d\tau\Bigg\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\le s\le t}\mathbf{E}\Bigg\|\int\limits_0^s\Bigg(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times\Bigg(\int\limits_{\mathbb{R}^d} b(\tau,\xi,\zeta)u(\tau-h)d\zeta-$$
$$-\int\limits_{\mathbb{R}^d} b(\tau,\xi,\zeta)v(\tau-h)d\zeta\Bigg)d\xi\Bigg)d\tau\Bigg\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\le s\le t}\mathbf{E}\Bigg\|\int\limits_0^s\Bigg(\Delta_x\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times\Bigg(\int\limits_{\mathbb{R}^d} b(\tau,\xi,\zeta)\big(u(\tau-h,\zeta)-$$
$$-v(\tau-h,\zeta)\big)d\zeta\Bigg)d\xi\Bigg)d\tau\Bigg\|^2_{L_2(\mathbb{R}^d)}\le$$
$$\le Ct\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\Bigg\|D_x^2\int\limits_{\mathbb{R}^d} b(\tau,x,\zeta)\times$$
$$\times\big(u(\tau-h,\zeta)-v(\tau-h,\zeta)\big)d\zeta\Bigg\|^2dxd\tau\le$$
$$\le Ct\mathbf{E}\int\limits_0^t\int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\|D_x^2 b(\tau,x,\zeta)\|\times$$
$$\times|u(\tau-h,\zeta)-v(\tau-h,\zeta)|d\zeta\Bigg)^2dxd\tau\le$$
$$\le Ct\int\limits_0^t\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\mathbf{E}\|u(\tau-h)-v(\tau-h)\|^2_{L_2(\mathbb{R}^d)}d\tau\le$$
$$\le Ct^2\sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau-h)-v(\tau-h)\|^2_{L_2(\mathbb{R}^d)}\le$$
$$\le Ct^2\sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\Bigg(\sup_{-h\le\tau\le 0}\mathbf{E}\|\phi(\tau)-\phi(\tau)\|^2_{L_2(\mathbb{R}^d)}+$$
$$+\sup_{0\le\tau\le t-h}\mathbf{E}\|u(\tau)-v(\tau)\|^2_{L_2(\mathbb{R}^d)}\Bigg)\le$$
$$\le Ct^2\sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(\tau,x,\zeta)d\zeta dx\right)\times$$
$$\times\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)-v(\tau)\|^2_{L_2(\mathbb{R}^d)}, \qquad (23)$$
$$\|I_3(s)(u)-I_3(s)(v)\|^2_{\mathfrak{B}_{2,t}}=$$
$$=\sup_{0\le s\le t}\mathbf{E}\|I_3(s)(u)-I_3(s)(v)\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\le s\le t}\mathbf{E}\Bigg\|\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times f(\tau,u(\tau-h),\xi)d\xi d\tau-$$
$$-\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times f(\tau,v(\tau-h),\xi)d\xi d\tau\Bigg\|^2_{L_2(\mathbb{R}^d)}=$$
$$=\sup_{0\le s\le t}\mathbf{E}\Bigg\|\int\limits_0^s\int\limits_{\mathbb{R}^d}\mathscr{K}(s-\tau,x-\xi)\times$$
$$\times\big(f(\tau,u(\tau-h),\xi)-$$

$$- f(\tau, v(\tau-h),\xi)\big)d\xi d\tau\Bigg\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le L^2ct^2\|u-v\|^2_{\mathfrak{B}_{2,t}}, \qquad (24)$$

$$\|I_4(s)(u) - I_4(s)(v)\|^2_{\mathfrak{B}_{2,t}} =$$

$$= \sup_{0\le s\le t} \mathbf{E}\|I_4(s)(u) - I_4(s)(v)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= \sup_{0\le s\le t} \mathbf{E}\Bigg\|\int_0^s \sum_{n=1}^\infty \sqrt{\lambda_n}\Bigg(\int_{\mathbb{R}^d} \mathscr{K}(s-\tau, x-\xi)\times$$

$$\times \sigma(\tau, u(\tau-h),\xi)e_n(\xi)d\xi\Bigg)d\beta_n(\tau) -$$

$$- \int_0^s \sum_{n=1}^\infty \sqrt{\lambda_n}\Bigg(\int_{\mathbb{R}^d} \mathscr{K}(s-\tau, x-\xi)\times$$

