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INVARIANT METHODS FOR STUDYING STABILITY OF UNPERTURBED MOTION IN TERNARY DIFFERENTIAL SYSTEMS WITH POLYNOMIAL NONLINEARITIES

The centro-affine invariant conditions for Lyapunov stability of unperturbed motion in ternary differential systems with polynomial nonlinearities were determined and the centro-affine invariant conditions when a ternary differential system of the Lyapunov-Darboux form with quadratic nonlinearities have a holomorphic integral were obtained. On the base of the integral the stability of unperturbed period motion was studied.

1. Centro-affine invariant polynomials in ternary differential systems

Let us consider the ternary differential system with polynomial nonlinearities of perturbed motion (see, for example, [1] or [2])

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + \sum_{i=1}^{l} a^j_{\alpha_1\cdots\alpha_{m_i}} x^{\alpha_1}\cdots x^{\alpha_{m_i}} \qquad (1)$$
$$(j,\alpha,\alpha_1,\alpha_2,\ldots,\alpha_{m_i} = \overline{1,3}; l<\infty),$$

where $a^j_{\alpha_1\alpha_2\ldots\alpha_{m_i}}$ is a symmetric tensor in lower indices in which the total convolution is done and $\Gamma = \{m_1, m_2, ..., m_l\}$ $(m_i \geq 2)$ is a finite set of distinct natural numbers.

We will examine the centro-affine group $GL(3,\mathbb{R})$ for system (1) given by transformations q:

$$\bar{x}^j = q^j_\alpha x^\alpha \ (\Delta = det(q^j_\alpha) \neq 0) \ (j,\alpha = \overline{1,3}). \qquad (2)$$

Coefficients and variables in (1) and (2) takes values from the field of real numbers $\mathbb{R}$.

Observe that the transformation (2) preserves the form of the system (1)

$$\frac{d\bar{x}^j}{dt} = \bar{a}^j_\alpha \bar{x}^\alpha + \sum_{i=1}^{l} \bar{a}^j_{\alpha_1\cdots\alpha_{m_i}} \bar{x}^{\alpha_1}\cdots \bar{x}^{\alpha_{m_i}} \qquad (3)$$
$$(j,\alpha,\alpha_1,\alpha_2,\ldots,\alpha_{m_i} = \overline{1,3}; l<\infty),$$

where the coordinates of the vector $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ are determined by relations (2) and the coefficients from the right-hand sides of (3) are some linear functions in the coefficients of system (1) and rational in the parameters q^j_α of the transformation (2).

The phase variable vector $x = (x^1, x^2, x^3)$ of system (1), which change by formulas (2), in the theory of invariants [3] is called **contravariant**. The vector $u = (u_1, u_2, u_3)$, which change by formulas $u_r = p^j_r u_j$ $(r, j = \overline{1,3})$, where $p^r_j q^j_s = \delta^r_s$ is the Kroniker symbol, is called **covariant**. Any vector $y = (y^1, y^2, y^3)$, which change by formulas (2) is called **cogradient** of the vector x.

We will denote the set of coefficients of system (1) by a and of the system (3) by $\bar{a}$.

Definition 1. *We say (see [3]) that a polynomial $\varkappa(x, u, a)$ of the coefficients of system* (1) *and of the coordinates of vectors x and u is called* ***mixt comitant*** *of the system* (1) *with respect to the group $GL(3,\mathbb{R})$, if the following holds*

$$\varkappa(\bar{x}, \bar{u}, \bar{a}) = \Delta^{-g}\varkappa(x, u, a)$$

for all q from $GL(3,\mathbb{R})$, for every coordinates of vectors x and u, as well, any coefficients of system (1).

Here g is an integer number called **the weight of comitant**.

If the mixt comitant $\varkappa$ does not depend on coordinates of vector u, then following [4], we call it **comitant**. If $\varkappa$ does not depend on coordinates of vector x it will be called, as in [3], **contravariant**. If $\varkappa$ does not depend of x and u, then we call it **invariant** of the system (1) with respect to the group $GL(3,\mathbb{R})$.

