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UNIFORMLY CONTINUOUS MAPPINGS ON PREMETRIC SPACES

We study the notion of uniformly continuous mapping between quasi-metric spaces and construct an example of the topological homeomorphism between two compact Hausdorff partially metric spaces such that the corresponding mapping between quasi-metric spaces is not uniformly continuous. This example shows, in particular, that Theorem 4.4 from [6] is not true. In addition, we prove an analogue of the classical Heine-Cantor theorem on the uniform continuity of any continuous mapping $f : X \to Y$ between a premetric space X, which satisfies a strengthened condition of the countable compactness, and a uniform space Y. We also give an example of a continuous mapping $f : X \to Y$ between a compact Hausdorff premetric space X and a uniform space Y, which is not uniformly continuous.

Key words and phrases: continuous mapping, uniformly continuous mapping, metric space, partial metric spaces, quasi-metric spaces, premetric space, uniform space.

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Introduction

According to the classical Heine-Cantor theorem, for any compact metric space (X, d) and any metric space (Y, ϱ) every continuous mapping $f : X \to Y$ is uniformly continuous [3, Theorem 4.3.32]. It is well known that an arbitrary metric d on a set X induces the uniformity $\mathcal{U}_d$ on X, which consists of all sets U for which there exists a number $\varepsilon > 0$ such that

$$\{(x, y) \in X^2 : d(x, y) < \varepsilon\} \subseteq U.$$

Moreover, for any metric spaces (X, d) and (Y, ϱ) the uniform continuity of a mapping $f : (X, d) \to (Y, \varrho)$ is equivalent to the uniform continuity of the corresponding mapping $f : (X, \mathcal{U}_d) \to (Y, \mathcal{U}_\varrho)$ (see, for example, [3, Exercise 8.1.A]).

On the other hand, for every compact Hausdorff space X there exists exactly one uniformity $\mathcal{U}$ on X which is compatible with the topology of X. This uniformity $\mathcal{U}$ consists of all neighbourhoods U of the diagonal $\Delta = \{(x, x) : x \in X\}$ in X^2 (see [1, Chapter II, § 4, Theorem 1]). So, the following theorem (see [1, Chapter II, § 4, Theorem 2]) is a generalization of the Heine-Cantor theorem.

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Theorem 1. *Every continuous mapping from a compact Hausdorff space X to a uniform space $(Y,\mathcal{U})$ is uniformly continuous.*

Notice that the metric version of the concept of uniformly continuous mapping can be naturally adapted to more general classes of spaces: quasi-metric, quasi-pseudometric and premetric. Since the corresponding metric analogues do not possess the symmetry property, the study of the uniform continuity of mappings between such spaces cannot be reduced to the consideration of uniform spaces. Therefore, analogs of Theorem 1 for mappings between spaces from such classes require separate study and are of independent interest.

The paper [6, Theorem 4.4] contains the following result (see Section 1 for corresponding definitions and denotations).

Theorem 2. *Let $f : (X,p_1) \to (Y,p_2)$ be a continuous mapping from a compact partial metric space (X,p_1) to a partial metric space (Y,p_2). Then f is uniformly continuous as mapping between the quasi-metric spaces (X,q_{p_1}) and (Y,q_{p_2}).*

In this article, we study the notion of uniformly continuous mapping between quasi-metric spaces and construct an example of the topological homeomorphism between two compact Hausdorff partially metric spaces such that the corresponding mapping between quasi-metric spaces is not uniformly continuous. This example shows, in particular, that Theorem 2 is not true. In addition, we prove an analogue of Theorem 1 on the uniform continuity of any continuous mapping $f : X \to Y$ between a premetric space X, which satisfies a strengthened condition of the countable compactness, and a uniform space Y. We also give an example of a continuous mapping $f : X \to Y$ between a compact Hausdorff premetric space X and a uniform space Y, which is not uniformly continuous.

1 Basic notions and denotations

A function $q : X^2 \to [0,+\infty)$ is called *a quasi-metric* on X (see [7]) if

(q_1) $q(x,x) = 0$;

(q_2) $q(x,z) \le q(x,y) + q(y,z)$;

(q_3) $x = y \Leftrightarrow q(x,y) = q(y,x) = 0$

for all $x,y,z \in X$.

