Huzyk N.M., Brodyak O.Ya.

COEFFICIENT INVERSE PROBLEM FOR PARABOLIC EQUATION WITH STRONG POWER DEGENERATION

In a domain with known boundaries it is investigated an inverse problem for a parabolic equation with strong degeneration. The degeneration of the equation is caused by power function with respect to time variable at the higher order derivative of unknown function. It is known that the minor coefficient of the equation is a polynomial of the first order for the space variable with two unknown functions with respect to time. The boundary conditions of the second kind and the means of heat moments as overdetermination conditions are given. We establish conditions of existence and uniqueness of the classical solution to the named inverse problem.

Key words and phrases: coefficient inverse problem, parabolic equation, strong power degeneration, minor coefficient.

Hetman Petro Sahaidachnyi National Army Academy, Lviv, Ukraine (Huzyk N.M.)
Lviv Polytechnic National University, Lviv, Ukraine (Brodyak O.Ya.)
e-mail: *hryntsiv@ukr.net (Huzyk N.M.), brodyakoksana1976@gmail.com (Brodyak O.Ya.)*

Introduction

The theory of inverse problems for parabolic equations is actively developing in recent decades due to its practical application. Unlike direct problems, coefficient inverse problems arise when it is necessary to determine some parameters of the equation in addition to its solution. One of the first papers that studied the inverse problem of determination of the time-dependent coefficient of thermal conductivity in a heat equation is the paper by B.F. Jones [17]. The conditions of existence and uniqueness of the classical solution to this problem are established in it applying the Schauder Fixed Point Theorem. The coefficient inverse problems for parabolic equations in a domain with fixed boundaries with different boundary and overdetermination conditions are well studied for today (see, for example, [5, 1, 18, 7, 9, 4, 24, 21, 19, 20] and bibliography in them). Note that among these papers there are some with unknown time-dependent major coefficients of the parabolic equation [5, 1, 18, 7, 9], and time-dependent [4, 24, 21] or space-dependent [19, 20] minor coefficients in it.

УДК 517.95
2010 *Mathematics Subject Classification:* 35R30, 35K65.
Information on some grant ...

When describing such processes as the movement of liquids and gases in a porous medium, desalination of sea water, the behavior of financial markets, population dynamics, problems arise for parabolic equations with degenerations. Inverse problems for determination of the function $a = a(t), a(t) > 0, t \in [0, T]$ in parabolic equation

$$u_t = a(t)t^\beta u_{xx} + b(x,t)u_x + c(x,t)u + f(x,t)$$

were investigated in [23, 16] for both cases of weak ($0 < \beta < 1$) and strong ($\beta \geq 1$) degeneration respectively. Coefficient inverse problems for degenerate parabolic equations were studied in [10, 11, 12, 8, 22, 6, 3]. Note that among them there are problems of identification of the time-dependent minor coefficient in equations with degenerations with respect to time variable [10, 11, 12, 8] and space variables [22, 6], and the problem with unknown space-dependent coefficient in the degenerate equation with respect to this variable [3]. The problems of determining the coefficients in the degenerate parabolic equations, which depend on both time and space variables, remain uninvestigated for today.

In this paper coefficient inverse problem for parabolic equation with degeneration caused by time-dependent power function at the higher order derivative is investigated. It is known that the minor coefficient of the equation is a a polynomial of the first order for the space variable with two unknown functions with respect to time. The boundary conditions of the second kind and the means of heat moments as overdetermination conditions are given. The case of strong degeneration is studied. The conditions of existence and uniqueness of the classical solution to the named problem are established. Note that the case of weak degeneration for the named problem is investigated in [13, 2] and the case of strong degeneration with the Dirichlet boundary condition in [14].

1 The statement of the problem and the main results

In a domain $Q_T = \{(x,t) : 0 < x < l, 0 < t < T\}$ we consider the coefficient inverse problem for the degenerate parabolic equation:

$$w_t = a(t)t^\beta w_{xx} + (b_1(t)x + b_2(t))w_x + c(x,t)w + f(x,t), \tag{1}$$

$$w(x,0) = \varphi(x), \quad x \in [0,l], \tag{2}$$

$$w_x(0,t) = \mu_1(t),\ w_x(l,t) = \mu_2(t),\ t \in [0,T], \tag{3}$$

$$\int_0^l w(x,t)dx = \mu_3(t), \quad t \in [0,T], \tag{4}$$

$$\int_0^l xw(x,t)dx = \mu_4(t), \quad t \in [0,T]. \tag{5}$$

It is known that $a(t) > 0, t \in [0,T]$ and the degeneration of the equation is caused by the power function t^β, where $\beta \geq 1$ (the case of strong degeneration). The coefficient at the first derivative of unknown function $w = w(x,t)$ in the equation (1) is a polynomial of the first order for the space variable with two unknown functions with respect to time variable $b_1 = b_1(t), b_2 = b_2(t)$.

Definition 1. *A triplet of functions* $(b_1, b_2, w) \in (C[0,T_0])^2 \times C^{2,1}(Q_{T_0}) \cap C^{1,0}(\overline{Q}_{T_0})$, $|b_1(t)| \le M_1 t^{\eta}$, $|b_2(t)| \le M_2 t^{\eta}$, *where* $\eta = \min\{\gamma, \beta\}$, $\gamma > \frac{\beta-1}{2}$ *is an arbitrary number,* M_1, M_2 *are the positive constants defined by the input data, which satisfies the equation* (1) *and conditions* (2)-(5) *point by point for all* $t \le T_0$ *is called the local solution to the problem* (1)-(5) *at* $T_0 < T$ *and the global solution to this problem at* $T_0 = T$.

The main result of the paper is contained in the following Theorem.

Theorem. *Suppose that the assumptions*

A1) $\varphi \in C^3[0,l]$, $a \in C[0,T]$, $c, f \in C^{1,0}(\overline{Q}_T)$, $\mu_i \in C^1[0,T], i = \{1,2,3,4\}$;

A2) $a(t) > 0$, $t \in [0,T]$, $\varphi'(x) > 0$, $x \in [0,l]$;

A3) $|f(x,t)| + |f_x(x,t)| \le A_1 t^{\gamma}$, $|c(x,t)| + |c_x(x,t)| \le A_2 t^{\gamma}, (x,t) \in \overline{Q}_T$, $|\mu_3'(t)| \le A_3 t^{\gamma}$, $|\mu_4'(t)| \le A_4 t^{\gamma}$, $t \in [0,T]$, *where* A_i, $i = 1,2,3,4$ *are arbitrary positive constants;*

A4) $\mu_1(0) = \varphi'(0)$, $\quad \mu_2(0) = \varphi'(l)$, $\int\limits_0^l \varphi(x)dx = \mu_3(0)$, $\int\limits_0^l x\varphi(x)dx = \mu_4(0)$

hold. Then there exists the unique local solution to the problem (1)-(5).

2 Existence of the solution

To prove the existence of the solution to the problem (1)-(5) we apply the Schauder Fixed Point Theorem. For this purpose, using the apparatus of Green's functions of boundary value problems for the heat equation, the inverse problem (1)-(5) is reduced to an equivalent system of equations and conditions of Schauder's theorem are ensured for it.