It was shown in [5] that the expressions

$$\varkappa_1 = x^\alpha u_\alpha,\ \varkappa_2 = a^\alpha_\beta x^\beta u_\alpha,$$
$$\varkappa_3 = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta,\ \theta_1 = a^\alpha_\alpha,\ \theta_2 = a^\alpha_\beta a^\beta_\alpha, \qquad (4)$$
$$\theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta,\ \delta_4 = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u_\beta u_r \varepsilon^{pqr}$$

in the coordinates of the vectors x, u and of the tensor a^j_α, compose a functional base of the mixt comitants of the linear part of differential system (1), where ε^{pqr} is the unit trivector with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ and $\varepsilon^{pqr} = 0 \ \ (p, q, r = \overline{1,3})$ for all other cases.

An important role in studying ternary systems with polynomial nonlinearities (1) has the comitant

$$\sigma_1 = a^\alpha_\mu a^\beta_\delta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu} \qquad (5)$$

($\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1$ and $\varepsilon_{\beta\gamma\nu} = 0 \ \ (\beta, \gamma, \nu = \overline{1,3})$) from [4], which is a particular integral of the system

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha \quad (j, \alpha = \overline{1,3}) \qquad (6)$$

of the first approximation ([1], [2]) for (1).

In [6] it was proved the following assertion.

Lemma 1. *The following equivalences hold*:

$$\sigma_1(x) \equiv 0 \Longleftrightarrow \delta_4(u) \equiv 0 \qquad (7)$$

and conversely

$$\sigma_1(x) \not\equiv 0 \Longleftrightarrow \delta_4(u) \not\equiv 0, \qquad (8)$$

where δ_4 is from (4) and σ_1 is from (5).

2. Centro-affine invariant conditions for stability of unperturbed motion

As it follows from [2], the zero values of the variables $x^j(t)$ $(j = \overline{1,3})$ correspond to the to unperturbed motion of perturbed system (1). As consequence, we have the following **definition of stability by Lyapunov [2]**:

If for any small positive value ε, however small, one can find a positive number δ such that at $t = t_0$, for all perturbation $x^j(t_0)$ satisfying

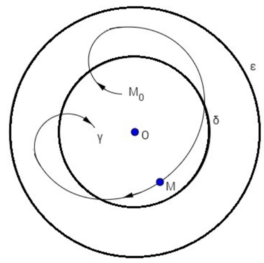
$$\sum_{j=1}^{n} (x^j(t_0))^2 \le \delta \qquad (9)$$

the inequality

$$\sum_{j=1}^{n} (x^j(t))^2 < \varepsilon$$

is valid, then the unperturbed motion $x^j = 0$ $(j = \overline{1,3})$ is called stable, otherwise it is called unstable.

Geometrically this definition has the following interpretation:



If the motion is stable, then for sphere ε one can find another sphere δ such that starting at any point M_0 inside or on the surface of the sphere δ, the image point M will always remain inside the sphere ε, never reaching its external surface.

If the perturbed motion is unstable, then irrespective of how close to the reference origin the point M_0 may be, in time, at least one trajectory of the representative point M will cross the sphere ε from inside to outside.

If the unperturbed motion is stable and the value δ can be found however small such that for any perturbed motions satisfying (9) the condition

$$\lim_{t\to\infty} \sum_{j=1}^{n} (x^j(t))^2 = 0$$

is valid, then the unperturbed motion is called *asymptotically stable.*

Lemma 2. The characteristic equation of system (1) and (6) is

$$\varrho^3 + L_{1,3}\varrho^2 + L_{2,3}\varrho + L_{3,3} = 0, \qquad (10)$$

where

$$L_{1,3} = -\theta_1, \quad L_{2,3} = (\theta_1^2 - \theta_2)/2,$$
$$L_{3,3} = -(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3)/6, \qquad (11)$$

and θ_i $(i = \overline{1,3})$ are from (4).

According to Lyapunov theorems on stability of unperturbed motion by sign of the eigenvalues of the differential system in the first approximation and Hurwitz theorem to the root analysis of the characteristic equation (see [2]), there were proved the following theorems:

Theorem 1. *If the centro-affine invariants of* (1) *satisfy the following conditions*

$$L_{1,3} > 0,\ L_{2,3} > 0,\ L_{3,3} > 0,\\ L_{1,3}L_{2,3} - L_{3,3} > 0,$$

then the unperturbed motion $x^1 = x^2 = x^3 = 0$ *of the system is asymptotically stable, where* $L_{i,3}$ $(i = \overline{1,3})$ *are the coefficients of the characteristic equation* (10) *of system* (1).

