УДК 517.95

©2016 р. Я.О. Баранецький, П.І. Каленюк, Л.І. Коляса

Національний університет "Львівська політехніка"

КРАЙОВА ЗАДАЧА ДЛЯ АБСТРАКТНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ОПЕРАТОРОМ ІНВОЛЮЦІЇ

Вивчається нелокальна задача для диференціально-операторних рівнянь з інволюцією. Встановлено спектральні властивості та умови існування і єдиності розв'язку. Наведено достатні умови при яких система кореневих функцій суттєво несамоспряженого оператора задачі утворює базис Рісса.

We study a nonlocal problem for differential-operator equations of order 2 with involution. The spectral properties of the operator of this problem are analyzed and the conditions for the existence and uniqueness of its solution are established. It is also proved that the system of root functions essentially a nonself-adjoint operator of the analyzed problem forms a Riesz basis.

Statement of the Problem.

Suppose that H is a separable Hilbert space, $A : H \to H$ is a positive self-adjoint operator with point spectrum $\sigma_p(A) = \{z_k \in \mathbf{R} : z_k \sim \beta k^{\alpha},\ \alpha, \beta > 0, k = 1, 2, ...\}$, $V(A) \equiv \{v_k \in H : k = 1, 2, ...\}$ is a system of eigenfunctions that form an orthonormalized basis in the space H, $H(A^s) = \{v \in H : A^s v \in H\}$, $s \geq 0$, $(u, v; H(A^s)) \equiv (A^s u, A^s v; H)$, $\|y; H(A^s)\| \equiv \|A^s y; H\|$, $H_1 \equiv L_2((0,1), H) = \{u(t) : (0,1) \to H,\ \|u(t); H\| \in L_2(0,1)\}$, $D_x : H_1 \to H_1$ is a strong derivative in the space H_1, i.e. $\left\|\dfrac{u(x+\Delta x) - u(x)}{\Delta x} - D_x u; H\right\| \to 0$, $(\Delta x \to 0)$, $I : L_2(0,1) \to L_2(0,1)$ is operator of involution: $Iu(x) \equiv u(1-x)$,$(u(x) \in L_2(0,1))$, $p_0 \equiv \frac{1}{2}(E + I)$, $p_1 \equiv \frac{1}{2}(E - I)$, $L_{2,j}(0,1) \equiv \{y \in L_2(0,1) : y = p_j y\}$, $H_{1,j} \equiv \{y(x) \in H_1 : y(x) \equiv p_j y(x)\}$,$j = 0, 1$, $H_2 \equiv \{y(x) \in H_1 : D_x^2 y \in H_1, A^2 y \in H_1\}$, $\|y; H_2\|^2 \equiv \|D_x^2 y; H_1\|^2 + \|A^2 y; H_1\|^2$, $L(H(A^m); H(A^q))$ is algebra of bounded linear operators $A : H(A^m) \to H(A^q)$, $(m, q \geq 0)$, $H(A^0) = H$, $L(H(A^m)) \equiv L(H(A^m); H(A^m))$, $H^1 \equiv H\left(A^{\frac{3}{2}}\right)$, $H^2 \equiv H\left(A^{\frac{1}{2}}\right)$, $B^j \in L(H^2), B(x) \equiv \sum_{r=1}^{q} B_r \left(x - \frac{1}{2}\right)^{2r-1}$, $B_r \in L(H_2)$, $B_r v_k = \frac{1}{r!} b_{r,k} v_k, B^j v_k = b_k^j v_k, B(x) v_k = b_k(x) v_k$, $b_{r,k}, b_k^j \in R, r = 1, 2, ...; q, j = 0, 1; k = 1, 2, ...$. Consider the following problem:

$$L(D_x, A)\, y \equiv -D_x^2 y(x) + A^2 y + B(x)(y(x) - -y(1-x)) = f(x), \tag{1}$$

$$l_1 y \equiv y(0) - y(1) = h_1, \tag{2}$$

$$l_2 y \equiv B^0 D_x y(0) + B^1 D_x y(1) = h_2, \tag{3}$$

$$f(x) \in H_1, h_1 \in H^1, h_2 \in H^2.$$

We interpret the solution [16,21] of problem (1), (3) as a function $y(x) \in H_2$ satisfying the equalities

$$\|L(D_x, A)\, y - f; H_1\| = 0, \left\|l_1 y - h_1; H^1\right\| = 0,$$

$$\left\|l_2 y - h_2; H^2\right\| = 0.$$

Differential equation (1) includes operator of involution. First study the properties of involution operator launched C.Babbage (see [5]). In paper [13] T.Carleman introduced the concept of operator shift – a generalization of the concept of involution $Iy(x) \equiv y(1-x)$, $x \in L_2(0,1)$. Exploration partial differential equations with involution are devoted [2, 3, 4, 7, 11, 13, 22, 32, 37, 37].