In the problem (1)-(5) we make the substitution

$$w(x,t) = \widetilde{w}(x,t) + w_0(x,t), \tag{6}$$

where the function $w_0(x,t)$ satisfies the given nonhomogeneous initial and boundary conditions (2), (3). It is easy to verify by direct inspection, that the function $w_0(x,t)$ is defined by the formula

$$w_0(x,t) = \varphi(x) + x(\mu_1(t) - \mu_1(0)) + \frac{x^2}{2l}\Big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big). \tag{7}$$

As a result of the substitution (6) we obtain the nonhomogeneous equation with respect to the function $\widetilde{w} = \widetilde{w}(x,t)$ with homogeneous initial and boundary conditions:

$$\begin{aligned}
\widetilde{w}_t &= a(t)t^{\beta}\widetilde{w}_{xx} + (b_1(t)x + b_2(t))\widetilde{w}_x + c(x,t)\widetilde{w} + f(x,t) - x\mu_1'(t) - \frac{x^2}{2l}(\mu_2'(t) - \mu_1'(t)) \\
&+ (b_1(t)x + b_2(t))\Big(\varphi'(x) + \mu_1(t) - \mu_1(0) + \frac{x}{l}\big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\big)\Big) \\
&+ c(x,t)\Big(\varphi(x) + x(\mu_1(t) - \mu_1(0)) + \frac{x^2}{2l}\big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\big)\Big) \\
&+ a(t)t^{\beta}\Big(\varphi''(x) + \mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big),
\end{aligned}$$

$$\widetilde{w}(x,0)=0, \quad x\in[0,l], \tag{8}$$

$$\widetilde{w}_x(0,t)=\widetilde{w}_x(l,t)=0, \quad t\in[0,T]. \tag{9}$$

Applying the Green's function $G_2=G_2(x,t,\xi,\tau)$ of the second initial-boundary problem for the heat equation

$$w_t=a(t)t^{\beta}w_{xx}, \tag{10}$$

we reduce the problem (2)-(9) to the equivalent integro-differential equation

$$\widetilde{w}(x,t)=\int_0^t\int_0^l G_2(x,t,\xi,\tau)\Bigg(\big(b_1(\tau)\xi+b_2(\tau)\big)\widetilde{w}_\xi(\xi,\tau)+c(\xi,\tau)\widetilde{w}(\xi,\tau)+f(\xi,\tau)$$
$$+a(\tau)\tau^{\beta}\big(\varphi''(\xi)+\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)\big)-\xi\mu_1'(\tau)-\frac{\xi^2}{2l}\big(\mu_2'(\tau)-\mu_1'(\tau)\big)$$
$$+(b_1(\tau)\xi+b_2(\tau))\bigg(\varphi'(\xi)+\mu_1(\tau)-\mu_2(0)+\frac{\xi}{l}\big(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)\big)\bigg)$$
$$+c(\xi,\tau)\bigg(\varphi(\xi)+\xi(\mu_1(\tau)-\mu_1(0))+\frac{\xi^2}{2l}(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0))\bigg)\Bigg)\,d\xi\,d\tau. \tag{11}$$

The Green's functions of the first ($k=1$) and the second ($k=2$) initial-boundary problems for the equation (10) are defined by the formulas [15, p. 12]

$$G_k(x,t,\xi,\tau)=\frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}}\sum_{n=-\infty}^{+\infty}\left(\exp\left(-\frac{(x-\xi+2nl)^2}{4(\theta(t)-\theta(\tau))}\right)\right.$$
$$\left.+(-1)^k\exp\left(-\frac{(x+\xi+2nl)^2}{4(\theta(t)-\theta(\tau))}\right)\right),\ k=1,2, \tag{12}$$

with $\theta(t)=\int_0^t a(\tau)\tau^{\beta}d\tau$. The estimates

$$\int_0^l|G_k(x,t,\xi,\tau)|d\xi\le 1,\ \int_0^l|G_{kx}(x,t,\xi,\tau)|\,d\xi\le\frac{C_1}{\sqrt{\theta(t)-\theta(\tau)}},\ k=1,2, \tag{13}$$

hold for them ([15, p. 12]), where C_1 is an arbitrary positive constant.

Put $v(x,t)\equiv w_x(x,t)$, $u(x,t)\equiv w_{xx}(x,t)$. Since $G_1(0,t,\xi,\tau)=G_1(l,t,\xi,\tau)=0$, $G_{2x}=-G_{1\xi}$, then, taking into account (6), (11), we replace the problem (1)-(3) by the system of equivalent integral equations

$$w(x,t)=w_0(x,t) \tag{14}$$
$$+\int_0^t\int_0^l G_2(x,t,\xi,\tau)\bigg((b_1(\tau)\xi+b_2(\tau))v(\xi,\tau)+c(\xi,\tau)w(\xi,\tau)+f(\xi,\tau)-\xi\mu_1'(\tau)$$
$$-\frac{\xi^2(\mu_2'(\tau)-\mu_1'(\tau))}{2l}+a(\tau)\tau^{\beta}\bigg(\varphi''(\xi)+\frac{\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)}{l}\bigg)\bigg)\,d\xi\,d\tau,$$

$$v(x,t) = w_{0x}(x,t) \tag{15}$$
$$+ \int_0^t \int_0^l G_1(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))u(\xi,\tau) + (b_1(\tau) + c(\xi,\tau))v(\xi,\tau)$$
$$+ c_\xi(\xi,\tau)w(\xi,\tau) + f_\xi(\xi,\tau) - \mu_1'(\tau) - \frac{\xi(\mu_2'(\tau) - \mu_1'(\tau))}{l} + a(\tau)\tau^\beta\varphi'''(\xi)\Big)\, d\xi\, d\tau,$$
$$u(x,t) = w_{0xx}(x,t) \tag{16}$$
$$+ \int_0^t \int_0^l G_{1x}(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))u(\xi,\tau) + (b_1(\tau) + c(\xi,\tau))v(\xi,\tau)$$
$$+ c_\xi(\xi,\tau)w(\xi,\tau) + f_\xi(\xi,\tau) - \mu_1'(\tau) - \frac{\xi(\mu_2'(\tau) - \mu_1'(\tau))}{l} + a(\tau)\tau^\beta\varphi'''(\xi)\Big)\, d\xi\ d\tau.$$

The equations with respect to functions $b_1 = b_1(t)$, $b_2 = b_2(t)$ we find multiplying (1) by $x^k, k = 0,1$ alternately and integrating them with respect to space variable from 0 to l:

$$b_1(t) = \triangle^{-1}\Bigg(\Big(\mu_3'(t) - a(t)t^\beta(\mu_2(t) - \mu_1(t)) - \int_0^l (c(x,t)w(x,t) + f(x,t))dx\Big)$$
$$\times \big(lw(l,t) - \mu_3(t)\big) - \Big(\mu_4'(t) - a(t)t^\beta(l\mu_2(t) - w(l,t) + w(0,t))$$
$$- \int_0^l x(c(x,t)w(x,t) + f(x,t))dx\Big)\big(w(l,t) - w(0,t)\big)\Bigg), \tag{17}$$
$$b_2(t) = \triangle^{-1}\Bigg(\Big(\mu_4'(t) - a(t)t^\beta(l\mu_2(t) - w(l,t) + w(0,t)) - \int_0^l x(c(x,t)w(x,t)$$
$$+ f(x,t))dx\Big)\big(lw(l,t) - \mu_3(t)\big) - \Big(\mu_3'(t) - a(t)t^\beta(\mu_2(t) - \mu_1(t))$$
$$- \int_0^l (c(x,t)w(x,t) + f(x,t))dx\Big)\big(l^2w(l,t) - 2\mu_4(t)\big)\Bigg), \tag{18}$$

where
$$\triangle(t) = (lw(l,t) - \mu_3(t))^2 - (w(l,t) - w(0,t))(l^2w(l,t) - 2\mu_4(t)). \tag{19}$$