Theorem 2. *If at least one of the centro-affine invariant expression* (11) *of system* (1) *is negative, then the unperturbed motion* $x^1 = x^2 = x^3 = 0$ *of this system is unstable.*

3. Lyapunov form of the ternary differential system

Let be given the system

$$\frac{dx^j}{dt} = \alpha^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta \equiv P^j(x), \qquad (12)$$

$(j = \overline{1,3})$, which can be obtained from (1) for $\Gamma = \{1,2\}$.

Lemma 3 [5]. *Suppose in* (5) *that* $\sigma_1 \equiv 0$. *Then by a centro-affine transformation*

$$\bar{x}^1 = x^2, \quad \bar{x}^2 = x^1 + \frac{a^3_2}{a^3_1}x^2, \quad \bar{x}^3 = x^3,$$

when $a^3_1 \neq 0$, *we obtain that the quadratic part of system* (12) *preserves the form, and the coefficients from the linear part of this system satisfy one of the following conditions:*

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0,\\ a^3_3 = a^2_2; \qquad (13)$$

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0,\\ a^3_3 = a^1_1; \qquad (14)$$

$$a^1_2 = a^1_3 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0,\\ a^2_2 = a^1_1; \qquad (15)$$

$$a^1_2 = a^1_3 = a^2_1 = a^3_1 = a^3_2 = 0,\\ a^2_3 \neq 0,\ a^3_3 = a^1_1; \qquad (16)$$

$$a^1_2 = a^1_3 = a^2_1 = a^3_1 = a^3_2 = 0,\\ a^2_2 = a^1_1,\ a^2_3 \neq 0; \qquad (17)$$

$$a^1_2 = a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0,\\ a^1_3 \neq 0,\ a^3_3 = a^2_2; \qquad (18)$$

$$a^1_2 = a^2_1 = a^3_1 = a^3_2 = 0,\\ a^1_3 \neq 0,\ a^2_2 = a^1_1; \qquad (19)$$

$$a^2_1 = a^2_3 = a^3_1 = a^3_2 = 0,\\ a^1_2 \neq 0,\ a^3_3 = a^2_2; \qquad (20)$$

$$a^2_1 = a^3_1 = a^3_2 = 0,\ a^1_2 \neq 0,\\ a^2_3 = \frac{a^1_3(a^2_2 - a^1_1)}{a^1_2},\ a^3_3 = a^1_1; \qquad (21)$$

$$a^3_1 = a^3_2 = 0,\ a^1_3 = \frac{a^2_3(a^1_1 - a^3_3)}{a^2_1},\\ a^2_1 \neq 0,\ a^1_2 = \frac{(a^1_1 - a^3_3)(a^2_2 - a^3_3)}{a^2_1}; \qquad (22)$$

$$a^2_1 = a^3_1 = 0,\ a^3_2 \neq 0,\ a^1_3 = \frac{a^1_2(a^3_3 - a^1_1)}{a^3_2},\\ a^2_3 = \frac{(a^1_1 - a^2_2)(a^1_1 - a^3_3)}{a^3_2}. \qquad (23)$$

Lemma 4. *Suppose that for the linear part of system* (12) $\sigma_1 \equiv 0$, *where* σ_1 *is from* (5). *Then the characteristic equation* (10) *of this system has real eigenvalues.*

Proof. The roots of the characteristic equation (10) of system (12), under conditions (13)-(23) are given in Table 1:

Table 1.

System (12) under conditions	Eigenvalues of (10)
(13)	$\varrho_1 = a^1_1,\ \ \varrho_{2,3} = a^2_2$
(14)	$\varrho_{1,2} = a^1_1,\ \ \varrho_3 = a^2_2$
(15)	$\varrho_{1,2} = a^1_1,\ \ \varrho_3 = a^3_3$
(16)	$\varrho_{1,2} = a^1_1,\ \ \varrho_3 = a^2_2$
(17)	$\varrho_{1,2} = a^1_1,\ \ \varrho_3 = a^3_3$
(18)	$\varrho_1 = a^1_1,\ \ \varrho_{2,3} = a^2_2$
(19)	$\varrho_{1,2} = a^1_1,\ \ \varrho_3 = a^3_3$
(20)	$\varrho_1 = a^1_1,\ \ \varrho_{2,3} = a^2_2$
(21)	$\varrho_{1,2} = a^1_1,\ \ \varrho_3 = a^2_2$
(22)	$\varrho_{1,2} = a^3_3,$ $\varrho_3 = a^1_1 + a^2_2 - a^3_3$
(23)	$\varrho_{1,2} = a^1_1,$ $\varrho_3 = -a^1_1 + a^2_2 + a^3_3$