Properties spectral problems for ordinary differential and functional- differential equations with involution investigated in the works [1, 9- 10, 17- 20, 23- 25, 29- 31, 34- 36, 38] and [5, 6, 8, 12, 27, 33, 39] respectively.

Solutions of spectral problem

$$L(D_x, A)\, y = \lambda y(x), \quad \lambda \in C, l_1 y = 0, l_2 y = 0, \tag{4}$$

consider as a product $y(x) = u(x)\, v_k$, $k = 1, 2, ...$, $u(x) \in W_2^2(0,1)$.

To determine the functions $u(x)$ obtain spectral problem

$$L_k(D_x)\, u \equiv -D_x^2 u(x) + z_k^2 u + b_k(x)(u(x) + u(1-x)) = \lambda u(x), \tag{5}$$

.

$$l_{1,k} u \equiv u(0) - u(1) = 0, \tag{6}$$

$$l_{2,k} u \equiv b_k^0 D_x u(0) + b_k^1 D_x(1) = 0. \tag{7}$$

Auxiliary spectral problems.

Consider the particular case the problem (5) – (7) if the specified conditions $B(x) = 0$, $b_k^0 = -b_k^1 = 1$.

$$-D_x^2 u(x) + z_k^2 u = \lambda u(x), \tag{8}$$

$$u(0) - u(1) = 0, \quad D_x u(0) - D_x u(1) = 0. \tag{9}$$

Lemma 1. *Let $B^0 = -B^1 = E$, $B(x) = 0$. Then problem (8), (9) have point spectrum $\sigma_k \equiv \left\{\lambda_{k,n} \in \mathbf{R} : \lambda_{k,n} = (2\pi n)^2 + z_k^2,\ n = 0, 1, ...,\right\}$ and system of eigenfunctions*

$$T \equiv \left\{ t_n^s \in L_2(0,1) : t_0^0(x) = 1, \quad t_n^0(x) = \sqrt{2}\cos 2\pi n x, \quad t_n^1(x) = \sqrt{2}\sin 2\pi n x, n \in N \right\}$$

forms a ortonormalized basis in spaces $L_2(0,1)$.

Property of spectral problem (6) – (8).

We now consider the operator $L_{0,k} : L_2(0,1) \to L_2(0,1)$ of the problem (6) – (8)

$$L_{0,k} u \equiv -D_x^2 u + z_k^2 u, u \in D(L_{0,k}), \quad D(L_{0,k}) \equiv \{u \in W_2^2(0,1) : l_{1,k} u = 0, l_{2,k} u = 0\}.$$

Let

$$v_{k,0}^{0,0}(x) \equiv 1, \ v_{k,n}^{0,0}(x) \equiv \sqrt{2}\cos 2\pi n x, \tag{10}$$

$$v_{k,n}^{1,0}(x) \equiv \sqrt{2}(1 + \beta_k(2x-1)) \sin 2\pi n x, \tag{11}$$

$$\beta_k \equiv \left(b_k^0 - b_k^1\right)^{-1}\left(b_k^0 + b_k^1\right). \tag{12}$$

You can check that

$$L_{0,k} v_{k,n}^{1,0}(x) = \lambda_{k,n} v_{k,n}^{1,0} + \xi_{k,n}^0 v_{k,n}^{0,0}, \quad \xi_{k,n}^0 = -8\pi n \beta_k, n = 0, 1, 2... . \tag{13}$$

Hence

$V(L_{0,k}) \equiv \left\{ v_{k,n}^{s,0}(x) ; s = 0, 1, n = 0, 1, 2, ...\right\}$ are the system of root functions of the operator $L_{0,k}$ in the sense of equality (13).

Lemma 2. *Let $b_k^0 \neq b_k^1$, $k = 1, 2, ...$. Then the operator $L_{0,k}$ of problem (8), (6), (7) have point spectrum σ_k, and system $V(L_{0,k})$ of root functions complete and minimal in $L_2(0,1)$.*

Proof. We now consider the adjoint spectral problem

$$-D_x^2 w(x) + z_k^2 w = \mu w(x), \mu \in C, \quad b_k^1 w(0) + b_k^0 w(1) = 0, \quad D_x w(0) - D_x w(1) = 0. \tag{14}$$

The operator of this adjoint problem (14) have point spectrum σ_k, and system of root functions

$$w_{k,0}^{1,0}(x) \equiv 1 - \beta_k(2x-1), \quad w_{k,n}^{0,0}(x) \equiv \sqrt{2}\sin 2\pi n x, \tag{15}$$

$$w_{k,n}^{1,0}(x) \equiv \sqrt{2}(1 - \beta_k(2x-1)) \cos 2\pi n x. \tag{16}$$

You can check that

$$L_{0,k}v^{1,0}_{k,n}(R_0)=\lambda_{k,n}v^{1,0}_{k,n}+\xi^0_{k,n}v^{0,0}_{k,n},\quad \xi^0_{k,n}=8\pi n\beta_k, n=1,2... . \tag{17}$$

Hence, the system of root functions $V(L_{0,k})$ of the operator $L_{0,k}$ possesses a unique biorthogonal system $W(L_{0,k})\equiv\left\{w^{s,0}_{k,n}(x); s=0,1; n=1,2,...\right\}$ are the system of root functions of the problem (14) in the sense of equality (17).

