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ON LOCALLY COMPACT SHIFT-CONTINUOUS TOPOLOGIES ON SEMIGROUPS $\mathscr{C}_+(A,B)$ AND $\mathscr{C}_-(A,B)$ WITH ADJOINED ZERO

Let $\mathscr{C}_+(a,b)$ and $\mathscr{C}_-(a,b)$ be upper and down subsemigroups of the bicyclic semigroup defined in [15]. Let $\mathscr{C}_+(p,q)^0$ and $\mathscr{C}_-(p,q)^0$ be the semigroups $\mathscr{C}_+(a,b)$ and $\mathscr{C}_-(a,b)$ with the adjoined zero. We show that the semigroups $\mathscr{C}_+(p,q)^0$ and $\mathscr{C}_-(p,q)^0$ admit continuum many different Hausdorff locally compact shift-continuous topologies up to topological isomorphism.

Key words and phrases: semitopological semigroup, topological semigroup, left topological semigroup, locally compact.

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In this paper we shall follow the terminology of [4, 5, 6, 18].

By ω we denote the set of all non-negative integers. Throughout these notes we always assume that all topological spaces involved are Hausdorff.

Definition 1 ([4, 18]). *Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation) $\mu\colon S\times S\to S$, $(x,y)\mapsto\mu(x,y)=xy$. Then the pair (S,μ) is called*

- (i) a right topological (left topological) semigroup *if all interior left (right) shifts $\lambda_s\colon S\to S$, $x\mapsto sx$ ($\rho_s\colon S\to S$, $x\mapsto xs$), are continuous maps, $s\in S$;*
- (ii) a semitopological semigroup *if the map μ is separately continuous;*
- (iii) a topological semigroup *if the map μ is jointly continuous.*

We usually omit the reference to μ and write simply S instead of (S,μ). It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;

УДК 512.536
2010 *Mathematics Subject Classification:* 22A15, 54D45, 54H10.

- a *shift-continuous* topology if (S,τ) is a semitopological semigroup;
- an *left-continuous* (*right-continuous*) topology if (S,τ) is a left (right) topological semigroup.

The bicyclic monoid $\mathscr{C}(a,b)$ is the semigroup with the identity 1 generated by two elements a and b subjected only to the condition $ab=1$. The semigroup operation on $\mathscr{C}(a,b)$ is determined as follows:

$$b^k a^l \cdot b^m a^n = \begin{cases} b^{k-l+m}a^n, & \text{if } l<m;\\ b^k a^n, & \text{if } l=m;\\ b^k a^{l-m+n}, & \text{if } l>m. \end{cases}$$

In [15] Makanjuola and Umar study algebraic property of the following anti-isomorphic subsemigroups

$$\mathscr{C}_{+}(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\leqslant j\right\} \quad \text{and} \quad \mathscr{C}_{-}(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\geqslant j\right\},$$

of the bicyclic monoid. In the paper [8] we prove that every Hausdorff left-continuous (right-continuous) topology on the monoid $\mathscr{C}_{+}(a,b)$ ($\mathscr{C}_{-}(a,b)$) is discrete and show that there exists a compact Hausdorff topological monoid S which contains $\mathscr{C}_{+}(a,b)$ ($\mathscr{C}_{-}(a,b)$) as a submonoid. Also, in [8] we constructed a non-discrete right-continuous (left-continuous) topology τ_p^{+} (τ_p^{-}) on the semigroup $\mathscr{C}_{+}(a,b)$ ($\mathscr{C}_{-}(a,b)$) which is not left-continuous (right-continuous).

Later by $\mathscr{C}_{+}(p,q)^0$ and $\mathscr{C}_{-}(p,q)^0$ we denote the semigroups $\mathscr{C}_{+}(a,b)$ and $\mathscr{C}_{-}(a,b)$ with the adjoined zero.