Every quasi-metric q on X induces a *conjugate* quasi-metric $q^{-1} : X^2 \to \mathbb{R}$ defined by $q^{-1}(x,y) = q(y,x)$ for every $x,y \in X$. Moreover, the function $d_q = q + q^{-1}$ is a metric on X.

Let (X,q) be a quasi-metric space. For every $x \in X$ the balls

$$B_q(x,\varepsilon) = \{y \in X : q(x,y) < \varepsilon\}, \quad \varepsilon > 0$$

form a base of the *quasi-metric topology* τ_q at the point x.

A function $p : X^2 \to [0,+\infty)$ is called *a partial metric* on X (see [7]) if

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(p_2) $p(x,x) \leq p(x,y)$;

(p_3) $p(x,y) = p(y,x)$;

(p_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

for all $x, y, z \in X$.

For any partial metric $p : X^2 \to [0, +\infty)$ the function $q_p : X^2 \to \mathbb{R}$,

$$q_p(x,y) = p(x,y) - p(x,x),$$

is a quasi-metric on X and the topology of the partial metric space (X, p) is the topology of the quasi-metric space (X, q_p) (see [7, Theorem 4.1]). Moreover, the function $d_p : X^2 \to \mathbb{R}$,

$$d_p(x,y) = d_{q_p}(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

For any partial metric space (X, p) we have that p is a metric on X if and only if $p(x,x) = 0$ for every $x \in X$. Moreover, $q_p = p$ and $d_p = 2p$ if p is a metric.

Let X be a nonempty set and $\Delta = \{(x,x) : x \in X\}$. A system $\mathcal{U} \subseteq 2^{X^2}$ is called a *uniformity* on X if it satisfies the following conditions:

(U_1) $\Delta \subseteq U$ for every $U \in \mathcal{U}$;

(U_2) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X^2$ then $V \in \mathcal{U}$;

(U_3) $U \cap V \in \mathcal{U}$ for every $U, V \in \mathcal{U}$;

(U_4) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$V \circ V = \{(x,z) : (\exists y \in X)\,((x,y),(y,z) \in V)\} \subseteq U;$$

(U_5) $U^{-1} = \{(x,y) : (y,x) \in U\} \in \mathcal{U}$ for every $U \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a *uniform* space and an element $U \in \mathcal{U}$ is called an *entourage*.

Let $(X, \mathcal{U})$ be a uniform space. For every $x \in X$ the sets

$$U[x] = \{y \in X : (x,y) \in U\}, \quad U \in \mathcal{U}$$

form the system of all neighbourhoods of x in some topology $\mathcal{T}_{\mathcal{U}}$. This topology is called the *topology induced by* $\mathcal{U}$ (see [1, Chapter II, § 1, Proposition 1 and Definition 3]). In particular, for a metric space (X, d) and corresponding uniformity $\mathcal{U}_d$ on X the topology $\mathcal{T}_{\mathcal{U}_d}$ coincides with the topology generated by d.

Let X be a topological space, $\mathcal{T}$ be the topology of X and $\mathcal{U}$ be a uniformity on the set X. We say that $\mathcal{U}$ is *compatible* with $\mathcal{T}$ if $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$.

Let X be a topological space. A point $x \in X$ is called a *cluster point* of a sequence $(x_n)_{n=1}^{\infty}$ of points $x_n \in X$ if for every neighborhood U of x the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite.

A topological space X is called *countably compact* if every countable open cover of X has a finite subcover, or equivalently, every sequence $(x_n)_{n=1}^{\infty}$ of points $x_n \in X$ has a cluster point $x \in X$.

2 Uniformly continuous mappings between quasi-metric spaces

Let (X, q) and (Y, r) be quasi-metric spaces. Following [6, Definition 4.1] we say that a mapping $f : X \to Y$ is *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ the inequality $q(x_1, x_2) < \delta$ implies $r(f(x_1), f(x_2)) < \varepsilon$.

Clearly, every uniformly continuous mapping between quasi-metric spaces is continuous.