$\left(v^{r,0}_{k,n}, w^{s,0}_{k,q}; L_2(0,1)\right)=\delta_{r,s}\delta_{k,q}$, $(r,s=0,1; q,n=0,1...)$.

Consider the operators $R_{0,k}, S_{0,k} : L_2(0,1) \to L_2(0,1)$, $R_{0,k}t^p_{k,n} \equiv v^{p,0}_{k,n}$, $R_{0,k}=E+S_{0,k}$, $p=0,1; n=0,1,...$.

From the definition of the operator $R_{0,k}$ and the completeness of system $V(L_{0,k})$ in space $L_2(0,1)$ we get $S_{0,k} : L_{2,0}(0,1) \to 0, S_{0,k} : L_{2,1}(0,1) \to L_{2,0}(0,1)$

To prove that the system $V(L_{0,k})$ forms a Riesz basis (see [15, 26]) in $L_2(0,1)$, it is sufficient, according to formula $R_{0,k}=E+S_{0,k}$, to show that the operator $S_{0,k} : L_2(0,1) \to L_2(0,1)$ is bounded.

Let ω be an arbitrary element from the space $L_2(0,1)$). We represent ω as a Fourier series in the system T.

$$\omega=\omega^0_0t^0_0+\sum_{m=1}^{\infty}\omega^0_mt^0_m+\omega^1_mt^1_m,$$

$$\omega^j_m=\left(\omega, t^j_m; L_2(0,1)\right).$$

According to the definition of the operator $S_{0,k}$, we find

$$S_{0,k}\omega=\beta_k(2x-1)\sum_{m=0}^{\infty}\omega^0_m\sqrt{2}\cos 2\pi mx.$$

Using the ratio

$$\|S_{0,k}\omega; L_2(0,1)\|=$$

$$=\left\|\beta_k(2x-1), \sum_{m=0}^{\infty}\omega^0_m\sqrt{2}\cos 2\pi mx; L_2(0,1)\right\|,$$

we estimate it

$$\|S_{0,k}\omega; L_2(0,1)\|\le|\beta_k|\,\|\omega; L_2(0,1)\|.$$

Hence, $R_{0,k}=E+S_{0,k}\in L(L_2(0,1))$ and $\left(R^{-1}_{0,k}\right)^*=E-S^*_{0,k}\in L(L_2(0,1))$.

So using theorem N.K.Bary (see theorem 6. 2.1 [15]) we obtain the following statement.

Theorem 1. *Let $b^0_k\neq b^1_k$, $k=1,2,...$. Then the operator $L_{0,k}$ of problem* (8), (6), (7) *have point spectrum σ_k , and system $V(L_{0,k})$ of root functions in the sense of equality* (12) *forms a Riesz basis in $L_2(0,1)$.*

Further, we introduce operator $L_{1,k} : L_2(0,1)\to L_2(0,1)$ of the problem (5) – (7):

$$L_{1,k}u\equiv L_k(D_x)u, u\in D(L_{1,k}),$$

$$D(L_{1,k})\equiv\left\{u\in W^2_2(0,1) : l_{1,k}u=0, l_{2,k}u=0\right\}.$$

By the direct substitution we can show that the functions $v^{0,1}_{k,n}(x)=\sqrt{2}\cos 2\pi nx$, is eigenfunctions of operator $L_{1,k}$:

$$v^{0,1}_{k,n}(x)=\sqrt{2}\cos 2\pi nx\in D(L_{1,k}),$$

$$L_{1,k}v^{0,1}_{k,n}(x)=\lambda_{k,n}v^{0,1}_{k,n}(x), k=1,2... .$$

Root function of operator $L_{1,k}$, defined by relation

$$v^{1,1}_{k,n}(x)\equiv\sqrt{2}\sin 2\pi nx+v^0_{k,n}(x)+ +v^1_{k,n}(x)+v^2_{k,n}(x), \tag{18}$$

$$v^0_{k,n}(x)\equiv\sqrt{2}\beta_k(2x-1)\sin 2\pi nx,\quad \beta_k\equiv\left(b^0_k-b^1_k\right)^{-1}\left(b^0_k+b^1_k\right), \tag{19}$$

$$v^1_{k,n}(x)\equiv\sqrt{2}\beta^1_{k,n}(2x-1)\sin 2\pi nx, \tag{20}$$

$$v_{k,n}^2(x) \equiv \sqrt{2}\sum_{j=1}^{q+1} c_{2j+1,n,k}\tfrac{1}{(2j+1)!}\left(x-\right.$$
$$\left.-\tfrac{1}{2}\right)^{2j+1}\sin 2\pi nx+ \tag{21}$$
$$+\sqrt{2}\sum_{j=0}^{q} c_{2j,n,k}\tfrac{1}{(2j)!}\left(x-\tfrac{1}{2}\right)^{2j}\cos 2\pi nx.$$