In [7] it is proved that every Hausdorff locally compact shift-continuous topology on the bicyclic monoid with adjoined zero is either compact or discrete. This result was extended by Bardyla onto the p-polycyclic monoid [1] and graph inverse semigroups [2], and by Mokrytskyi onto the monoid of order isomorphisms between principal filters of $\mathbb{N}^n$ with adjoined zero [17]. In [9] the results of paper [7] onto the monoid $\mathbf{I}\mathbb{N}_{\infty}$ of all partial cofinite isometries of positive integers with adjoined zero are extended. In [12] the similar dichotomy was proved for so called bicyclic extensions $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ when a family $\mathscr{F}$ consists of inductive non-empty subsets of ω. Algebraic properties on a group G such that if the discrete group G has these properties, then every locally compact shift continuous topology on G with adjoined zero is either compact or discrete studied in [16]. The above results are extended in [10] to the bicyclic extension $\boldsymbol{B}_{[0,\infty)}$ of the additive group of reals with adjoined zero (see [13]) in the cases when on the semigroup $\boldsymbol{B}_{[0,\infty)}$ the usual topology, the discrete topology or the topology determined by the natural partial order is defined. Also, in [11] it is proved that the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}^0$ with adjoined zero admits distinct $\mathfrak{c}$-many shift-continuous topologies, however every Hausdorff locally compact semigroup topology on $\mathscr{C}_{\mathbb{Z}}^0$ is discrete. In [3] Bardyla proved that a Hausdorff locally compact semitopological semigroup McAlister Semigroup $\mathcal{M}_1$ is either compact or discrete. However, this dichotomy does not hold for the McAlister Semigroup $\mathcal{M}_2$ and moreover, $\mathcal{M}_2$ admits continuum many different Hausdorff locally compact inverse semigroup topologies [3].

In this paper we show that the semigroups $\mathscr{C}_{+}(p,q)^0$ and $\mathscr{C}_{-}(p,q)^0$ admit continuum many different Hausdorff locally compact shift-continuous topologies up to topological isomorphism.

Lemma 1. *Every locally compact Hausdorff shift-continuous topology τ on the additive semigroup of non-negative integers $(\omega, +)$ is discrete.*

Proof. Fix any $n_0 \in \omega$. The Hausdorffness of the space (ω, τ) implies that $n_0^{\downarrow} = \{k \in \omega \colon k \leqslant n\}$ is a closed subset of (ω, τ). Then $\omega \setminus n_0^{\downarrow}$ is an open subset of (ω, τ), and by Corollary 3.3.10 of [6], $\omega \setminus n_0^{\downarrow}$ is locally compact, and hence, Baire. By Proposition 1.30 of [14] the space $\omega \setminus n_0^{\downarrow}$ contains an isolated point n_1, which is isolated in (ω, τ) because $\omega \setminus n_0^{\downarrow}$ is an open subset of (ω, τ). This and the condition $n_0 < n_1$ imply that n_0 is an isolated point in (ω, τ), because n_0 is the full preimage of n_1 under the continuous right shift $\rho_{n_1 - n_0} \colon (\omega, +, \tau) \to (\omega, +, \tau)$, $i \mapsto i + (n_1 - n_0)$. This completes the proof of the lemma. □

Later by $(\omega, +)^0$ we denote the additive semigroup of non-negative integers $(\omega, +)$ with adjoined zero. Without loss of generality we may assume that $(\omega, +)^0 = \omega \cup \{\infty\}$ with the extended semigroup operation $n + \infty = \infty + n = \infty + \infty = \infty$ for all $n \in \omega$, i.e., ∞ is the zero of $(\omega, +)^0$.

Proposition 1. *Every Hausdorff locally compact shift-continuous topology on the semigroup $(\omega, +)^0$ is either compact or discrete.*

Proof. Let τ_{lc} be an arbitrary non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $(\omega, +)^0$. The Hausdorffness of $((\omega, +)^0, \tau_{\mathrm{lc}})$ implies that ω is an open subset of $((\omega, +)^0, \tau_{\mathrm{lc}})$. Then by Corollary 3.3.10 of [6], ω is locally compact, and by Lemma 1 is a discrete subspace of $((\omega, +)^0, \tau_{\mathrm{lc}})$.

Since all point from ω are open-and-closed subsets of the locally compact space $((\omega, +)^0, \tau_{\mathrm{lc}})$, there exists a base $\mathscr{B}_{\tau_{\mathrm{lc}}}(\infty)$ of the topology τ_{lc} at the point ∞ which consists of compact-and-open subsets of $((\omega, +)^0, \tau_{\mathrm{lc}})$. Hence, for any $U, V \in \mathscr{B}_{\tau_{\mathrm{lc}}}(\infty)$ the set $U \setminus V$ is finite.