Let us establish the behavior of the integrals on the right hand sides of the formulas (14)-(16). Denote $W(t) = \max\limits_{(x,\tau)\in[0,l]\times[0,t]} |w(x,\tau)|$, $V(t) = \max\limits_{(x,\tau)\in[0,l]\times[0,t]} |v(x,\tau)|$, $U(t) = \max\limits_{(x,\tau)\in[0,l]\times[0,t]} |u(x,\tau)|$, $t \in [0,T]$. Using condition $(A3)$ of the Theorem and (13), from the equations (14)-(18) we obtain

$$W(t) \le C_2 + C_3\int_0^t \Big((|b_1(\tau)| + |b_2(\tau)|)V(\tau) + \tau^\gamma W(\tau)\Big)d\tau,\ t \in [0,T], \tag{20}$$

$$V(t) \le C_4 + C_5 \int\limits_0^t \Big((|b_1(\tau)| + |b_2(\tau)|)U(\tau) + (|b_1(\tau)| + \tau^\gamma)V(\tau) + \tau^\gamma W(\tau) \Big) d\tau, \ t \in [0,T], \tag{21}$$

$$U(t) \le \frac{C_6}{t^{\frac{\beta-1}{2}}} + C_7 \int\limits_0^t \frac{(|b_1(\tau)| + |b_2(\tau)|)U(\tau) + (|b_1(\tau)| + \tau^\gamma)V(\tau) + \tau^\gamma W(\tau)}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}} d\tau, \ t \in (0,T], \tag{22}$$

$$|b_1(t)| \le C_8\big(t^\gamma + t^\beta\big) + C_9\big(t^\gamma + t^\beta\big)W(t) + C_{10}\big(t^\gamma + t^\beta\big)W^2(t), \ t \in [0,T], \tag{23}$$

$$|b_2(t)| \le C_{11}\big(t^\gamma + t^\beta\big) + C_{12}\big(t^\gamma + t^\beta\big)W(t) + C_{13}\big(t^\gamma + t^\beta\big)W^2(t), \ t \in [0,T]. \tag{24}$$

We conclude from the (20)-(24) that the functions $w = w(x,t), v(x,t)$ are continuous in $\overline{Q}_T$, $u = u(x,t)$ has the singularity $t^{\frac{1-\beta}{2}}$ at $t \to 0$ and the functions $b_1 = b_1(t), b_2 = b_2(t)$ tend to zero when $t \to 0$ as the power function t^η with $\eta = \min\{\gamma, \beta\}$. Besides, we note that the integrals on the right hand sides of (14), (15) tend to zero when $t \to 0$. It yields that the sum of all the summands of (15) except the first term of the function $w_{0x} = w_{0x}(x,t)$, is infinitely small when $t \to 0$. It means that we can indicate such number $t_1, 0 < t_1 \le T$ that

$$\Bigg| \mu_1(t) - \mu_1(0) + \frac{x}{l}\Big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big) + \int\limits_0^t \int\limits_0^l G_1(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))u(\xi,\tau) + (b_1(\tau) + c(\xi,\tau))v(\xi,\tau) + c_\xi(\xi,\tau)w(\xi,\tau) + f_\xi(\xi,\tau) - \mu_1'(\tau) - \frac{\xi(\mu_2'(\tau) - \mu_1'(\tau))}{l} + a(\tau)\tau^\beta \varphi'''(\xi)\Big) d\xi\, d\tau \Bigg| \le \frac{\min\limits_{[0,l]} \varphi'(x)}{2}, (x,t) \in \overline{Q}_{t_1}. \tag{25}$$

As a result from the equation (15) we get

$$v(x,t) \ge \frac{\min\limits_{[0,l]} \varphi'(x)}{2} > 0, (x,t) \in \overline{Q}_{t_1}. \tag{26}$$

Since we can rewrite $\triangle(t)$ in a view

$$\triangle(t) = \Big(\int\limits_0^l xv(x,t)dx\Big)^2 - \int\limits_0^l v(x,t)dx \int\limits_0^l x^2 v(x,t)dx = -\frac{1}{2}\int\limits_0^l \int\limits_0^l (y_2 - y_1)^2 v(y_1,t)v(y_2,t)dy_1 dy_2, \tag{27}$$

then

$$\min_{t\in[0,T]} |\triangle(t)| \ge \frac{l^4 \Big(\min\limits_{[0,l]} \varphi'(x)\Big)^2}{48} > 0, (x,t) \in \overline{Q}_{t_1}, \tag{28}$$

that is the condition (A2) of the Theorem ensures the difference from the zero of the denominator $\triangle(t)$ on the segment $[0, t_1]$.

Thus, the problem (1)-(5) is reduced to the system of equations (14)-(18). Under the solution to this system we will understand such set of the functions (w, v, u, b_1, b_2), that $(w, v, u, b_1, b_2) \in (C(\overline{Q}_{t_1}))^2 \times C([0, l] \times (0, t_1]) \times (C[0, t_1])^2$, $|b_1(t)| \leq M_1 t^\eta$, $|b_2(t)| \leq M_2 t^\eta$ and satisfy (14)-(18).

The problem (1)-(5) and the system of equations (14)-(18) are equivalent in the following sense: if a triplet of finctions (b_1, b_2, w) is a local solution to the problem (1)-(5) in $\overline{Q}_{t_1}$, then (w, v, u, b_1, b_2) is a solution to the system of equations (14)-(18) and contrary. The first part of this claim emerges from the way of reduction of system of these equations. We prove that if $(w, v, u, b_1, b_2) \in (C(\overline{Q}_{t_1}))^2 \times C([0, l] \times (0, t_1]) \times (C[0, t_1])^2$, $|b_1(t)| \leq M_1 t^\eta$, $|b_2(t)| \leq M_2 t^\eta$ is a solution to the system of the equations (14)-(18), then (b_1, b_2, w) belong to $(C[0, t_1])^2 \times C^{2,1}(Q_{t_1}) \cap C^{1,0}(\overline{Q}_{t_1})$, and satisfy the conditions (1)-(5) and estimations $|b_1(t)| \leq M_1 t^\eta$, $|b_2(t)| \leq M_2 t^\eta$.