Proposition 1. *Let (X, q) and (Y, r) be quasi-metric spaces and $f : X \to Y$. Then the following conditions are equivalent.*

(i) $f : (X, q) \to (Y, r)$ *is uniformly continuous.*

(ii) $f : (X, q^{-1}) \to (Y, r^{-1})$ *is uniformly continuous.*

Proof. It follows immediately from the equalities

$$q(x_1, x_2) = q^{-1}(x_2, x_1) \quad \text{and} \quad r(f(x_1), f(x_2)) = r^{-1}(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$. □

Proposition 2. *Let (X, q) and (Y, r) be quasi-metric spaces and $f : (X, q) \to (Y, r)$ be a uniformly continuous mapping. Then $f : (X, d_q) \to (Y, d_r)$ is uniformly continuous.*

Proof. Fix any $\varepsilon > 0$ and choose $\delta > 0$ such that for every $x_1, x_2 \in X$ the inequality $q(x_1, x_2) < \delta$ implies $r(f(x_1), f(x_2)) < \frac{\varepsilon}{2}$. Then for every $x_1, x_2 \in X$ with $d_q(x_1, x_2) < \delta$ we have that

$$\max\{q(x_1, x_2), q(x_2, x_1)\} \le d_q(x_1, x_2) < \delta$$

and therefore,

$$d_r(f(x_1), f(x_2)) = r(f(x_1), f(x_2) + r(f(x_2), f(x_1) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

□

The following example shows that the converse implication is not true.

Proposition 3. *There exist quasi-metrics q and r on the set $X = \mathbb{R}$ such that the identity mapping $f : (X, q) \to (X, r)$, $f(x) = x$, is everywhere discontinuous and $d_q = d_r$, in particular, $f : (X, d_q) \to (X, d_r)$ is uniformly continuous.*

Proof. Consider the function $q : X^2 \to \mathbb{R}$ defined by

$$q(x, y) = \begin{cases} 1, \text{ if } y < x, \\ y - x, \text{ if } y \ge x. \end{cases}$$

According to [5, Example 2], q is a quasi-metric on X and q generates the topology of Sorgenfrey line on X, that is, for every $x \in X$ the family $([x, x + \varepsilon) : \varepsilon > 0)$ forms a base of neighbourhoods of x in (X, q). Notice that

$$d_q(x, y) = \begin{cases} 0, \text{ if } x = y, \\ 1 + |y - x|, \text{ if } x \ne y. \end{cases}$$

Now let $r = \frac{1}{2}d_q$. Since $d_q(x,y) = d_q(y,x)$,

$$d_r(x,y) = r(x,y) + r(y,x) = d_q(x,y)$$

for every $x, y \in X$. Moreover, for any $x, y \in X$ with $d_p(x,y) < 1$ we have that $x = y$ and, in particular, $d_r(x,y) = 0 < \varepsilon$ for every $\varepsilon > 0$. □

It follows from the following example that Theorem 2 is not true.

Theorem 3. *There exist a compact metric space (X,d), a compact partial metric space (Y,p) and a homeomorphism $f : X \to Y$ such that $f : (X,d) \to (Y,q_p)$ is not uniformly continuous.*

Proof. Let $x_0 = 0$, $x_n = \frac{1}{n}$ for every $n \in \mathbb{N}$, $X = \{x_n : n \geq 0\}$ and $d(x,y) = |x-y|$.

Now $Y = \{y_n : n \geq 0\}$ where all elements y_n are distinct and

$$p(x,y) = p(y,x) = \begin{cases} 1, \text{ if } x = y = y_0, \\ 0, \text{ if } x = y = y_n, n \in \mathbb{N}, \\ 1 + \frac{1}{n}, \text{ if } x = y_0, y = y_n, n \in \mathbb{N}, \\ 1, \text{ if } x = y_n, y = y_m, n, m \in \mathbb{N}, n \neq m. \end{cases}$$

Notice that p is a partial metric on Y. Conditions $(p_1) - (p_3)$ are obvious. It remains to verify (p_4) for distinct points $x, y, z \in X$. If $y \neq y_0$, then

$$p(x,z) + p(y,y) = p(x,z) \leq 2 \leq p(x,y) + p(y,z).$$

If $y = y_0$, then $p(x,z) = p(y,y) = 1$ and

$$p(x,z) + p(y,y) = 2 \leq p(x,y) + p(y,z).$$

Thus, (Y,p) is a partial metric space.