From Table 1, it follows that all eigenvalues ϱ_i ($i=\overline{1,3}$) are real. Lemma 4 is proved.

Lemma 5. *Suppose that in* (5) $\sigma_1 \not\equiv 0$. *Then system* (12)*, by means of a centro-affine transformation, can be brought to the form*

$$\begin{aligned} \dot{x}^1 &= x^2 + a^1_{\alpha\beta}x^\alpha x^\beta, \\ \dot{x}^2 &= x^3 + a^2_{\alpha\beta}x^\alpha x^\beta, \\ \dot{x}^3 &= -L_{3,3}x^1 - L_{2,3}x^2 - \\ &\quad -L_{1,3}x^1 + a^3_{\alpha\beta}x^\alpha x^\beta, \end{aligned} \tag{24}$$

where $L_{i,3}$ ($i=\overline{1,3}$) *are from* (11).

Proof. Consider the substitution

$$\overline{x}^1 = \varkappa_1, \quad \overline{x}^2 = \varkappa_2, \quad \overline{x}^3 = \varkappa_3, \tag{25}$$

with $\varkappa_i$ ($i=\overline{1,3}$) given in (4).

From (25) we have

$$Det(\varkappa_1,\varkappa_2,\varkappa_3) \equiv \delta_4 = $$
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ a_1^{\alpha_1}u_{\alpha_1} & a_2^{\alpha_1}u_{\alpha_1} & a_3^{\alpha_1}u_{\alpha_1} \\ a_1^\alpha a_\alpha^\beta u_\beta & a_2^\alpha a_\alpha^\beta u_\beta & a_3^\alpha a_\alpha^\beta u_\beta \end{vmatrix},$$

where δ_4 is from (4) and

$$\begin{aligned} x^1 = {} & [(a_2^{\alpha_1}a_3^\alpha a_\alpha^\beta u_{\alpha_1}u_\beta - a_3^{\alpha_1}a_2^\alpha a_\alpha^\beta u_{\alpha_1}u_\beta)\bar{x}^1 + \\ & +(a_2^\alpha a_\alpha^\beta u_\beta u_3 - a_3^\alpha a_\alpha^\beta u_\beta u_2)\bar{x}^2 + \\ & +(a_3^{\alpha_1}u_{\alpha_1}u_2 - a_2^{\alpha_1}u_{\alpha_1}u_3)\bar{x}^3]/\delta_4, \\ x^2 = {} & [(a_3^{\alpha_1}a_1^\alpha a_\alpha^\beta u_{\alpha_1}u_\beta - a_1^{\alpha_1}a_3^\alpha a_\alpha^\beta u_{\alpha_1}u_\beta)\bar{x}^1 + \\ & +(a_3^\alpha a_\alpha^\beta u_\beta u_1 - a_1^\alpha a_\alpha^\beta u_\beta u_3)\bar{x}^2 + \\ & +(a_1^{\alpha_1}u_{\alpha_1}u_3 - a_3^{\alpha_1}u_{\alpha_1}u_1)\bar{x}^3]/\delta_4, \\ x^3 = {} & [(a_1^{\alpha_1}a_2^\alpha a_\alpha^\beta u_{\alpha_1}u_\beta - a_2^{\alpha_1}a_1^\alpha a_\alpha^\beta u_{\alpha_1}u_\beta)\bar{x}^1 + \\ & +(a_1^\alpha a_\alpha^\beta u_\beta u_2 - a_2^\alpha a_\alpha^\beta u_\beta u_1)\bar{x}^2 + \\ & +(a_2^{\alpha_1}u_{\alpha_1}u_1 - a_1^{\alpha_1}u_{\alpha_1}u_2)\bar{x}^3]/\delta_4. \end{aligned} \tag{26}$$

Substituting (25) and (26) in (12) and taking into account Lemma 1 ($\delta_4 \not\equiv 0 \Leftrightarrow \sigma_1 \not\equiv 0$), we obtain the system (24). The initial notation of variables and coefficients in the quadratic parts of (24) are preserved. Lemma 5 is proved.