Let us differentiate (15) with respect to x. The right hand side of the expression

$$v_x(x,t) = w_{0xx}(x,t) + \int_0^t \int_0^l G_{1x}(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))u(\xi,\tau) + (b_1(\tau) + c(\xi,\tau))v(\xi,\tau)$$
$$+ c_\xi(\xi,\tau)w(\xi,\tau) + f_\xi(\xi,\tau) - \mu_1'(\tau) - \frac{\xi(\mu_2'(\tau) - \mu_1'(\tau))}{l} + a(\tau)\tau^\beta \varphi'''(\xi) \Big)\, d\xi\, d\tau$$

and the equality (16) coincide, so $u(x,t) \equiv v_x(x,t),\ (x,t) \in [0,l] \times (0, t_1]$. Then we differentiate the equality (14) with respect to x. Using the known properties of the Green's functions we deduce

$$w_x(x,t) = w_{0x}(x,t) + \int_0^t \int_0^l G_1(x,t,\xi,\tau)\Big((b_1(\tau)\xi + b_2(\tau))v_\xi(\xi,\tau) + b_1(\tau)v(\xi,\tau) + c(\xi,\tau)$$
$$\times w_\xi(\xi,\tau) + c_\xi(\xi,\tau)w(\xi,\tau) + f_\xi(\xi,\tau) - \mu_1'(\tau) - \frac{\xi(\mu_2'(\tau) - \mu_1'(\tau))}{l} + a(\tau)\tau^\beta \varphi'''(\xi) \Big)\, d\xi\, d\tau.$$

Subtracting the corresponding parts of the obtained equality and (15), we obtain the homogeneous integral Volterra equation of second kind

$$w_x(x,t) - v(x,t) = \int_0^t \int_0^l G_1(x,t,\xi,\tau)c(\xi,\tau)\big(w_\xi(\xi,\tau) - v(\xi,\tau)\big)\, d\xi\, d\tau.$$

In a virtue of (13) and uniqueness of the solution to these equations it yields that $v(x,t) \equiv w_x(x,t),\ (x,t) \in [0,l] \times (0,t_1]$. Furthermore, taking into account the behavior of the functions $b_1 = b_1(t), b_2 = b_2(t)$, $v(x,t)$ we can state that the products $b_i(t)v(x,t),\ i = 1,2$ are continuous in a rectangle $[0,l] \times [0,t_1]$. Then, considering the equation (14) as integro-differential one with respect to $w = w(x,t)$, we can assert that $w \in C^{2,1}(Q_{t_1}) \cap C^{1,0}(\overline{Q}_{t_1})$ and satisfies (1)-(3).

We multiply the equality (17) by $lw(l,t)-\mu_3(t)$ and (18) by $w(l,t)-w(0,t)$ respectively. Summing up the obtained equalities, we find

$$b_1(t)(lw(l,t)-\mu_3(t))+b_2(t)(w(l,t)-w(0,t))=\mu_3'(t)-a(t)t^{\beta}(\mu_2(t)-\mu_1(t))$$
$$-\int_0^l(c(x,t)w(x,t)+f(x,t))dx.$$

Using (1)-(3), this equality we rewrite in the form

$$b_1(t)\Big(\int_0^l w(x,t)dx-\mu_3(t)\Big)=-\Big(\int_0^l w_t(x,t)dx-\mu_3'(t)\Big).$$

Put $z(t)\equiv\int_0^l w(x,t)dx-\mu_3(t)$. Then $z'(t)=-b_1(t)z(t)$, and respectively $z(t)=z(0)e^{-\int_0^t b_1(\tau)d\tau}$. Since $z(0)=0$ according to the condition $(A4)$ of the Theorem, so $z(t)\equiv 0$, that is the condition (4) is fulfilled.

As a similar way we multiply the equation (17) by $l^2w(l,t)-2\mu_4(t)$, and (18) by $lw(l,t)-\mu_3(t)$. After summing up we find

$$b_1(t)(l^2w(l,t)-2\mu_4(t))+b_2(t)(lw(l,t)-\mu_3(t))=\mu_4'(t)-a(t)t^{\beta}(l\mu_2(t)$$
$$-w(l,t)+w(0,t))-\int_0^l x(c(x,t)w(x,t)+f(x,t))dx.$$

Then the compatibility condition $(A4)$ of the Theorem yields (5). It means that the equivalence of the inverse problem (1)-(5) and the system of equations (14)-(18) is proved.

Denote $p_1(t)=b_1(t)t^{-\eta}, p_2(t)=b_2(t)t^{-\eta}, \widetilde{u}(x,t)=t^{\frac{\beta-1}{2}}u(x,t)$. The system of equations (14)-(18) we represent in the form

$$w(x,t)=w_0(x,t) \tag{29}$$
$$+\int_0^t\int_0^l G_2(x,t,\xi,\tau)\Big((p_1(\tau)\xi+p_2(\tau))\tau^{\eta}v(\xi,\tau)+c(\xi,\tau)w(\xi,\tau)+f(\xi,\tau)-\xi\mu_1'(\tau)$$
$$-\frac{\xi^2(\mu_2'(\tau)-\mu_1'(\tau))}{2l}+a(\tau)\tau^{\beta}\Big(\varphi''(\xi)+\frac{\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)}{l}\Big)\Big)d\xi\,d\tau,$$

$$v(x,t)=w_{0x}(x,t) \tag{30}$$
$$+\int_0^t\int_0^l G_1(x,t,\xi,\tau)\Big((p_1(\tau)\xi+p_2(\tau))\tau^{\eta-\frac{\beta-1}{2}}\widetilde{u}(\xi,\tau)+(p_1(\tau)\tau^{\eta}+c(\xi,\tau))v(\xi,\tau)$$
$$+c_{\xi}(\xi,\tau)w(\xi,\tau)+f_{\xi}(\xi,\tau)-\mu_1'(\tau)-\frac{\xi(\mu_2'(\tau)-\mu_1'(\tau))}{l}+a(\tau)\tau^{\beta}\varphi'''(\xi)\Big)d\xi\,d\tau,$$

$$\widetilde{u}(x,t)=w_{0xx}(x,t) \tag{31}$$

$$+ t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{1x}(x,t,\xi,\tau)\Big((p_1(\tau)\xi + p_2(\tau))\tau^{\eta-\frac{\beta-1}{2}} \widetilde{u}(\xi,\tau) + (p_1(\tau)\tau^{\eta} + c(\xi,\tau))v(\xi,\tau)$$

$$+ c_\xi(\xi,\tau)w(\xi,\tau) + f_\xi(\xi,\tau) - \mu_1'(\tau) - \frac{\xi(\mu_2'(\tau)-\mu_1'(\tau))}{l} + a(\tau)\tau^{\beta}\varphi'''(\xi)\Big)\, d\xi\, d\tau,$$

$$p_1(t) = \triangle^{-1}\tau^{-\eta}\Bigg(\Big(\mu_3'(t) - a(t)t^{\beta}(\mu_2(t)-\mu_1(t)) - \int_0^l (c(x,t)w(x,t) + f(x,t))dx\Big)$$

$$\times \big(lw(l,t) - \mu_3(t)\big) - \Big(\mu_4'(t) - a(t)t^{\beta}(l\mu_2(t) - w(l,t) + w(0,t))$$

$$- \int_0^l x(c(x,t)w(x,t) + f(x,t))dx\Big)\big(w(l,t) - w(0,t)\big)\Bigg), \tag{32}$$

$$p_2(t) = \triangle^{-1}\tau^{-\eta}\Bigg(\Big(\mu_4'(t) - a(t)t^{\beta}(l\mu_2(t) - w(l,t) + w(0,t)) - \int_0^l x(c(x,t)w(x,t)$$

$$+ f(x,t))dx\Big)\big(lw(l,t) - \mu_3(t)\big) - \Big(\mu_3'(t) - a(t)t^{\beta}(\mu_2(t)-\mu_1(t))$$

$$- \int_0^l (c(x,t)w(x,t) + f(x,t))dx\Big)\big(l^2 w(l,t) - 2\mu_4(t)\big)\Bigg), \tag{33}$$

where $\triangle(t)$ is defined by the formula (19). Note, that this system is considered in $\overline{Q}_{t_1}$, so the difference from zero of the denominators in the formulas (32), (33) is argued in (28).