Notice that

$$q_p(x,y) = \begin{cases} 0, \text{ if } x = y, \\ \frac{1}{n}, \text{ if } x = y_0, y = y_n, n \in \mathbb{N}, \\ 1 + \frac{1}{n}, \text{ if } y = y_0, x = y_n, n \in \mathbb{N}, \\ 1, \text{ if } x = y_n, y = y_m, n, m \in \mathbb{N}, n \neq m. \end{cases}$$

Clearly, all points $y \neq y_0$ are isolated in (Y,p) and $y_n \to y_0$. Therefore, the mapping $f : X \to Y$, $f(x_n) = y_n$, is a homeomorphism. Moreover, for any distinct $n, m \in \mathbb{N}$ we have that $d(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}|$ and $q_p(y_n, y_m) = 1$. So, $f : (X,d) \to (Y,q_p)$ is not uniformly continuous. □

Remark 1. *In the proof of Theorem 2 the authors use the following inequality*

$$\sup\{q(y,z) : y, z \in B_q(x,\varepsilon)\} \leq 2\varepsilon,$$

which may not hold for quasi-metric space (X,q).

3 Uniformly continuous mappings on premetric spaces

In this section we study uniformly continuity of mappings from a premetric space to a uniform space.

A nonnegative function $p : X^2 \to [0, +\infty)$ is called *a premetric* on X (see [2]) if $p(x, x) = 0$ for every $x \in X$.

The following statement has an obvious proof.

Proposition 4. *Let (X, p) be a premetric space. Then the system $\mathcal{T}_p$ of all sets $G \subseteq X$ such that for every $x \in G$ there exists $\varepsilon > 0$ such that*

$$\{y \in X : p(x, y) < \varepsilon\} \subseteq G$$

forms an topology on X.

The topology $\mathcal{T}_p$ from Proposition 4 is called a *topology of premetric space* (X, p).

Let X be a topological space, $\mathcal{T}$ be the topology of X and p be a premetric on the set X. We say that p is *compatible* with $\mathcal{T}$ if $\mathcal{T}_p = \mathcal{T}$.

Clearly, any quasi-metric $q : X \to \mathbb{R}$ is a premetric on X and the topologies τ_q of the quasi-metric space (X, q) and $\mathcal{T}_q$ of the premetric space (X, q) coincides.

Notice that, in general, a ball

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$$

might not be open in a premetric space (X, p). Moreover, $B_p(x, \varepsilon)$ might not be a neighbourhood of x (see [2, Section 2]).

Nevertheless, the following characterization of continuous mapping on premetric spaces follows immediately from the characterization of continuity in the terms of open sets.

Proposition 5. *Let (X, p) be a premetric space, Y be a topological space and $f : X \to Y$. Then the following conditions are equivalent.*

(i) *f is continuous.*

(ii) *For every $x \in X$ and every neighborhood V of $f(x)$ in Y there exists $\delta > 0$ such that $f(u) \in V$ for every $u \in X$ with $p(x, u) < \delta$.*

Proof. $(i) \Rightarrow (ii)$. Let V be an open neighbourhood of $f(x)$ in Y. Then the set $U = f^{-1}(V)$ is open in X by (i). Therefore, there exists $\delta > 0$ such that $B_p(x, \delta) \subseteq U$.

$(ii) \Rightarrow (i)$. Let G be an open set in Y. Then the set $f^{-1}(G)$ is open in X by (ii). □

We consider the following generalization of the uniform continuity of mappings between metric spaces.

Definition 1. *Let (X, p) be a premetric space, $(Y, \mathcal{U})$ be a uniform space and $f : X \to Y$. We say that f is uniformly continuous if for every $U \in \mathcal{U}$ there exists $\delta > 0$ such that $(f(x), f(y)) \in U$ for very $x, y \in X$ with $p(x, y) < \delta$.*

The following property follows immediately from Proposition 5.

Proposition 6. *Let* (X, p) *be a premetric space,* $(Y, \mathcal{U})$ *be a uniform space and* $f : X \to Y$ *be uniformly continuous. Then* f *is continuous.*

We say that a premetric space (X, p) satisfies $(*)$ if for any sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ of points $x_n \in X$ and $y_n \in X$ with $\lim\limits_{n\to\infty} p(x_n, y_n) = 0$ the sequence $(z_n)_{n=1}^{\infty}$ of points $z_n = (x_n, y_n) \in X^2$ has a cluster point $z \in \Delta = \{(x, x) : x \in X\}$.

The following statement shows that $(*)$ is a strengthened condition of the countable compactness.

Proposition 7. *If a premetric space* (X, p) *has* $(*)$*, then* (X, p) *is countably compact.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points $x_n \in X$. Since $p(x_n, x_n) = 0$ for every $n \in \mathbb{N}$, the sequence of points $(x_n, x_n) \in X^2$ has a cluster point $(x, x) \in X^2$. Then the point $x \in X$ is a cluster point of the sequence $(x_n)_{n=1}^{\infty}$. So, X is a countably compact space. □

For quasi-metric spaces (X, q) the condition $(*)$ is equivalent to the countable compactness.