Support functions $v_{k,n}^1(x)$, $v_{k,n}^2(x)$ so choose

$$(L_k(D_x)-\lambda_{k,n})\,v_{k,n}^{1,1}(x)=\xi_{k,n}^1 v_{k,n}^{0,1}(x)=0, \tag{22}$$

$$l_{1,k}\left(v_{k,n}^1(x)+v_{k,n}^2(x)\right)=0, \tag{23}$$

$$l_{2,k}\left(v_{k,n}^1(x)+v_{k,n}^2(x)\right)=0. \tag{24}$$

Substitute equity (21) ratio (22) and equal define parameters $c_{r,n,k}$

$$c_{2s,n,k}=\tfrac{1}{2\pi n}\left(b_{s-1,k}\pm\frac{1}{(4\pi n)^2}b_{s-1,k}+...\pm\right.$$
$$\left.\pm\frac{1}{(4\pi n)^{2q-2s+2}}b_{q,k}\right), \tag{25}$$

$$c_{2s+1,n,k}=-\tfrac{1}{2(\pi n)^2}\left(b_{s,k}\pm\frac{1}{(4\pi n)^2}b_{s+1,k}+...\pm\right.$$
$$\left.\pm\frac{1}{(4\pi n)^{2q-2s}}b_{q,k}\right). \tag{26}$$

Taking into account that $v_{k,n}^1(x)+v_{k,n}^2(x)\in L_{2,0}(0,1)$, $\sqrt{2}\sin 2\pi nx+v_{k,n}^0(x)=v_{k,n}^{1,0}(x)\in D(L_{0,k})$ obtain equality

$$l_{1,k}v_{k,n}^{1,1}=l_{1,k}\left(v_{k,n}^{1,0}+v_{k,n}^1+v_{k,n}^2\right)=0.$$

If

$$D_x v_{k,n}^1(0)+D_x v_{k,n}^2(0)\equiv 2\sqrt{2}\beta_{k,n}^1+$$
$$+2\sqrt{2}\sum_{j=1}^{q+1}c_{2j+1,n,k}\tfrac{1}{(2j)!}\left(\tfrac{1}{2}\right)^{2j}=0,$$

$\beta_{k,n}^1=-\sum_{j=1}^{q+1}c_{2j+1,n,k}\frac{1}{(2j)!}\left(\frac{1}{2}\right)^{2j}$ then

$$l_{2,k}v_{k,n}^{1,1}=l_{2,k}v_{k,n}^{1,0}+l_{2,k}\left(v_{k,n}^1+v_{k,n}^2\right)=$$
$$=l_{2,k}\left(v_{k,n}^1+v_{k,n}^2\right)=0.$$

Hence, $v_{k,n}^{1,1}(x)\in D(L_{1,k})$,

$$L_{1,k}v_{k,n}^{1,1}(x)\equiv\lambda_{k,n}v_{k,n}^{1,1}(x)+\xi_{k,n}^1 v_{k,n}^{0,1}(x), \tag{27}$$

here

$$\xi_{k,n}^1=8\pi n\left(\beta_n+\beta_n^1\right), k=1,2,...,n=0,1,...$$

.

Consider the system functions $V_k\equiv\left\{v_{k,n}(x), v_{k,n}^{0,1}(x)\in L_2(0,1),\ k=1,2,...,n=0,1,...\right\}$,

here

$$v_{k,n}(x)\equiv\sqrt{2}\cos 2\pi nx+v_{k,n}^0(x)+v_{k,n}^1(x),$$
$$v_{k,n}^{0,1}(x)=\sqrt{2}\sin 2\pi nx, k=1,2,...,$$
$$n=0,1,\dots\ . \tag{28}$$

Lemma 3. *Let* $b_k^0\neq b_k^1$, $k=1,2,...$ *Then the system functions (28) forms a Riesz basis in space* $L_2(0,1)$.

Lemma proved just as Theorem 1.

Lemma 4. *Let* $b_k^0\neq b_k^1$,$k=1,2,\dots$. *Then the system functions* V_k and $V(L_{1,k})$ *a square close in space* $L_2(0,1)$.

Proof. A formula (19), (25), (26) that follows.