We represent the system of equations (29)-(33) as an operator equation

$$\omega = P\omega, \tag{34}$$

where $\omega = (w, v, \widetilde{u}, p_1, p_2)$ and the operator $P = (P_1, P_2, P_3, P_4, P_5)$ is defined y the right hand sides of equations (29)-(33) respectively.

Assume that $|w(x,t)| \le M_3$, $0 < M_4 \le v(x,t) \le M_5$, $|\widetilde{u}(x,t)| \le M_6, (x,t) \in \overline{Q}_{t_1}$, where M_3, M_4, M_5, M_6 are some positive constants. We will define them below. Using these estimates and (28) in (32)-(33), we find

$$|P_4\omega| \le \frac{C_{14}(t^{\gamma-\eta} + t^{\beta-\eta})(1 + M_3 + M_3^2)}{\min\limits_{t\in[0,t_1]} |\triangle(t)|} \equiv M_1,\ t \in [0, t_1], \tag{35}$$

$$|P_5\omega| \le \frac{C_{15}(t^{\gamma-\eta} + t^{\beta-\eta})(1 + M_3 + M_3^2)}{\min\limits_{t\in[0,t_1]} |\triangle(t)|} \equiv M_2,\ t \in [0, t_1], \tag{36}$$

where the numbers C_{14}, C_{15} are determined by the input data.

Let us consider the equations (29)-(31). Taking into account (35), (36), we obtain

$$|P_1\omega| \le \Big|\int_0^t \int_0^l G_2(x,t,\xi,\tau)\Big((M_1 l + M_2)t^{\eta} M_5 + \max_{(x,t)\in\overline{Q}_{t_1}} |c(\xi,\tau)| M_3$$

$$+\max_{(\xi,\tau)\in\overline{Q}_{t_1}}\left|f(\xi,\tau)-\xi\mu_1'(\tau)+a(\tau)\tau^{\beta}\left(\varphi''(\xi)+\frac{\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)}{l}\right)\right.$$

$$\left.\left.-\frac{\xi^2(\mu_2'(\tau)-\mu_1'(\tau))}{2l}\right|\right)d\xi\,d\tau\Bigg|+\max_{(x,t)\in\overline{Q}_{t_1}}|w_0(x,t)|$$

$$\le C_{16}t^{\eta+1}+C_{17}t^{\gamma+1}+C_{18}t+\max_{(x,t)\in\overline{Q}_{t_1}}|w_0(x,t)|, \tag{37}$$

$$P_2\omega\ge\min_{x\in[0,l]}\varphi'(x)+\min_{(x,t)\in\overline{Q}_{t_1}}\Bigg|\mu_1(t)-\mu_1(0)+\frac{x}{l}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big)$$

$$+\int_0^t\int_0^l G_1(x,t,\xi,\tau)\Big((p_1(\tau)\xi+p_2(\tau))\tau^{\eta-\frac{\beta-1}{2}}\widetilde{u}(\xi,\tau)+(p_1(\tau)\tau^{\eta}+c(\xi,\tau))v(\xi,\tau)$$

$$+c_\xi(\xi,\tau)w(\xi,\tau)+f_\xi(\xi,\tau)-\mu_1'(\tau)-\frac{\xi(\mu_2'(\tau)-\mu_1'(\tau))}{l}+a(\tau)\tau^{\beta}\varphi'''(\xi)\Big)\,d\xi\,d\tau\Bigg|, \tag{38}$$

$$P_2\omega\le\max_{x\in[0,l]}\varphi'(x)+\max_{(x,t)\in\overline{Q}_{t_1}}\Bigg|\mu_1(t)-\mu_1(0)+\frac{x}{l}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big)$$

$$+\int_0^t\int_0^l G_1(x,t,\xi,\tau)\Big((p_1(\tau)\xi+p_2(\tau))\tau^{\eta-\frac{\beta-1}{2}}\widetilde{u}(\xi,\tau)+(p_1(\tau)\tau^{\eta}+c(\xi,\tau))v(\xi,\tau)$$

$$+c_\xi(\xi,\tau)w(\xi,\tau)+f_\xi(\xi,\tau)-\mu_1'(\tau)-\frac{\xi(\mu_2'(\tau)-\mu_1'(\tau))}{l}+a(\tau)\tau^{\beta}\varphi'''(\xi)\Big)\,d\xi\,d\tau\Bigg|, \tag{39}$$

$$|P_3\omega|\le\Bigg|t^{\frac{\beta-1}{2}}\int_0^t\int_0^l G_{1x}(x,t,\xi,\tau)\Big((M_1l+M_2)t^{\eta-\frac{\beta-1}{2}}M_6+\big(M_1\tau^{\eta}+\max_{(x,t)\in\overline{Q}_{t_1}}|c(\xi,\tau)|\big)M_5$$

$$+\max_{(x,t)\in\overline{Q}_{t_1}}|c_\xi(\xi,\tau)|M_3+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|f(\xi,\tau)-\mu_1'(\tau)-\frac{\xi}{l}(\mu_2'(\tau)-\mu_1'(\tau))$$

$$+a(\tau)\tau^{\beta}\varphi''(\xi)\Big|\Big)\,d\xi\,d\tau\Bigg|+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|t^{\frac{\beta-1}{2}}w_{0xx}(x,t)\Big|$$

$$\le C_{19}t^{\eta-\frac{\beta-1}{2}}+C_{20}t^{\gamma}+C_{21}t^{\eta}+C_{22}+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|t^{\frac{\beta-1}{2}}w_{0xx}(x,t)\Big|. \tag{40}$$

Now we estimate the expresion

$$\Bigg|\int_0^t\int_0^l G_1(x,t,\xi,\tau)\Big((p_1(\tau)\xi+p_2(\tau))\tau^{\eta-\frac{\beta-1}{2}}\widetilde{u}(\xi,\tau)+(p_1(\tau)\tau^{\eta}+c(\xi,\tau))v(\xi,\tau)$$

$$+c_\xi(\xi,\tau)w(\xi,\tau)+f_\xi(\xi,\tau)-\mu_1'(\tau)-\frac{\xi(\mu_2'(\tau)-\mu_1'(\tau))}{l}+a(\tau)\tau^{\beta}\varphi'''(\xi)\Big)\,d\xi\,d\tau$$

$$+\mu_1(t)-\mu_1(0)+\frac{x}{l}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big)\Bigg|$$

$$\le\Bigg|\int_0^t\int_0^l G_1(x,t,\xi,\tau)\Big((M_1l+M_2)t^{\eta-\frac{\beta-1}{2}}M_6+\big(M_1\tau^{\eta}+\max_{(x,t)\in\overline{Q}_{t_1}}|c(\xi,\tau)|\big)M_5$$

$$+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|c_\xi(\xi,\tau)|M_3+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|f(\xi,\tau)-\mu_1'(\tau)-\frac{\xi}{l}(\mu_2'(\tau)-\mu_1'(\tau))$$
$$+a(\tau)\tau^\beta\varphi''(\xi)\Big|\Big)\,d\xi\,d\tau\Big|+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|\frac{x}{l}\Big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big)\Big|$$
$$+\max_{t\in[0,t_1]}\big|\mu_1(t)-\mu_1(0)\big|\le C_{23}t^{\eta-\frac{\beta-3}{2}}+C_{24}t^{\eta+1}+C_{25}t^{\gamma+1}+C_{26}t. \tag{41}$$