We state that for any $U \in \mathscr{B}_{\tau_{\mathrm{lc}}}(\infty)$ the set $\omega \setminus U$ is finite. Suppose to the contrary that there exists $U \in \mathscr{B}_{\tau_{\mathrm{lc}}}(\infty)$ the set $\omega \setminus U$ is infinite. The separate continuity of the semigroup operation in $((\omega, +)^0, \tau_{\mathrm{lc}})$ implies that there exists $V \in \mathscr{B}_{\tau_{\mathrm{lc}}}(\infty)$ such that $V \subseteq U$ and $1 + V \subseteq U$. Since $\omega \setminus U$ is infinite, there exists a sequence $\{x_n\}_{n \in \omega} \subseteq U$ such that $1 + x_i \neq U$ for any $i \in \omega$. This implies that $x_i \neq V$ for any $i \in \omega$, and hence, the set $U \setminus V$ is infinite, a contradiction. The obtained contradiction implies that τ_{lc} is a compact topology on $(\omega, +)^0$. □

Later by τ_{lc} we denote a Hausdorff locally compact shift-continuous topology on the semigroup $\mathscr{C}_+(p, q)^0$.

Since every Hausdorff shift-continuous topology on the semigroup $\mathscr{C}_+(p, q)$ is discrete (see [8, Theorem 6]), the following statements holds.

Lemma 2. *If U and V are any compact-and-open neighbourhoods of the zero in $(\mathscr{C}_+(p, q)^0, \tau_{\mathrm{lc}})$, then the set $U \setminus V$ is finite.*

For any $i \in \omega$ we denote

$$\mathscr{C}_+^i(p, q) = \{b^i a^{i+s} \in \mathscr{C}_+(p, q) \colon s \in \omega\}.$$

The semigroup operation of $\mathscr{C}_+(p,q)$ implies that $\mathscr{C}_+^i(p,q)$ is a subsemigroup of $\mathscr{C}_+(p,q)$, and moreover, $\mathscr{C}_+^i(p,q)$ is isomorphic to the additive semigroup of non-negative integers $(\omega,+)$ for any $i\in\omega$ [8].

Lemma 3. *For any compact-and-open neighbourhood U of the zero in $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ there exists $i\in\omega$ such that the set $U\cap\mathscr{C}_+^i(p,q)$ is infinite.*

Proof. Suppose to the contrary that there exists a compact-and-open neighbourhood U of the zero in $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ such that $\left|U\cap\mathscr{C}_+^i(p,q)\right|<\infty$ for any $i\in\omega$. Then there exists a sequence $\{i_j\}_{j\in\omega}\subseteq\omega$ such that $U\cap\mathscr{C}_+^{i_j}(p,q)\neq\varnothing$ for any $j\in\omega$. The separate continuity of the semigroup operation in $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ and local compactness of τ_{lc} imply that there exists a compact-and-open neighbourhood V of zero in $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ such that $V\subseteq U$ and $V\cdot a\subseteq U$. By the definition of the semigroup operation in $\mathscr{C}_+(p,q)$ we get that $\mathscr{C}_+^i(p,q)\cdot a\subseteq\mathscr{C}_+^i(p,q)$ for all $i\in\omega$. Since for any $j\in\omega$ the set $U\cap\mathscr{C}_+^{i_j}(p,q)$ is non-empty and finite, there exists maximal non-negative integer s_j such that $b^{i_j}a^{i_j+s_j}\in U$ but $b^{i_j}a^{i_j+s_j}\notin V$. This implies that the set $U\setminus V$ is infinite, which contradicts Lemma 2. The obtained contradiction implies the statement of the lemma. □

Lemma 4. *For any compact-and-open neighbourhood U of the zero in $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ there exists $i_0\in\omega$ such that $\mathscr{C}_+^{i_0}(p,q)\cup\{0\}$ is a compact subset of $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$.*

Proof. By Lemma 3 for any compact-and-open neighbourhood U of the zero in $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ there exists $i_0\in\omega$ such that the set $U\cap\mathscr{C}_+^{i_0}(p,q)$ is infinite. Since $\mathscr{C}_+(p,q)$ is a discrete subspace of $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$, $\mathscr{C}_+^{i_0}(p,q)\cup\{0\}$ is a closed subset of $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$. By Corollary 3.3.10 of [6], $\mathscr{C}_+^{i_0}(p,q)\cup\{0\}$ is locally compact. Since the semigroup $\mathscr{C}_+^{i_0}(p,q)\cup\{0\}$ is isomorphic to the additive semigroup of non-negative integers with adjoined zero $(\omega,+)^0$, by Proposition 1 the semigroup $\mathscr{C}_+^{i_0}(p,q)\cup\{0\}$ is a compact subsemigroup of $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$. □