So get inequality

$$\sum_{n=0}^{\infty}\left\|v_{k,n}^{1,1}-v_{k,n},L_2(0,1)\right\|^2=$$
$$=\sum_{n=0}^{\infty}\left\|v_{k,n}^2,L_2(0,1)\right\|^2<\infty.$$

Of the results [14] follows that the system V_k is complete and minimal in space $L_2(0,1)$.

Hence, the system of root functions $V(L_{1,k})$ of the operator $L_{1,k}$ possesses a unique biorthogonal system $W(L_{1,k})$

$$\left(v_{k,n}^{r,1},w_{q,n}^{s,1};L_2(0,1)\right)=\delta_{r,s}\delta_{k,q},$$
$$(r,s=0,1;q,k=0,1...)$$

.

So using theorem N.K.Bary (see theorem 6. 2.3 [15]) we obtain the following statement.

Theorem 2. *Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator $L_{1,k}$ of problem (5) – (7) have point spectrum σ_k, and system $V(L_{1,k})$ of root functions in the sense of equality (27) forms a Riesz basis in space $L_2(0,1)$.*

The spectral problem (1)**,** (2)**.**

We now consider the operator $L : H_1 \to H_1$ of problem (1), (2):

$$Ly \equiv L(D_x, A)\, y, y \in D(L),$$

$$D(L) \equiv \{y \in H_2 : l_1 y = 0, l_2 y = 0\}.$$

Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator L of problem (1), (2) have point spectrum

$$\sigma \equiv \left\{\lambda_{k,n} \in R : \lambda_{k,n} \equiv 4n^2\pi^2 + z_k^2,\right.$$

$$\left. n = 0, 1 \dots; k = 1, 2, \dots\right\},$$

and system of root functions

$$V(L) \equiv \left\{v_{k,n}^s(L) \in H_1 : v_{k,n}^s(L) = v_{k,n}^{s,1}(x)\, v_k,\right.$$

$$\left. s = 0, 1; n = 0, 1, 2, \dots; k = 1, \dots\right\},$$

$$v_{k,0}^0(L) \equiv v_k, v_{k,n}^0(L) \equiv \sqrt{2}\sin 2\pi n x v_k,$$

$$v_{k,n}^1(L) \equiv v_{k,n}^{1,1}(x)\, v_k, \ n, k = 1, 2, \dots .$$

System $V(L)$ of root functions of the operator L possesses a unique biorthogonal system $W(L) \equiv \left\{w_{p,m}^s \in H_1 v : w_{p,m}^s \equiv w_{p,m}^{s,1}(x)\, v_m,\right.$ $\left. p = 1, 2, \dots; m = 0, 1, \dots; s = 0, 1\right\}$ in the sense of equality

$$\left(v_{k,m}^j, w_{p,n}^s; H_1\right) = \delta_{j,s}\delta_{k,p}\delta_{m,n}.$$

Hence, we obtain the following statement.

Lemma 5. *Let $b_k^0 \neq b_k^1$, $k = 1, 2, \dots$. Then the operator L of problem* (1), (2) *have complete and minimal in H_1 system of root functions $V(L_0)$.*

Then $\|R_{1,k}\omega; L_2(0,1)\| \leq C\|\omega; L_2(0,1)\|$, $\left\|\left(R_{1,k}^{-1}\right)\omega; L_2(0,1)\right\| \leq C\|\omega; L_2(0,1)\|, C > 0$.

Further, we introduce operator $B \equiv (B^0 + B^1)(B^0 - B^1)^{-1}$.

Theorem 3. *Let $B \in L(H^2)$, $B_r \in L(H_2)$, $r = 1, 2, \dots, q$. Then the operator L of problem* (1), (2)*, have system of root functions $V(L)$ forms a Riesz basis in H_1.*

Proof. Let

$$T_1 \equiv \left\{t_{k,n}^s \in H_1 : t_{k,n}^j \equiv t_n^j v_k, t_n^j \in T,\right.$$
$$\left. v_k \in V(A), j = 0, 1, n = 0, 1, \dots, k = 1, 2, \dots\right\}.$$

Consider the operators $R_1, S_1 : H_1 \to H_1$, $R_1 t_{k,n}^s \equiv v_{k,n}^{s,1}$, $R_1 = E + S_1$.

From the definition of the operator R_1 for any $g = \sum_{s,k,m} g_{k,m}^s t_{k,m}^s \in H_1$, $g_{k,m}^s = \left(g, t_{k,m}^s; H_1\right)$ we get

$$R_1 g = \sum_{j,k,m} g_{k,m}^j v_{k,m}^{j,1} \in H_1,$$

$$\left(R_1^{-1}\right)^* \sum_{j,k,m} g_{k,m}^j t_{k,m}^j = \sum_{j,k,m} g_{k,m}^j w_{k,m}^{j,1}$$

$$\|R_1 g, H_1\| \leq \max\|E + S_{1,k}, L(H_1)\|\,\|g, H_1\| =$$

$$= C_1\|g, H_1\|,$$

$$\left\|\left(R_1^{-1}\right)^* g, H_1\right\| \leq \max\left\|E - (S_{1,k})^*, L(H_1)\right\|$$

$$\|g, H_1\| = C_2\|g, H_1\|, \ k = 1, 2, \dots .$$

So using theorem N.K.Bary (see theorem 6. 2.1 [15]) we obtain the following statement of the theorem 3.