Choose the constants M_3-M_6 such that

$$M_3>\max_{(x,t)\in\overline{Q}_{t_1}}|w_0(x,t)|,\ M_4=\frac{1}{2}\min_{x\in[0,l]}\varphi'(x),\ M_5=\max_{x\in[0,l]}\varphi'(x)+\frac{1}{2}\min_{x\in[0,l]}\varphi'(x),$$

$$M_6>C_{22}+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|t^{\frac{\beta-1}{2}}w_{0xx}(x,t)\Big|.$$

Fix the number $T_0, 0<T_0\le t_1$ ia a such way

$$C_{16}T_0^{\eta+1}+C_{17}T_0^{\gamma+1}+C_{18}T_0+\max_{(x,t)\in\overline{Q}_{t_1}}|w_0(x,t)|\le M_3, \tag{42}$$

$$C_{19}T_0^{\eta-\frac{\beta-1}{2}}+C_{20}T_0^{\gamma}+C_{21}T_0^{\eta}+C_{22}+\max_{(x,t)\in\overline{Q}_{t_1}}\Big|t^{\frac{\beta-1}{2}}w_{0xx}(x,t)\Big|\le M_6, \tag{43}$$

$$C_{23}T_0^{\eta-\frac{\beta-3}{2}}+C_{24}T_0^{\eta+1}+C_{25}T_0^{\gamma+1}+C_{26}T_0\le\frac{1}{2}\min_{x\in[0,l]}\varphi'(x). \tag{44}$$

As a result we obtain

$$|P_1\omega|\le M_3,\ 0<M_4\le P_2\omega\le M_5,\ |P_3\omega|\le M_6,\ (x,t)\in\overline{Q}_{T_0}. \tag{45}$$

We consider the operator equation (34) on a convex closed set $N\equiv\{(w,v,\widetilde{u},p_1,p_2)\in(C(\overline{Q}_{T_0}))^3\times(C[0,T_0])^2:\ |w(x,t)|\le M_3, 0<M_4\le v(x,t)\le M_5, |\widetilde{u}(x,t)|\le M_6, |p_1(t)|\le M_1, |p_2(t)|\le M_2\}$ in the Banach space $\mathcal{B}\equiv(C(\overline{Q}_{T_0}))^3\times(C[0,T_0])^2$. The estimates (35), (36), (45) guarantee that the operator P maps the set N into itself. To prove the compactness of the operator P on the set N we apply the Arzela-Ascoli theorem. For this aim we have to show that the set PN is uniformly bounded and equicontinuous. The latter means that

$$\forall\varepsilon\ \exists\varrho:|P\omega(x_2,t_2)-P\omega(x_1,t_1)|<\varepsilon$$

for all $|x_2-x_1|<\varrho,\ |t_2-t_1|<\varrho,\ \omega(x,t)\in N$.

Let us consider the operator $P_4\omega(t)$. Using (32) we represent it in the form

$$P_4\omega=\frac{F(t)}{\triangle(t)t^\delta},$$

where $\triangle(t)$ is defined by (19) and

$$F(t)\equiv\Big(\mu_3'(t)-a(t)t^\beta(\mu_2(t)-\mu_1(t))-\int_0^l(c(x,t)w(x,t)+f(x,t))dx\Big)$$

$$\times \big(lw(l,t) - \mu_3(t)\big) - \Bigg(\mu_4'(t) - a(t)t^{\beta}(l\mu_2(t) - w(l,t) + w(0,t))$$

$$- \int_0^l x(c(x,t)w(x,t) + f(x,t))dx\Bigg)\big(w(l,t) - w(0,t)\big).$$

Taking into account the conditions of the Theorem and the definition of the set N we deduce that $F(t)$ is continuous on $[0, T_0]$ and $F(t) \le C_{27} t^{\delta}$. The constant C_{27} in last inequality is determined by the input data.

Fix an arbitrary number $\varepsilon > 0$. Since $\lim\limits_{t\to 0} \frac{F(t)}{\triangle(t)t^{\delta}} = \kappa_1$, so we can indicate such number t^*, that

$$\left|\frac{F(t)}{\triangle(t)t^{\delta}} - \kappa_1\right| < \frac{\varepsilon}{2}$$

for $0 < t \le t^*$. As a result we obtain

$$|P_4\omega(t_2) - P_4\omega(t_1)| \le |P_4\omega(t_2) - \kappa_2| + |\kappa_2 - P_4\omega(t_1)| < \varepsilon$$

for $0 < t_1, t_2 \le t^*$.

For the case $t_1, t_2 > t^*$ we find

$$|P_4\omega(t_2) - P_4\omega(t_1)| \le \left|\frac{F(t_1)}{t_1^{\delta}}\left(\frac{1}{\triangle(t_1)} - \frac{1}{\triangle(t_2)}\right)\right| + \left|\frac{F(t_1)}{\triangle(t_2)}\left(\frac{1}{t_1^{\delta}} - \frac{1}{t_2^{\delta}}\right)\right| + \left|\frac{F(t_1) - F(t_2)}{\triangle(t_2)t_2^{\delta}}\right|$$

$$\le \frac{|F(t_1)||\triangle(t_2) - \triangle(t_1)|}{(t^*)^{\delta}\big(\min\limits_{\tau\in[0,T_0]} |\triangle(\tau)|\big)^2} + \frac{|F(t_1)||t_1^{\delta} - t_2^{\delta}|}{(t^*)^{2\delta}\big(\min\limits_{\tau\in[0,T_0]} |\triangle(\tau)|\big)} + \frac{|F(t_1) - F(t_2)|}{(t^*)^{\delta}\big(\min\limits_{\tau\in[0,T_0]} |\triangle(\tau)|\big)}.$$

In a virtue of continuity of input data and the mean value theorem we conclude

$$|P_4\omega(t_2) - P_4\omega(t_1)| \le \varepsilon,\ |t_2 - t_1| < \varrho.$$

The case $t_1 < t^*, t_2 > t^*$ combines two previous ones because

$$|P_4\omega(t_2) - P_4\omega(t_1)| \le |P_4\omega(t_2) - P_4\omega(t^*)| + |P_4\omega(t^*) - P_4\omega(t_1)|.$$

We prove the equicontinuous of the set P_5N in a similar way.

The compactness of operators $P_1 - P_3$, whose kernels are Green's functions can be proved according to the scheme given in [15, p. 27] adapted to the case of strong degeneration [16]. Applying the Schauder Fixed Point Theorem we state that there exists the solution to the system of equations (29)-(33) in $\overline{Q}_{T_0}$ and therefore to inverse problem (1)-(5) in $\overline{Q}_{T_0}$.