Lemma 6. *The characteristic equation* (10) *of system* (24) *with* $\sigma_1 \not\equiv 0$ *has purely imaginary eigenvalues if and only if the system has the form*

$$\begin{aligned} \dot{x}^1 &= x^2 + a^1_{\alpha\beta}x^\alpha x^\beta, \\ \dot{x}^2 &= x^3 + a^2_{\alpha\beta}x^\alpha x^\beta, \\ \dot{x}^3 &= -L_{1,3}L_{2,3}x^1 - L_{2,3}x^2 - \\ &\quad - L_{1,3}x^3 + a^3_{\alpha\beta}x^\alpha x^\beta \quad (L_{2,3} > 0), \end{aligned} \tag{27}$$

where $L_{i,3}$ ($i=\overline{1,3}$) *are of the form* (11).

Proof. By Lemma 3, it is necessary to examine only the case when $\sigma_1 \not\equiv 0$.

Assume the characteristic equation (10) has purely imaginary eigenvalues $\varrho_1 = \alpha i, \varrho_2 = -\alpha i$ ($\alpha \neq 0$ is real), then the third root, evidently is real $\varrho_3 = \beta$. By means of the *Viète* theorem for eigenvalues of (10), we can write

$$\begin{aligned} &\varrho_1 + \varrho_2 + \varrho_3 = -L_{1,3}, \\ &\varrho_1\varrho_2 + \varrho_1\varrho_3 + \varrho_2\varrho_3 = L_{2,3}, \\ &\varrho_1\varrho_2\varrho_3 = -L_{3,3}. \end{aligned}$$

Taking into account the last equalities, we get

$$\beta = -L_{1,3}, \quad \alpha^2 = L_{2,3}, \quad L_{3,3} = L_{1,3}L_{2,3}.$$

Since α is real and nonzero $\alpha \neq 0$, we have $L_{2,3} > 0$. Substituting the last conditions in (24), we obtain (27).

The sufficiency of these conditions is confirmed as follows. Assume system (24) is of the form (27), then the characteristic equation (10) can be written as

$$(\varrho^2 + L_{2,3})(\varrho + L_{1,3}) = 0.$$

Because $L_{2,3} > 0$, the equation has two purely imaginary eigenvalues and one real eigenvalue. Lemma 6 is proved.

Lemma 7. *By a centro-affine transformation the system* (27) *with* $\sigma_1 \not\equiv 0$ *can be brought to the Lyapunov form [1, §33]*

$$\begin{aligned} \dot{x}^1 &= -\lambda x^2 + a^1_{\alpha\beta}x^\alpha x^\beta, \\ \dot{x}^2 &= \lambda x^1 + a^2_{\alpha\beta}x^\alpha x^\beta, \\ \dot{x}^3 &= x^2 - L_{1,3}x^3 + a^3_{\alpha\beta}x^\alpha x^\beta, \end{aligned} \tag{28}$$

where $L_{1,3}$ *is from* (11)*, and*

$$\lambda^2 = L_{2,3} \ (L_{2,3} > 0).$$

Proof. We will examine the linear part of the ternary differential system (28) in the Lyapunov form. According to [1, §33], the linear part of this system must have the form

$$\begin{aligned} \dot{X}^1 &= -\lambda X^2 + ..., \\ \dot{X}^2 &= \lambda X^1 + ..., \\ \dot{X}^3 &= aX^1 + bX^2 + cX^3 + ..., \end{aligned} \tag{29}$$

where by dots we mean the quadratic part of the system. The coefficients λ, a, b, c are expressions in $L_{i,3}$ $(i=\overline{1,3})$ and the new variables X^1, X^2, X^3 have the form

$$\begin{aligned} X^1 &= \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3, \\ X^2 &= \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3, \\ X^3 &= \gamma_1 x^1 + \gamma_2 x^2 + \gamma_3 x^3, \end{aligned} \tag{30}$$

where

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0. \tag{31}$$

In these conditions, we observe that substitution (30) form a centro-affine transformation. Substituting (30) and (31) in the Lyapunov form (29) and comparing with the system (27), we obtain a system of nine algebraic equation in 12 unknowns.