3. Property of problem (1)**,** (2)**.**

Replaced condition (2) on equivalent terms

$$\begin{array}{c} l_1 y \equiv y(0) - y(1) = h_1, \\ l_3 y \equiv D_x y(0) - D_x y(1) + \\ + B(D_x y(0) + D_x y(1)) = h_3. \end{array} \qquad (29)$$

Here $h_3 \equiv 2(B^0 - B^1)^{-1} h_1$.

Consider the particular case the problem (1), (29) if the specified conditions $B = 0$, $B_0 = 0$

$$-D_x^2 y(x) + A^2 y = g(x), \qquad (30)$$

$$\begin{array}{c} y(0) - y(1) = g_1, \\ D_x y(0) - D_x y(1) = g_2, \ g_j \in H^j, j = 1, 2. \end{array} \qquad (31)$$

Theorem 4. *Let $B = 0$, $B_0 = 0$. Then for any $g \in H_1, g_1 \in H^1, g_2 \in H^2$, there exists a unique solution of problem (30), (31).*

Proof. We seek the solution of this problem in the form $y = u + v$, there u is the solution of the problem

$$-D_x^2 u(x) + A^2 u = g(x), y(0) - y(1) = 0, \quad D_x y(0) - D_x y(1) = 0 \tag{32}$$

and v is the solution of the problem

$$-D_x^2 v(x) + A^2 v(x) = 0, v(0) - v(1) = g_1, \quad D_x v(0) - D_x v(1) = g_2. \tag{33}$$

Consider the problem (32). We expand the functions $u(x), g(x)$ in a series in the orthonormalized T_1 basis in the space H_1 :

$$u = \sum_{s,k,m} u_{k,m}^s t_{k,m}^s, \quad u_{k,m}^s = \left(u, t_{k,m}^s; H_1\right),$$

$$g = \sum_{s,k,m} g_{k,m}^s t_{k,m}^s, \quad g_{k,m}^s = \left(g, t_{k,m}^s; H_1\right).$$

We estimate a number

$$-D_x^2 u = \sum_{s,k,m} (2\pi m)^2 \left((2\pi m)^2 + z_k^2\right)^{-1} g_{k,m}^s t_{k,m}^s,$$

$$\left\|D_x^2 u; H_1\right\| \le \|g; H_1\|,$$

$$A^2 u = \sum_{s,k,m} z_k^2 \left((2\pi m)^2 + z_k^2\right)^{-1} g_{k,m}^s t_{k,m}^s,$$

$$\left\|A^2 u; H_1\right\| \le \|g; H_1\|.$$

Hence

$$\|u; H_2\| \le \sqrt{2}\,\|g; H_1\|. \tag{34}$$

Consider the problem (33). Further, we introduce operators, $Y_j(x, A) \equiv e^{Ax} + (-1)^j e^{A(1-x)} \in L(H^2; H_2)$, $j = j = 0, 1$. The solution of the differential equation (33) has the form

$$v(x) = Y_0(x, A)\varphi_0 + Y_1(x, A)\varphi_1 \tag{35}$$

where φ_0, φ_1 are unknown.

To determine the, $\varphi_0, \varphi_1 \in H^1$ we substitute expression (35) in the condition (33) and obtain

$$\varphi_1 = \tfrac{1}{2} W_1(0, A)^{-1} g_1, \varphi_0 = \tfrac{1}{2} W_1(0, A)^{-1} A^{-1} g_2.$$

Hence,

$$v = \tfrac{1}{2} W_1(x, A) W_1(0, A)^{-1} g_1 + \tfrac{1}{2} W_0(x, A) W_1(0, A)^{-1} A^{-1} g_2,$$

$$\|v; H_2\|^2 \le C\left(\left\|g_1; H^1\right\|^2 + \left\|g_2; H^2\right\|^2\right). \tag{36}$$

Therefore follows from inequalities (34), (36) inequality

$$\|y; H_2\|^2 \le C_1\left(\|g; H_1\|^2 + \left\|g_1; H^1\right\|^2 + \left\|g_2; H^2\right\|^2\right).$$

We now return to the original problem (1), (2). Consider in connection problem as the sum $y = y_0 + y_1$, $y_j \in H_{1,j} \bigcap H_{2,j}$, $j = 0, 1$.