Proposition 8. *A quasi-metric space* (X, q) *has* $(*)$ *if and only if* (X, q) *is countably compact.*

Proof. According to Proposition 7, it is enough to verify that every countably compact quasi-metric space (X, q) has $(*)$.

Now let (X, q) be countably compact, $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of points $x_n \in X$ and $y_n \in X$ with $\lim\limits_{n\to\infty} q(x_n, y_n) = 0$. Since (X, q) is countably compact, the sequence $(x_n)_{n=1}^{\infty}$ has a cluster point $x \in X$. Show that the point (x, x) is a cluster point of the sequence of points (x_n, y_n) in X^2. Fix any neighbourhood W of (x, x) in X^2.There exists $\varepsilon > 0$ such that $B_q(x, \varepsilon) \times B_q(x, \varepsilon) \subseteq W$. Since $\lim\limits_{n\to\infty} q(x_n, y_n) = 0$, there exists $n_0 \in \mathbb{N}$ such that $q(x_n, y_n) < \frac{\varepsilon}{2}$ for every $n \geq n_0$. Since x is a cluster point of $(x_n)_{n=1}^{\infty}$, the set

$$N = \{n \geq n_0 : x_n \in B_q(x, \tfrac{\varepsilon}{2})\}$$

is infinite. Then for every $n \in N$ we have that

$$q(x, y_n) \leq q(x, x_n) + q(x_n, y_n) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

and therefore,

$$(x_n, y_n) \in B_q(x, \tfrac{\varepsilon}{2}) \times B_q(x, \varepsilon) \subseteq W.$$

□

The following example shows that the premetric analog of Proposition 8 is not true.

Proposition 9. *There exists a compact Hausdorff premetric space* (X, p) *which has no* $(*)$*.*

Proof. Let $X = [0,1]\times\{0,1\}$ be the linearly ordered compact with the lexicographical order, i.e. $(y,i) < (z,j)$ if $y < z$ or $y = z$ and $i < j$ (the space X is known as the *two arrow space*, [3, Exercise 3.10.C]). Notice that for any $x = (y,0) \in X$ the sets

$$B(x,\varepsilon) = \{x\} \cup \{(z,i) \in X : 0 < y - z < \varepsilon\}, \quad \varepsilon > 0$$

form a base of the neighbourhoods of x in X and for any $x = (y,1) \in X$ the sets

$$B(x,\varepsilon) = \{x\} \cup \{(z,i) \in X : 0 < z - y < \varepsilon\}, \quad \varepsilon > 0$$

form a base of the neighbourhoods of x in X.

For any $x_1 = (y,i), x_2 = (z,j) \in X$ we set

$$p(x_1,x_2) = \begin{cases} 1, \text{ if } & i = 1 \text{ and } z < y; \\ 1, \text{ if } & i = 0 \text{ and } z > y; \\ 1, \text{ if } & y = z \text{ and } i \neq j; \\ |y-z|, & \text{otherwise.} \end{cases}$$

Clearly, the function $p : X^2 \to \mathbb{R}$ is a premetric on X.

Fix any $x_0 = (i,y) \in X$ and $\varepsilon \in (0,1)$. If $i = 0$ then

$$\{x \in X : p(x_0,x) < \varepsilon\} = \{x_0\} \cup \{(j,z) \in X : z < y \text{ and } |z-y| < \varepsilon\} = B(x_0,\varepsilon).$$

Analogously, $\{x \in X : p(x_0,x) < \varepsilon\} = B(x_0,\varepsilon)$ if $i = 1$. Therefore, p is compatible with the topology of X.

It remains to show that the premetric space (X,p) has no $(*)$. Consider the sequences $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ of points

$$u_n = (\tfrac{1}{2} - \tfrac{1}{2^n}, 1) \qquad \text{and} \qquad v_n = (\tfrac{1}{2} + \tfrac{1}{2^n}, 0).$$

Notice that $p(u_n,v_n) = \frac{1}{2^{n-1}}$ for every $n \in \mathbb{N}$ and $(u_n,v_n) \to (x_1,x_2)$ where $x_1 = (\frac{1}{2},0)$ and $x_2 = (\frac{1}{2},1)$. Thus the compact space (X,p) has no $(*)$. □

Now we give a variant of the theorem on the uniform continuity of a continuous mapping on a compact premetric space.