Solving this system, we have

$$\begin{aligned} X^1 &= -L_{1,3}^2 \lambda x^1 + \lambda x^3, \\ X^2 &= L_{1,3} L_{2,3} x^1 + (L_{1,3}^2 + L_{2,3}) x^2 + L_{1,3} x^3, \\ X^3 &= 2L_{2,3} x^1 + L_{1,3} x^2 + x^3, \end{aligned}$$

where $\lambda^2 = L_{2,3}$, and

$$\Delta = -2L_{2,3}\lambda(L_{1,3}^2 + L_{2,3}) \neq 0 \quad (L_{2,3} > 0).$$

This transformation brings the system (27) to a system with the linear part in the Lyapunov form

$$\begin{aligned} \dot{X}^1 &= -\lambda X^2 + ..., \\ \dot{X}^2 &= \lambda X^1 + ..., \\ \dot{X}^3 &= X^2 - L_{1,3} X^3 + ..., \end{aligned}$$

for which, preserving in the quadratic part the initial notations of variables and coefficients, we obtain (28). Lemma 7 is proved

4. Invariant conditions for stability of unperturbed periodic motions

We will use the following $GL(3,\mathbb{R})$-invariant polynomials for system (12) from [5–6]:

$$\begin{aligned} \eta = a^{\alpha}_{\beta\gamma} x^{\beta} x^{\gamma} x^{\delta} y^{\mu} \varepsilon_{\alpha\delta\mu}, \; P_1 = a^{\alpha}_{\alpha\beta} x^{\beta}, \\ P_2 = a^{\alpha}_{\beta} a^{\beta}_{\alpha\gamma} x^{\gamma}, \;\; P_3 = a^{\alpha}_{\gamma} a^{\beta}_{\alpha} a^{\gamma}_{\beta\delta} x^{\delta}, \end{aligned} \tag{32}$$

where η is a comitant of two cogradient vectors $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$ which are linear independent.

It is easy to verify

Lemma 8 [6]. *Suppose that in system* (12) $\eta \equiv 0$. *Then the quadratic parts of this system have a common linear factor.*

The differential systems (12), with property stated in Lema 8, will be called the differential systems *of the Darboux form* [6]. If the linear part of system (12) has the Lyapunov form and the quadratic one - the Darboux form, then such systems will be called *Lyapunov-Darboux* differential systems.

In [6] it was proved the following assertion

Theorem 3. *Let* $\eta \equiv 0$. *Then system* (12) *has a* $GL(3,\mathbb{R})$*-invariant integrating factor* $\mu^{-1} = \sigma_1 \varphi$ *with* σ_1 *from* (5)*, where*

$$\varphi = -2L_{3,3} + 3L_{2,3}P_1 + 4L_{1,3}P_2 + 4P_3 \tag{33}$$

are $GL(3,\mathbb{R})$*-invariant particular integrals of this system with* $L_{i,3}$ $(i=\overline{1,3})$ *from* (11) *and* P_i $(i=\overline{1,3})$ *from* (32).

Lemma 9. *By a centro-affine transformation, the system* (12) *can be brought to the Lyapunov-Darboux form*

$$\begin{aligned} \frac{dx^1}{dt} &= -\lambda x^2 + 2x^1(gx^1 + \\ &\quad + hx^2 + kx^3) \equiv P^1, \\ \frac{dx^2}{dt} &= \lambda x^1 + 2x^2(gx^1 + \\ &\quad + hx^2 + kx^3) \equiv P^2, \\ \frac{dx^3}{dt} &= x^2 - L_{1,3}x^3 + 2x^3(gx^1 + \\ &\quad + hx^2 + kx^3) \equiv P^3, \end{aligned} \tag{34}$$

if and only if the following centro-affine invariant conditions hold

$$\begin{aligned} \sigma_1 \not\equiv 0, \quad \eta \equiv 0, \quad L_{1,3}L_{2,3} = L_{3,3} \\ (\lambda^2 = L_{2,3}, \; L_{2,3} > 0, \; L_{1,3} > 0), \end{aligned} \tag{35}$$

where σ_1 *is from* (5)*,* $L_{i,3}$ $(i=\overline{1,3})$ *is from* (11) *and* η *is from* (32).