To determine the unknowns $y_j \in H_{1,j}$ get the problem

$$-D_x^2 y_1(x) + A^2 y_1(x) = f_1(x), f_1(x) \in H_{1,1},$$

$$y_1(0) - y_1(1) = h_1, D_x y_1(0) - D_x y_1(1) = 0,$$

$$-D_x^2 y_0(x) + A^2 y_0(x) = f_0(x) - 2B(x) y_1(x), \quad f_0(x) \in H_{1,0},$$

$$y_0(0) - y_0(1) = 0, D_x y_0(0) - D_x y_0(1) = = h_3 - B\left(D_x y_1(0) + D_x y_1(1)\right),$$

$$\|y; H_2\|^2 \le C\left(\|f; H_1\|^2 + \left\|h_1; H^1\right\|^2 + \left\|h_2; H^2\right\|^2\right), (C > 0).$$

For unknown functions $y_j \in H_{1,j}$ get that problem is a particular case of the problem (21), (22). Hence the statement is correct.

Theorem 5. Let $B \in L(H^1)$, $B(x) \in L(H_2)$. *Then for any* $f \in H_1, h_1 \in H^1, h_2 \in H^2$, *there exists a unique solution of problem* (1), (2) *and fair inequality*

$$\|y; H_2\|^2 \leq C\left(\|f; H_1\|^2 + \left\|h_1; H^1\right\|^2 + \right.$$

$$\left. + \left\|h_2; H^2\right\|^2\right), (C > 0).$$

Conclusion.

Investigated the spectral properties essentially a nonself-adjoint operator nonlocal problems for abstract differential equation with involution.

Studied the problem solution is built on a number of root functions

References

1. *Aftabizadeh A. R., Huang Y. K., Wiener J.* Bounded solutions for differential equations with reection of the argument // J. Math. Anal. Appl. – 1988. –**135**, – P. 31–37.

2. *Aliev B. D., Aliev R. M.* Properties of the solutions of elliptic equations with deviating arguments // Izdat. Akad. Nauk Azerbaijan. SSR, Baku, – 1968. – P. 15-25. (Russian)

3. *Andreev A. A., Shindin I. P.* On the well-posedness of boundary value problems for a partial differential equation with deviating argument // Kuibyshev. Gos. Univ. – 1987. – P. 3–6. (Russian)

4. *Ashyralyev A. , Sarsenbi A. M.* Well– posedness of an elliptic equations with an involution// Electron. J. Diff. Equ. – 2015. – **284**, – P. 1–8.

5. *Babbage C.* An essay towards the calculus of calculus of functions // Philos. Trans. Roy. Soc. London. – 1816. – **106**, Part II. – P. 179–256.

6. *Baranetskij Ya. O.* Boundary value problems with irregular conditions for differential-operator equations // Bukov. Mathemat.Journ. – 2015. – **3**, N3-4. – P. 33–40. (Ukraine)

7. *Baranetskij Ya., Yarka Y.* The existence of izospectral perturbation of Dirichlet problem of infinite order differential operator // Visnyk Derzh. Univ. "L'viv. Politekhnika,"Ser. Prykl. Matem. – 1997. – **320**, – P. 15–18. (Ukraine)

8. *Baranetskij Ya., Yarka Y.* One class of boundary value probems for differential- operator equatons of even probem // Mat. Metody Fiz.- Mekh. Polya. – 1999. – **42**, N4. – P. 64-67. (Ukraine)

9. *Baranetskij Ya., Kalenyuk P., Yarka Y.* Perturbation boundary value problems for ordinary differential equations of second order // Visnyk Derzh. Univ. "L'viv. Politekhnika,"Ser. Prykl. Matem.– 1998. – **337**, – P. 70–73. (Ukraine)

10. *Baranetskij Ya., Yarka Y., Fedushko S.* Izospectral perturbation the differential operator Dirichlet. Spectral property // Sci. Bull. Uzhgor. Univer. – 2012. – **23**, N1. – P. 12–16. (Ukraine)

11. *Burlutskaya M. Sh., Khromov A. P.* Initial-boundary value problems for first-order hyperbolic equations with involution // Doklady Math. – 2011. – **84**, N3. – P. 783–786.

12. *Busenberg S. N., Travis C. C.* On the use of reducible-functional-differential equations in biological models // J. Math. Anal. Appl. – 1982. – **89**, – P. 46–66.

13. *Carleman T.* Sur la the'orie des e'quations inte'grales et ses applications // Verh. Internat. Math. Kongr. – 1932. – **1**, – P. 138–151.

14. *Gomilko A. M., Radzievskii G. V.* Equivalence in Lp[0,1] of the system expi2πkx (k=0,$\pm$1,. . .)and the system of the eigenfunctions of an ordinary functional-differential operator // Mathematical Notes, – 1991, – **49**, N1. – P. 34–40.