Lemma 5. *$\mathscr{C}_+^i(p,q)\cup\{0\}$ is a compact subset of $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$ for any $i\in\omega$.*

Proof. By Lemma 4 there exists $i_0\in\omega$ such that $\mathscr{C}_+^{i_0}(p,q)\cup\{0\}$ is a compact subset of $(\mathscr{C}_+(p,q)^0,\tau_{\mathrm{lc}})$. We fix an arbitrary $i\in\omega$. The semigroup operation in $\mathscr{C}_+(p,q)^0$ implies the following:

(1) if $i<i_0$, then

$$\begin{aligned}
a^{i_0-i}\cdot\mathscr{C}_+^{i_0}(p,q)&=\left\{a^{i_0-i}\cdot b^{i_0}a^{i_0+s}\colon s\in\omega\right\}=\\
&=\left\{b^{i_0-(i_0-i)}a^{i_0+s}\colon s\in\omega\right\}=\\
&=\left\{b^{i}a^{i_0+s}\colon s\in\omega\right\}=\\
&=\mathscr{C}_+^{i}(p,q)\setminus\left\{b^ia^i,\ldots,b^ia^{i_0-1}\right\};
\end{aligned}$$

(2) if $i>i_0$, then

$$\begin{aligned}
b^ia^i\cdot\mathscr{C}_+^{i_0}(p,q)&=\left\{b^ia^i\cdot b^{i_0}a^{i_0+s}\colon s\in\omega\right\}=\\
&=\left\{b^ia^i\cdot b^{i_0}a^{i_0}\cdot a^s\colon s\in\omega\right\}=\\
&=\left\{b^ia^i\cdot a^s\colon s\in\omega\right\}=\\
&=\left\{b^ia^{i+s}\colon s\in\omega\right\}=\\
&=\mathscr{C}_+^{i}(p,q).
\end{aligned}$$

Hence, if $i > i_0$, then $\mathscr{C}_+^i(p,q) \cup \{0\}$ is a compact subset of $(\mathscr{C}_+(p,q)^0, \tau_{\mathrm{lc}})$ as a continuous image of compact space $\mathscr{C}_+^{i_0}(p,q) \cup \{0\}$ under the left shift $\lambda_{b^ia^i} \colon x \mapsto b^ia^i \cdot x$ in $(\mathscr{C}_+(p,q)^0, \tau_{\mathrm{lc}})$. By a similar way, in the case when $i < i_0$ we obtain that $\mathscr{C}_+^i(p,q) \cup \{0\}$ is compact, because it is the union of the finite family of compact subsets $\left\{a^{i_0-i} \cdot \mathscr{C}_+^{i_0}(p,q), \{b^ia^i\}, \ldots, \{b^ia^{i_0-1}\}\right\}$. □

Example 1. *Let $\{x_i\}_{i\in\omega}$ be any non-decreasing sequence of non-negative integers. We define the topology $\tau_{\{x_i\}}$ on the semigroup $\mathscr{C}_+(p,q)^0$ in the following way. Put*

$$U^n_{\{x_i\}}(0) = \{0\} \cup \left\{b^k a^{k+x_k+s} \colon k, s \in \omega \quad \text{and} \quad k + x_k + s > n\right\}.$$

We suppose that all points of the set $\mathscr{C}_+(p,q)$ are isolated in $(\mathscr{C}_+(p,q)^0, \tau_{\{x_i\}})$, and the family $\mathscr{B}_{\{x_i\}}(0) = \left\{U^n_{\{x_i\}}(0) \colon n \in \omega\right\}$ is the base of the topology $\tau_{\{x_i\}}$ at zero 0 of the semigroup $\mathscr{C}_+(p,q)^0$.

It is obvious that the space $(\mathscr{C}_+(p,q)^0, \tau_{\{x_i\}})$ is Hausdorff and locally compact.

Next we show that the semigroup operation in $(\mathscr{C}_+(p,q)^0, \tau_{\{x_i\}})$ is separately continuous.

Suppose $b^m a^{m+x_m+s} \in U^n_{\{x_i\}}(0)$ and $m \leqslant n = i + j \geqslant i$. Then $m + x_m + s > n$, and hence, $i + j + x_m + s \geqslant n$, which implies that $b^i a^{i+j++x_m+s} \in U^n_{\{x_i\}}(0)$.