3 Uniqueness of the solution

Suppose that the system of equations (29)-(33) has two solutions $(w_i, v_i, \widetilde{u}_i, p_{1i}, p_{2i}), i = 1, 2$. Denote $w(x,t) = w_1(x,t) - w_2(x,t), v(x,t) = v_1(x,t) - v_2(x,t), \widetilde{u}(x,t) = \widetilde{u}_1(x,t) - \widetilde{u}_2(x,t),\ p_1(t) = p_{11}(t) - p_{12}(t), p_2(t) = p_{21}(t) - p_{22}(t)$. Using (29)-(33), we find

$$w(x,t) = \int_0^t \int_0^l G_2(x,t,\xi,\tau)\Bigg((p_{11}(\tau)\xi + p_{21}(\tau))\tau^{\beta} v(\xi,\tau)$$

$$+ (p_1(\tau)\xi + p_2(\tau))\tau^{\beta} v_2(\xi,\tau) + c(\xi,\tau)w(\xi,\tau)\Big) d\xi\, d\tau,\ (x,t) \in \overline{Q}_{T_0}, \quad (46)$$

$$v(x,t) = \int_0^t \int_0^l G_1(x,t,\xi,\tau)\Big((p_{11}(\tau)\xi + p_{21}(\tau))\tau^{\eta - \frac{\beta-1}{2}} \widetilde{u}(\xi,\tau)$$

$$+ (p_1(\tau)\xi + p_2(\tau))\tau^{\eta - \frac{\beta-1}{2}} \widetilde{u}_2(\xi,\tau) + (p_{11}(\tau)\tau^{\eta} + c(\xi,\tau))v(\xi,\tau)$$

$$+ p_1(\tau)\tau^{\eta} v_2(\xi,\tau) + c_{\xi}(\xi,\tau)w(\xi,\tau)\Big) d\xi\, d\tau,\ (x,t) \in \overline{Q}_{T_0}, \quad (47)$$

$$\widetilde{u}(x,t) = t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{1x}(x,t,\xi,\tau)\Big((p_{11}(\tau)\xi + p_{21}(\tau))\tau^{\eta - \frac{\beta-1}{2}} \widetilde{u}(\xi,\tau)$$

$$+ (p_1(\tau)\xi + p_2(\tau))\tau^{\eta - \frac{\beta-1}{2}} \widetilde{u}_2(\xi,\tau) + (p_{11}(\tau)\tau^{\eta} + c(\xi,\tau))v(\xi,\tau)$$

$$+ p_1(\tau)\tau^{\eta} v_2(\xi,\tau) + c_{\xi}(\xi,\tau)w(\xi,\tau)\Big) d\xi\, d\tau,\ (x,t) \in \overline{Q}_{T_0}, \quad (48)$$

$$p_1(t) = \triangle^{-1} t^{-\eta}\Bigg(\Big(\mu_3'(t) - a(t)t^{\beta}(\mu_2(t) - \mu_1(t)) - \int_0^l (c(x,t)w_1(x,t) + f(x,t))dx\Big)$$

$$\times lw(x,t) - \big(lw_2(l,t) - \mu_3(t)\big) \int_0^l c(x,t)w(x,t)dx - \Big(\mu_4'(t) - a(t)t^{\beta}(l\mu_2(t)$$

$$- w_1(l,t) + w_1(0,t)) - \int_0^l x(c(x,t)w_1(x,t) + f(x,t))dx\Big)\big(w(l,t) - w(0,t)\big)$$

$$- \Big(a(t)t^{\beta}(w(l,t) - w(0,t)) - \int_0^l xc(x,t)w(x,t)dx\Big)\big(w_2(l,t) - w_2(0,t)\big)\Bigg),\ t \in [0, T_0], \quad (49)$$

$$p_2(t) = \triangle^{-1} t^{-\eta}\Bigg(\Big(\mu_4'(t) - a(t)t^{\beta}(l\mu_2(t) - w_1(l,t) + w_1(0,t))$$

$$- \int_0^l x(c(x,t)w_1(x,t) + f(x,t))dx\Big) lw(l,t) + \Big(a(t)t^{\beta}(w(l,t) - w(0,t))$$

$$- \int_0^l xc(x,t)w(x,t)dx\Big)\big(lw_2(l,t) - \mu_3(t)\big) - \Big(\mu_3'(t) - a(t)t^{\beta}(\mu_2(t) - \mu_1(t))$$

$$- \int_0^l (c(x,t)w_1(x,t) + f(x,t))dx\Big) l^2 w(l,t) - \big(l^2 w_2(l,t) - 2\mu_4(t)\big)$$

$$\times \int_0^l c(x,t)w(x,t)dx\Bigg),\ t \in [0, T_0]. \quad (50)$$

Substituting (49)-(50) into (46)-(48), we obtain the system of homogeneous integral

Voltera equations of second kind with respect to unknowns $w = w(x,t), v = v(x,t)$, $\widetilde{u} = \widetilde{u}(x,t)$ for every $x \in [0,l]$:

$$w(x,t) = \int_0^t (K_{11}(t,\tau)v(x,\tau) + K_{12}(t,\tau)w(x,\tau))d\tau,\ t \in [0,T_0], \tag{51}$$

$$v(x,t) = \int_0^t (K_{21}(t,\tau)\widetilde{u}(x,\tau) + K_{22}(t,\tau)v(x,\tau) + K_{23}(t,\tau)w(x,\tau))d\tau,\ t \in [0,T_0], \tag{52}$$

$$\widetilde{u}(x,t) = \int_0^t (K_{31}(t,\tau)\widetilde{u}(x,\tau) + K_{32}(t,\tau)v(x,\tau) + K_{33}(t,\tau)w(x,\tau))d\tau,\ t \in [0,T_0]. \tag{53}$$

Taking into account (13), (45), we can state that the kernels of this system has integrable singularities. It means that the system has only trivial solution

$$w(x,t) \equiv 0,\ v(x,t) \equiv 0,\ \widetilde{u}(x,t) \equiv 0,\ (x,t) \in \overline{Q}_{T_0}. \tag{54}$$

Substituting (54) into (49), (50), we find

$$p_1(t) \equiv 0,\ p_2(t) \equiv 0,\ t \in [0,T_0]. \tag{55}$$

It completes the proof of the Theorem.

4 Conclusions

In the paper it is investigated the inverse problem of determination of two time-dependent functions in a first order polynomial with respect to space variable. It is a minor coefficient in a parabolic equation with strong power degeneration.

1. It is established conditions of existence and uniqueness of the local solution to the named problem.

2. It is proved that for the case of strong degeneration for parabolic equation with Neumann boundary conditions the unknown function $w = w(x,t)$ and its first derivative with respect to space variable are continuous in $\overline{Q}_{T_0}$. The second derivative of this function has the singularity $t^{\frac{1-\beta}{2}}$ at $t \to 0$ unlike both cases of weak degeneration when all these functions are continuous in $\overline{Q}_{T_0}$ and strong degeneration of parabolic equation with Dirichlet boundary conditions when the first derivative of unknown function behaves as $t^{\frac{1-\beta}{2}}$ when $t \to 0$.

3. It is established that the unknown functions $b_1 = b_1(t)$, $b_2 = b_2(t)$ behave at $t \to 0$ as t^{δ} with $\delta = \min\{\gamma, \beta\}$ in contrast to the case of strong degeneration for parabolic equation with Dirichlet boundary conditions when $\delta = \min\{\gamma, \frac{\beta+1}{2}\}$ with $\gamma > \frac{\beta-1}{2}$.

4. The system of equations (29)-(33) which is obtained in the paper can served the base for application some numerical methods for construction the approximate solutions to the named problem.

5. Results of this paper can be used in research of inverse problems of identification the younger coefficients in parabolic equation which depend on both space and time variables. Besides, they are the first step in investigation parabolic equations with general strong degeneration or multidimensional degenerate parabolic equations.