15. *Gokhberg I. Ts., Krein M. G.* Introduction to the Theory of Linear Not Self-Adjoint Operators.— M.: Nauka, 1965. — 448 pp.(Russian)

16. *Gorbachuk V. L.,Gorbachuk M.L.* Boundary Value Problems for Differential- Operator Equations.— K.: Nauk. dumka, 1984. — 320 pp.(Russian)

17. *Gupta C. P.* Boundary value problems for differential equations in Hilbert spaces involving reflection of the argument // J. Math. Anal. Appl. – 1987. – **128**, – P. 375–388.

18. *Gupta C. P.* Two-point boundary value problems involving reflection of the argument // Int. J., Math. Math. Sci. – 1987. – **10**, N2. – P. 361–371.

19. *Gupta C. P.* Existence and uniqueness theorems for boundary value problems involving refection of the argument // Nonlinear Anal. – 1987. – **11**, – P. 1075–1083.

20. *Fernandez A.E. , Araujo J.A.E., Tojo F.A.F., Villamarin D.M.* Existence results for a linear equation with reflection, non-constant coeficient and periodic boundary conditions // Journal of Mathematical Analysis and Applications. – 2014. – **412**, N1. – P. 529-546.

21. *Kalenyuk P. I., Baranetskij Ya. E., Nitrebich Z. N.* Generalized Method of the Separation of Variables. — K.: Nauk. dumka, 1993. — 231 pp. (Russian)

22. *Kirane M., Al-Salti N.* Inverse problems for a nonlocal wave equation with an involution perturbation // J.Nonlinear Sci. Appl. – 2016. – **9**, – P. 1243–1251.

23. *Kopzhassarova A. A., Lukashov A. L., Sarsenbi A. M.* Spectral properties of non-self-

adjoint perturbations for a spectral problem with involution// Abstr. Appl. Anal. – 2012. – P. 1–5. doi:10.1155/2012/576843

24. *Kritskov L. V. , Sarsenbi A. M.* Spectral properties of a nonlocal problem for the differential equation with involution // Differ. Equ. – 2015. – **51**, N8. – P. 984–990. (Russian)

25. *Kurdyumov V. P.* On Rieszc bases of eigenfunction of 2-nd order differential operator with involution and integral boundary conditions // Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform. – 2015. – **15**, N4. – P. 392–405 (Russian)

26. *Naimark M. A.* Linear differential operators.— M.: Nayka, 1969.—528 pp. (Russian)

27. *Przeworska-Rolewicz D.* Sur les a equations involutives et leurs applications // Stud. Math. – 1961. – **20**, – P. 95–117.

28. *Przeworska- Rolewicz D.* Equations with Transformed Argument. An Algebraic Approach, (W.: Polish Scientific Publishers, 1973. — 354 pp.

29. *O'Regan D.* Existence results for differential equations with reflection of the a reflection of the argument// J. Aust. Math. Soc. – 1994. – **57**, N2. – P. 237–260.

30. *Sadybekov M. A., Sarsenbi A. M.* Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution // Differ. Equ. – 2012. – **48**, N8. – P. 1112–1118. (Russian)

31. *Sadybekov M. A., Sarsenbi A. M.* Mixed problem for a differential equation with involution under boundary conditions of general form. In:A. Ashyralyev, A. Lukashov // (eds.) First International Conference on Analysis and Applied Mathematics: ICAAM 2012. AIP Conference Proceedings. – 2012. – **1470**, – P. 225–227.

32. *Sarsenbi A. M., Tengaeva A. A.* On the basis properties of root functions of two generalized eigenvalue problems // Differ. Equ. – 2012. – **48**, N2. – P. 306–308. (Russian)

33. *Sharkovsky A. N.* Functional-differential equations with a finite group of argument transformations, Asymptotic Behavior of Solutions of Functional-Differential Equations // Coll. sci. Works, Akad. Nauk Ukraine SSR, Inst. Mat., K.: – 1978. – P. 118–142.(Russian)

34. *Shevelo V. N., GritsaўI A. G.* Some approaches to the study of the properties of solutions of differential equations with involutions, Methods for Investigating Differential and Functional-Differential Equations // Akad. Nauk Ukraine SSR, Inst. Mat., K.: – 1990. – P. 110–117. (Russian)

35. *Silberstein L.* Solution of the equation f _(x) = f(1/x) // Philos. Mag. – 1940. – **30,** N7. – P. 185–186.

36. *Wiener J.* Differential equations with involutions // Differ. Equ. – 1969. – **5**, N 6. – P. 1131–1137. (Russian)

37. *Wiener J.* Partial differential equations with involutions // Differ. Equ. – 1970. – **6**, N 7. – P. 1320–1322. (Russian)

38. *Wiener J., Aftabizadeh A. R.* Boundary value problems for differential equations with reflection of the argument // Int. J. Math. Math. Sci. – 1985. – **8**, N1. – P. 151–163.

39. *Wiener J.* Generalized solutions of functional differential equations // Singapore Singapore World Sci, – 1993. – P. 160–215.