If $m > n = i + j \geqslant i$, then $b^ia^{i+j} \cdot b^m a^{m+x_m+s} = b^{m-j} a^{m+x_m+s}$. In the case when $m - j \leqslant n$ we have that $m + x_m + s \geqslant n$ and $b^{m-j}a^{m+x_m+s} \in U^n_{\{x_i\}}(0)$. In the case when $m - j > n$ we have that

$$m - j + x_{m-j} + s \leqslant m - j + x_m + s \leqslant m + x_m + s,$$

because $\{x_i\}_{i\in\omega}$ is a non-decreasing sequence of non-negative integers, and hence $b^{m-j}a^{m+x_m+s} \in U^n_{\{x_i\}}(0)$. Therefore, the inclusion $b^ia^{i+j} \cdot U^n_{\{x_i\}}(0) \subseteq U^n_{\{x_i\}}(0)$ holds for any $n \geqslant i + j$.

If $m \geqslant n$, then $m + x_m + s > n$, and hence, we have that

$$b^m a^{m+x_m+s} \cdot b^i a^{i+j} = b^m a^{m+x_m+s-i+i+j} = b^m a^{m+j+x_m+s}.$$

This implies that for any $n \geqslant i + j$ the following inclusion $U^n_{\{x_i\}}(0) \cdot b^ia^{i+j} \subseteq U^n_{\{x_i\}}(0)$ holds.

Therefore, the semigroup operation in $(\mathscr{C}_+(p,q)^0, \tau_{\{x_i\}})$ is separately continuous.

Since there exist continuum many non-decreasing sequence of non-negative integers in ω, Lemma 5 and Example 1 imply the main theorem of this paper.

Theorem 1. *On the semigroup $\mathscr{C}_+(p,q)^0$ ($\mathscr{C}_-(p,q)^0$) there exist continuum many Hausdorff locally compact shift-continuous topologies up to topological isomorphism.*

Since for any non-decreasing sequence of non-negative integers $\{x_i\}_{i\in\omega}$ in ω and any $n \in \omega$ the set $\mathscr{C}_+(p,q)^0 \setminus U^n_{\{x_i\}}(0)$ is either finite or infinite, we get the following corollary.

Corollary 1. *On the semigroup $\mathscr{C}_+(p,q)^0$ ($\mathscr{C}_-(p,q)^0$) there exist exactly three Hausdorff locally compact shift-continuous topologies up to homeomorphism.*

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Received 08.09.2024

Гутік О.В. *Про локально компактні трансляційно неперервні топології на напівгрупах $\mathscr{C}_+(a,b)$ і $\mathscr{C}_-(a,b)$ з приєднаним нулем* // Буковинський матем. журнал — 2024. — Т.12, №1. — С. 14–20.

У праці [15] Маканюола та Умар вивчали алгебричні властивості верхньої $\mathscr{C}_+(a,b) = \left\{q^ip^j \in \mathscr{C}(p,q) \colon i \leqslant j\right\}$ та нижньої $\mathscr{C}_-(a,b) = \left\{q^ip^j \in \mathscr{C}(p,q) \colon i \geqslant j\right\}$ піднапівгруп біциклічного моноїда $\mathscr{C}(a,b)$. Приймемо $\mathscr{C}_+(p,q)^0$ і $\mathscr{C}_-(p,q)^0$ – напівгрупи $\mathscr{C}_+(a,b)$ і $\mathscr{C}_-(a,b)$ з приєднаним нулем. Відомо [7], що на біциклічній напівгрупі з приєднаним нулем $\mathscr{C}(p,q)^0$ кожна гаусдорфова локально компактна трансляційно неперервна топологія є або компактною, або дискретною. У цій праці описано всі гаусдорфові локально компактні трансляційно неперервні топології на адитивній напівгрупі невід'ємних цілих чисел з приєднаним нулем $(\omega,+)^0$ і на напівгрупах $\mathscr{C}_+(p,q)^0$ і $\mathscr{C}_-(p,q)^0$. Зокрема доведено, що на напівгрупах $\mathscr{C}_+(p,q)^0$ і $\mathscr{C}_-(p,q)^0$ існує континуум різних гаусдорфових локально компактних трансляційно неперервних топологій з точністю до топологічного ізоморфізму, причому таких топологій на напівгрупах $\mathscr{C}_+(p,q)^0$ і $\mathscr{C}_-(p,q)^0$ існує рівно три з точністю до гомеоморфізму.