Theorem 4. *Let (X,p) be a premetric space with $(*)$, $(Y,\mathcal{U})$ be an uniform space and $f : X \to Y$ be a continuous mapping. Then f is uniformly continuous.*

Proof. Assume that there exists $U \in \mathcal{U}$ such that for every $n \in \mathbb{N}$ there exists $x_n, y_n \in X$ with $p(x_n,y_n) \leq \frac{1}{n}$ and $(f(x),f(y)) \notin U$. Since (X,p) has $(*)$, the sequence of points (x_n,y_n) has a cluster point (x_0,x_0). Let $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \circ V \subseteq U$. Using the continuity of f at x_0 choose a neighborhood W of x_0 in X such that $(f(x_0),f(x)) \in V$ for every $x \in W$. Since (x_0,x_0) is a cluster point of the sequence $(x_n,y_n)_{n=1}^{\infty}$, the set $\{n \in \mathbb{N} : (x_n,y_n) \in W^2\}$ is nonempty. Therefore, there exists $n \in \mathbb{N}$ with $x_n,y_n \in W$. Then $(f(x_n),f(x_0)) \in V^{-1} = V$, $(f(x_0),f(y_n)) \in V$ and

$$(f(x_n),f(y_n)) \in V \circ V \subseteq U,$$

– a contradiction. □

Corollary 1. *Let (X, q) be a countable compact quasi-metric space and $(Y, \mathcal{U})$ be a uniform space. Then every continuous mapping $f : (X, q) \to (Y, \mathcal{U})$ is uniformly continuous.*

Remark 2. *Notice that countable compactness of a quasi-metric space is not equivalent to the compactness (see [4]). But for partial metric spaces compactness and countable compactness are equivalent [8, Theorem 5.7].*

The following example shows that for premetric space the analog of Corollary 1 is not true.

Theorem 5. *There exist a compact Hausdorff X, a compatible premetric p on X and a compatible uniformity $\mathcal{U}$ on X such that the identity homeomorphism $f : (X, p) \to (X, \mathcal{U})$ is not uniformly continuous.*

Proof. Consider the premetric space (X, p) from Proposition 9. According to [1, Chapter II, § 4, Theorem 1], there exists a uniformity $\mathcal{U}$ on X which is compatible with $\mathcal{T}_p$. Verify that the identity mapping $f : (X, p) \to (X, \mathcal{U})$ is not uniformly continuous.

Let $x_1 = (\frac{1}{2}, 0)$, $x_2 = (\frac{1}{2}, 1)$. Since the uniformity $\mathcal{U}$ consists of all neighbourhoods U of the diagonal $\Delta = \{(x, x) : x \in X\}$ in X^2, there exists a closed in X^2 entourage $U \in \mathcal{U}$ such that $(x_1, x_2) \notin U$. There exist open neighbourhoods V_1 and V_2 of x_1 and x_2 in X such that $(V_1 \times V_2) \cap U = \varnothing$. According to the proof of Proposition 9, there exist sequences $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ of $u_n, v_n \in X$ such that $u_n \to x_1$, $v_n \to x_2$ and $p(u_n, v_n) \to 0$. Then for every $\delta > 0$ there exists $N \in \mathbb{N}$ such that $u_N \in V_1$, $v_N \in V_2$ and $p(u_N, v_N) < \delta$. Therefore, $u_N, v_N) \notin U$ and f is not uniformly continuous. □

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Вивчаються рівномірно неперервні відображення між квазіметрчними просторами і побудовано топологічний гомеоморфізм між двома компактними гаусдорфовими частково метричними просторами такий, що відображенн між відповідними квазіметричними просторами не є рівномірно неперервним. Цей приклад, зокрема, показує, що теорема 4.4 з [6] є хибною. Крім того, доводиться аналог теореми Гейне-Кантора про рівномірну неперервність довільного неперервного відображення $f : X \to Y$, визначеного на преметричному просторі X, який задовольняє деяку підсилену умову зліченної компактності, і набуває значень у рівномірному просторі Y. Також подано приклад неперервного відображення $f : X \to Y$, визначеного на компактному гаусдорфовому преметричному просторі X, і зі значеннями у рівномірному просторі Y, яке не є рівномірно неперервним.