5 An example

It can be shown by direct calculation that a triplet of function $b_1(t) = b_2(t) = (\beta+1)t^\beta, w(x,t) = (x+1)e^{t^{\beta+1}}$ is the solution to the inverse problem

$$w_t = t^\beta w_{xx} + (b_1(t)x + b_2(t))w_x,\ (x,t) \in (0,1)\times(0,T),$$

$$w(x,0) = x+1,\ x\in[0,1],$$

$$w_x(0,t) = w_x(1,t) = e^{t^{\beta+1}},\ t\in[0,T],$$

$$\int_0^1 w(x,t)dx = \frac{3}{2}e^{t^{\beta+1}},\ t\in[0,T],$$

$$\int_0^1 xw(x,t)dx = \frac{5}{6}e^{t^{\beta+1}},\ t\in[0,T].$$

The input data of this problem satisfy the requirements of the Theorem given in the paper.

References

[1] Azari H., Li C., Nie Y. and Shang S. *Determination of an unknown coefficient in a parabolic inverse problem.* Dynamics of Continuos, Discrete and Impulsive Systems. Series A: Math. Analysis 2004, **11**, 665-674. doi: 124328203.

[2] Brodyak O., Huzyk N. *Coefficient inverse problems for parabolic equation with general weak degeneartion.* Bukovinian Math.Journal 2021, **9** (1), 91-106. doi: 10.31861/bmj2021.01.08. (in Ukrainian)

[3] Cannarsa Piermarco, Doubova Anna and Yamamoto Masahiro *Inverse problem of reconstruction of degenerate diffusion coefficient in a parabolic equation.* Inverse problems 2021, **37** (12), 1-39. doi: arXiv:2106.06832.

[4] Cannon J.R. and Peres-Esteva S. *Determination of the coefficient of u_x in a linear parabolic equation.* Inverse Problems 1994, **10** (3), 521-531. doi: 10.1088/0266-5611/10/3/002.

[5] Cannon J.R. and Rundell W. *Recovering a time-dependent coefficient in a parabolic differential equation.* J. Math. Anal. Appl. 1991, **160**, 572-582. doi: https://core.ac.uk/reader/82169960.

[6] Deng Zui-Cha, Yang Liu. *An inverse problem of identifying the coefficient of first order in a degenerate parabolic equation.* J. of Computational and Applied Math 2011, **235**, 4407-4417. doi: arXiv:1309.7421.

[7] Hazanee A. and Lesnic D. *Determination of a time-dependent coefficient in the bioheat eqaution.* International Journal of Mechanical Sciences 2014, **88**, 259–266. doi: 10.1016/j.ijmecsci.2014.05.017.

[8] Huntul M.J. and Lesnic D. *Determination of Time-Dependent Coefficients for a Weakly Degenerate Heat Equation.* Computer Modeling in Engineering and Sciences 2020, **123** (2), 475-494. https://doi.org/10.32604/cmes.2020.08791.

[9] Hussein M.S. and Lesnic D. *Identification of the time-dependent conductivity of an inhomogeneous diffusive material.* Applied Mathematics and Computation 2015, **269**, 35–58. doi: 10.1016/j.amc.2015.07.039.

[10] Huzyk N. *Inverse problem of determining the coefficients in a degenerate parabolic equation.* Electronic Journal of Differential Equations 2014, **172**, 1-11. doi: 10.7153/dea-2021-13-14.

[11] Huzyk N. *Inverse free boundary problems for a generally degenerate parabolic equation.* Journal of Inverse and Ill-Posed Problems 2015, **23**, 103-119. doi: 10.1515/jiip-2011-0016.

[12] Huzyk N. *Determination of the lower coefficient in a parabolic equation with strong degeneration.* Ukrainian Mathematical Journal 2016, **68** (7), 1049-1061. doi:10.1007/s11253-016-1276-4 (Translation from Ukrains'kyi Matematychnyi Zhurnal 2006, **68** (7), 922-932, https://umj.imath.kiev.ua/index.php/umj/article/view/1886. (in Ukrainian))

[13] Huzyk N. *Coefficient inverse problem for the degenerate parabolic equation.* Differential Equations and Applications 2021, **13** (3), 243-255. doi: 10.7153/dea-2021-13-14.

[14] Huzyk N.M., Pukach P.Y., Vovk M.I. *Coefficient inverse problem for the strongly degenerate parabolic equation.* Carpathian Math. Publ. 2023, **15** (1), 52-65. doi:10.15330/cmp.15.1.52-65.

[15] Ivanchov M. Inverse problems for equations of parabolic type. VNTL Publishers, Lviv, 2003.

[16] Ivanchov M.I., Saldina N.V. *Inverse problem for a parabolic equation with strong power degeneration.* Ukr Math J 2006, **58** (11), 1685–1703. doi: 10.1007/s11253-006-0162-x (Translation from Ukrains'kyi Matematychnyi Zhurnal 2006, **58** (11), 1487–1500. (in Ukrainian))

[17] Jones B.F. *The determination of a coefficient in a parabolic differential equation. I. Existence and uniqueness.* J. Math. Mech. 1962, **11** (5), 907-918. doi: http://www.jstor.org/stable/24900744.

[18] Lesnic D., Yousefi S.A., and Ivanchov M.*Determination of a timedependent diffusivity from nonlocal conditions.* Journal of Applied Mathematics and Computing 2013, **41** (1-2), 301–320. doi: 10.1007/s12190-012-0606-4.

[19] Lowe Bruce D. and Rundell William *The determination of a coefficient in a parabolic equation from input sources.* Journal of Applied Mathematics 1994, **52** (1), 31-50. doi: 10.1093/imamat/52.1.31.

[20] Nguyen Duc Phuong, Dumitru Baleanu, Tran Thanh Phong, Le Dinh Long. *Recovering the source term for parabolic equation with nonlocal integral condition.* Mathematical Methods in Applied Sciences 2021, **44** (11), 9026-9041. doi: 10.1002/mma.7331.

[21] Pabyrivska N. and Pabyrivskyy V. *On the determination of an unknown source in the parabolic equation.* Math modeling and computing 2017, **4** (2), 171-176. doi: 10.23939/mmc2017.02.171

[22] Ranran Li and Zhiyuan Li. *Identifying unknown source in degenerate parabolic equation from final observation.* Inverse Problems in Science and Engineering 2021, **29** (7), 1012-1031. doi: 10.1080/17415977.2020.1817005.

[23] Saldina N. *Inverse problem for parabolic equation with weak degeneration.* Mathematical Methods And Physicoмechanical Fields 2006, **49** (3), 7-17. (in Ukrainian)

[24] Trong D.D. and Ang D.D.*Coefficient identification for a parabolic equation.* Inverse Problems 1994, **10** (3), 733-752. doi: 10.1088/0266-5611/10/3/015.

Received 21.08.2024

Гузик Н.М., Бродяк О.Я. *Коефіцієнтна обернена задача для параболічного рівняння з сильним степеневим виродженням* // Буковинський матем. журнал — 2024. — Т.12, №2. — С. 10–26.

В області з відомими межами досліджується обернена задача для параболічного рівняння з сильним виродженням. Виродження рівняння спричинене степеневою функцією від часу при старшій похідній невідомої функції. Відомо, що молодший коефіцієнт рівняння є поліномом першого степеня за просторовою змінною з двома невідомими коефіцієнтами від часу. Задано крайові умови другого роду та значення теплових моментів у якості умов перевизначення. Встановлено умови існування та єдиності класичного розв'язку вказаної задачі.