The proof of Lemma 9 follows directly from Lemmas 6–8 and [1].

We determinate the Lie algebra of the operators admissible by system (34) [5]. We assume that the coordinate of the operator

$$X = \xi^i \frac{\partial}{\partial x^i} \quad (i=\overline{1,3}),$$

have the form

$$\xi^i = A^i_\beta x^\beta + A^i_{\beta\gamma} x^\beta x^\gamma \ \ (\beta,\gamma=\overline{1,3}),$$

and satisfy the determinant equations

$$\begin{aligned}(\xi^1)_{x^1}P^1 + (\xi^1)_{x^2}P^2 + (\xi^1)_{x^3}P^3 = \\ = \xi^1 P^1_{x^1} + \xi^2 P^1_{x^2} + \xi^3 P^1_{x^3}, \\ (\xi^2)_{x^1}P^1 + (\xi^2)_{x^2}P^2 + (\xi^2)_{x^3}P^3 = \\ = \xi^1 P^2_{x^1} + \xi^2 P^2_{x^2} + \xi^3 P^2_{x^3}, \\ (\xi^3)_{x^1}P^1 + (\xi^3)_{x^2}P^2 + (\xi^3)_{x^3}P^3 = \\ = \xi^1 P^3_{x^1} + \xi^2 P^3_{x^2} + \xi^3 P^3_{x^3}.\end{aligned}$$

Then solving this system for (34), we obtain the following operators

$$\begin{aligned}X_1 = \{&\lambda L_{1,3}x^1 - \lambda^2 x^2 + \\ &+2[(-k - hL_{1,3} + g\lambda)(x^1)^2 + \\ &+(gL_{1,3} + h\lambda)x^1x^2]\}\frac{\partial}{\partial x^1} + \\ &+\{\lambda^2 x^1 + \lambda L_{1,3}x^2 + 2(-k - hL_{1,3} + \\ &+g\lambda)x^1x^2 + 2(gL_{1,3} + h\lambda)(x^2)^2\}\frac{\partial}{\partial x^2} + \\ &+\{\lambda x^2 + 2(-k - hL_{1,3} + g\lambda)x^1x^3 + \\ &+2(gL_{1,3} + h\lambda)x^2x^3\}\frac{\partial}{\partial x^3}, \\ X_2 = [&\lambda^2 x^1 + \lambda L_{1,3}x^2 + \\ &+2(-gL_{1,3} - h\lambda)(x^1)^2 + \\ &+2(g\lambda - k - hL_{1,3})x^1x^2]\frac{\partial}{\partial x^1} + \\ &+[\lambda L_{1,3}x^1 + \lambda^2 x^2 - 2(gL_{1,3} + h\lambda)x^1x^2 + \\ &+2(g\lambda - k - hL_{1,3})(x^2)^2]\frac{\partial}{\partial x^2} + \\ &+[-\lambda x^1 + 2(-gL_{1,3} - h\lambda)x^1x^3 - \\ &-2(k + hL_{1,3} - g\lambda)x^2x^3]\frac{\partial}{\partial x^3}, \\ X_3 = \{&-\lambda L_{1,3} + 2[(k + hL_{1,3})x^1 - \\ &-gL_{1,3}x^2 + k\lambda x^3]\}(x^1\frac{\partial}{\partial x^1} + \\ &+x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}).\end{aligned} \tag{36}$$

By means of the commutators $[X_i, X_j] = X_iX_j - X_jX_i$ we can verify that these operators form a Lie three-dimensional commutative algebra.

According to [7] and using the operators $X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i}$ $(\alpha = 1,2;\ i=\overline{1,3})$ (ignoring a constant factor) we obtain the Lie integrating factor of the form

$$\mu^{-1} = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 \\ P^1 & P^2 & P^3 \end{vmatrix}$$

or

$$\begin{aligned}\mu^{-1} = [\lambda L_{1,3} - 2((k + hL_{1,3})x^1 - \\ -gL_{1,3}x^2 + k\lambda x^3)]\sigma_1.\end{aligned} \tag{37}$$

Using this expression and the Theorem on integrating factor, we have

Theorem 4. *One of the first integrals of system* (34) *has the form*

$$F_1 \equiv \frac{f_1}{f_2^2} = C_1, \tag{38}$$

where

$$\begin{aligned}f_1 &= (x^1)^2 + (x^2)^2, \\ f_2 &= -\lambda L_{1,3} + 2(k + hL_{1,3})x^1 - \\ &-2gL_{1,3}x^2 + 2\lambda k x^3.\end{aligned} \tag{39}$$

Corollary 1. *For system* (34) *we have*

$$\varphi = 12\lambda f_2, \tag{40}$$

where φ is of the form (33).