$$\times \sigma(\tau, v(\tau-h),\xi)e_n(\xi)d\xi\Bigg)d\beta_n(\tau)\Bigg\|^2_{L_2(\mathbb{R}^d)} =$$

$$= \sup_{0\le s\le t} \mathbf{E}\Bigg\|\int_0^s \sum_{n=1}^\infty \sqrt{\lambda_n}\times$$

$$\times\Bigg(\int_{\mathbb{R}^d} \mathscr{K}(s-\tau, x-\xi)\big(\sigma(\tau, u(\tau-h),\xi)-$$

$$- \sigma(\tau, v(\tau-h),\xi)\big)e_n(\xi)d\xi\Bigg)d\beta_n(\tau)\Bigg\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le L^2c\Bigg(\sum_{n=1}^\infty \lambda_n\Bigg)t\|u-v\|^2_{\mathfrak{B}_{2,t}}. \qquad (25)$$

Estimates (22) — (25) yield

$$\|\Psi u - \Psi v\|^2_{\mathfrak{B}_{2,t}} = \sup_{0\le s\le t} \mathbf{E}\Bigg\|\sum_{j=1}^4 \big(I_j(s)(u)-$$

$$- I_j(s)(v)\big)\Bigg\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le 4\sup_{0\le s\le t} \mathbf{E}\sum_{j=1}^4 \|I_j(s)(u) - I_j(s)(v)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= 4\sup_{0\le s\le t} \sum_{j=1}^4 \mathbf{E}\|I_j(s)(u) - I_j(s)(v)\|^2_{L_2(\mathbb{R}^d)} \le$$

$$\le 4\sum_{j=1}^4 \sup_{0\le s\le t} \mathbf{E}\|I_j(s)(u) - I_j(s)(v)\|^2_{L_2(\mathbb{R}^d)} =$$

$$= 4\sum_{j=1}^4 \|I_j(s)(u) - I_j(s)(v)\|^2_{\mathfrak{B}_{2,t}} \le$$

$$\le 4\Bigg(\sup_{0\le s\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b^2(s,x,\xi)d\xi dx +$$

$$+ Ct^2\sup_{0\le \tau\le t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi^2(\tau,x,\zeta)d\zeta dx +$$

$$+ L^2ct^2 + L^2c\Bigg(\sum_{n=1}^\infty \lambda_n\Bigg)t\Bigg)\|u-v\|^2_{\mathfrak{B}_{2,t}} =$$

$$= \gamma(t)\|u-v\|^2_{\mathfrak{B}_{2,t}}, \{u,v\}\subset \mathfrak{B}_{2,t}.$$

Due to (15), the first term of γ is less, than one. Therefore, by choosing small $0 \le t_1 \le \le T$, one concludes that $0 \le \gamma(t_1) \le 1$. It means that operator Ψ, defined in Banach space $\mathfrak{B}_{2,t_1}$, is contractive, and, according to Banach theorem on a contractive mapping, has a unique fixed point — mild solution $u \in \mathfrak{B}_{2,t_1}$ of (1) on the interval $[0, t_1]$. This procedure can be repeated finitely many steps on other sufficiently small intervals $[t_1, t_2]$, $[t_2, t_3]$, $\ldots$, $[t_{n-2}, t_{n-1}]$, $[t_{n-1}, T]$, — components of the entire interval $[0, T]$, — and, as a result, the solution is obtained as union of solutions on these small intervals. Thus, the theorem is proved.

REFERENCES

1. *Mahmudov N. I.* Existence, Uniqueness and Controllability Results for Neutral FSDEs in Hilbert Spaces // Dynamic Systems and Applications. – 2008. – **17**. – P. 53 – 70.

2. *Tessitore G., Zabczyk J.* Invariant Measures for Stochastic Heat Equations // Probability and Mathematical Statistics. – 1998. – **18**. – P. 271 – 287.

3. *Zabczyk J., Da Prato G.* Ergodicity for Infinite Dimensional Systems // Dynamic Systems and Applications. – Cambridge University Press. – 1996. – 449 p.

4. *Станжицкий А. Н., Цуканова А. О.* Существование и единственность решения задачи Коши для стохастического дифференциального уравнения реакции-диффузии нейтрального типа // Нелінійні коливання. – 2016. – **3**, № 3. – С. 408 – 430.