Corollary 2. *The first integral* (38) *with $f_2 \not\equiv 0$ ($\varphi \not\equiv 0$) can be written as a holomorphic integral of the form*

$$\widetilde{F_1} = (x^1)^2 + (x^2)^2 + F(x^1, x^2, x^3), \tag{41}$$

where $F(x^1,x^2,x^3)$ contains terms of degree at least two in variables x^1, x^2, x^3.

By Lemma 9, Theorem 4, Corollaries 1 and 2, there were established the centro-affine invariant conditions for the existence of a holomorphic integral (41) for differential system (12). Taking into account this, the Lyapunov Theorem on stability of unperturbed motion [1, §40, p. 160] and the holomorphic integral (41) we obtain the following main result.

Theorem 5. *Suppose for system* (12) *the centro-affine invariant conditions* (35) *hold, the comitant* φ *from* (33) *is a non-constant function and is not identically zero, and condition* $L_{1,3} > 0$ *from* (11) *is satisfied. Then the system has a periodic solution containing an arbitrary constant and varying the constant one can obtain a continuous sequences of periodic motions which describe the studied unperturbed motion. This motion is stable and any other unperturbed motion will tend asymptotically to one of the periodic motions.*

Assume that $L_{1,3} = 0$, then the differential system (34) admits the following operators

$$\begin{aligned} X_1 = [2hx^1x^2 + 2(k-g\lambda)x^1x^3 - \\ -\lambda x^2]\frac{\partial}{\partial x^1} + \\ +[2h(x^2)^2 + \lambda x^1 + \\ +2(k-g\lambda)x^2x^3]\frac{\partial}{\partial x^2} + [2hx^2x^3 + \\ +2(k-g\lambda)(x^3)^2 + x^2]\frac{\partial}{\partial x^3}, \\ X_2 = [2(k-g\lambda)x^1x^2 - 2h\lambda^2x^1x^3 - \\ -\lambda^2x^1]\frac{\partial}{\partial x^1} + \\ +[2(k-g\lambda)(x^2)^2 - \lambda^2x^2 - \\ -2h\lambda^2x^2x^3]\frac{\partial}{\partial x^2} + [2(k-g\lambda)x^2x^3 - \\ -2h\lambda^2(x^3)^2 + \lambda x^1]\frac{\partial}{\partial x^3}, \\ X_3 = (x^1 + \lambda x^3)(x^1\frac{\partial}{\partial x^1} + \\ +x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}), \end{aligned} \qquad (42)$$

which form a Lie commutative algebra.

Using the first two operators and ignoring a constant factor, similarly to the previous case, we obtain the Lie integrating factor

$$\mu^{-1} = ((x^1)^2 + (x^2)^2)(x^1 + \lambda x^3)^2. \qquad (43)$$

By means of this expression and the Lie Theorem on integrating factor [7], we have the following assertion.

Theorem 6. *The differential system* (34) *with* $L_{1,3} = 0$ *has a general integral composed of the following two first integrals*

$$\begin{aligned} F_1 \equiv \frac{(x^1)^2 + (x^2)^2}{(x^1 + \lambda x^3)^2} = C_1, \\ F_2 \equiv \frac{\lambda^2 + 2(\lambda g - k)x^2 + 2\lambda^2hx^3}{x^1 + \lambda x^3} + \\ +2k\, arctg\frac{x^2}{x^1} = C_2. \end{aligned} \qquad (44)$$

Remark 1. *The Lie algebra* (42)*, the Lie integrating factor* (43)*, the first integral* (44) *of the system* (34)*, ignoring constant factor, can be obtained from the Lie algebra* (36)*, the first integrating factor* (37) *and the first integral* (38)*, respectively, by substituting* $L_{1,3} = 0$.

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