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ON SOLVABILITY AND WELL-POSEDNESS OF $(N+1)$-TIMES INTEGRATED CAUCHY PROBLEM

For a closed operator A in a Banach space X, the $(n+1)$-times integrated Cauchy problem $C_{n+1}[\tau]$, $0<\tau<\infty$, of finding a solution $v(t)$ of the problem $v'(t)=Av(t)+\frac{t^n}{n!}x, v(0)=0, (t\in[0,\tau], x\in X)$ is considered. In the case where the operator A is normal in a Hilbert space, all its solutions are described. The necessary and sufficient conditions on the spectrum of A under which this problem is well-posed are established.

Key words and phrases: Banach and Hilbert spaces, closed operator, normal operator, Cauchy problem, $(n+1)$-times integrated Cauchy problem, well-posed solvability..

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Introduction

Let A be a closed linear operator in a Banach space X with norm $\|\cdot\|$ and $0<\tau<\infty$. By $C_0[\tau]$ we mean the Cauchy problem

$$\begin{cases} u\in C([0,\tau];\mathcal{D}(A))\cap C^1([0,\tau];X), \\ \quad u'(t)=Au(t), \quad t\in[0,\tau], \\ \quad u(0)=x, \end{cases} \qquad C_0[\tau]$$

where $\mathcal{D}(\cdot)$ is the domain of an operator, $C([0,\tau];\mathcal{D}(A))$ $(C^1([0,\tau];X))$ is the space of all continuous (continuously differentiable) vectop-valued functions $u(t):[0,\tau]\mapsto\mathcal{D}(A)$ $(u(t):[0,\tau]\mapsto X)$, $\mathcal{D}(A)$ is considered with the graph norm $\|x\|_A=\|x\|+\|Ax\|$.

If $u(t)$ is a solution of $C_0[\tau]$, then the vector-valued function

$$v(t)=\int\limits_0^t\frac{(t-s)^n}{n!}u(s)\,ds=\int\limits_0^t\int\limits_0^{t_1}\dots\int\limits_0^{t_n}v(t_{n+1})\,dt_{n+1}dt_n\dots dt_1$$

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is a solution of the problem

$$\begin{cases} v \in C([0,\tau];\mathcal{D}(A)) \cap C^1([0,\tau];X), \\ v'(t) = Av(t) + \dfrac{t^n}{n!}x, \quad t \in [0,\tau], \\ v(0) = 0. \end{cases} \qquad C_{n+1}[\tau]$$

Really, by differentiating $\int\limits_0^t \frac{(t-s)^n}{n!}u(s)\,ds$ in the parameter t and taking into account that not only the integrand but the upper limit of the integral depends on t, we obtain

$$v'(t) = \left(\int\limits_0^t \frac{(t-s)^n}{n!}u(s)\,ds\right)' = \int\limits_0^t \left(\frac{(t-s)^n}{n!}u(s)\right)'_t ds + t'\frac{(t-s)^n}{n!}u(s)\Big|_{s=t} =$$

$$= \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!}u(s)\,ds = u(s)\frac{(t-s)^n}{n!}\Big|_{s=0}^t + \int\limits_0^t \frac{(t-s)^n}{n!}u'(s)\,ds =$$

$$= \frac{t^n}{n!}x + A\int\limits_0^t \frac{(t-s)^n}{n!}u(s)\,ds = Av(t) + \frac{t^n}{n!}x.$$

In accordance with [1], $C_{n+1}[\tau]$ is called the $(n+1)$-times integrated Cauchy problem. By the definition, $C_n[\tau], n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is well-posed if for any $x \in X$ it has a unique solution.

It should be noted that if A is the generator of a C_0-semigroup of linear operators in X, then for all $x \in \mathcal{D}(A)$ there exists a unique solution of $C_0[\tau]$ (see, for example, [2]). As has been shown in [3], the converse, generally, is not true. However, $C_1[\tau]$ is well-posed if and only if A generates a C_0-semigroup.

In this paper, all the solutions of the problem $C_{n+1}[\tau]$ are described in the case where A is a normal operator in a Hilbert space, and the criterion of its well-posedness is presented. The results without proof were announced in [4].

1 PRELIMINARIES

For $\lambda \in \mathbb{C},\ n \in \mathbb{N}_0$, we put

$$\Phi_n(\lambda,t) = \frac{1}{\lambda^{n+1}}\left(e^{\lambda t} - \sum_{k=0}^{n}\frac{(t\lambda)^k}{k!}\right).$$

It is obvious that

$$\Phi_n(\lambda,t) = \sum_{k=n+1}^{\infty}\frac{t^k\lambda^{k-(n+1)}}{k!} = \sum_{p=0}^{\infty}\frac{t^{p+n+1}\lambda^p}{(p+n+1)!} = \frac{t^{n+1}}{(n+1)!}\sum_{p=0}^{\infty}\frac{t^p\lambda^p}{(n+2)\dots(n+p+1)}. \quad (1)$$

The function $\Phi_n(\lambda,t)$ possesses the following properties:

1) for a fixed λ (a fixed t), $\Phi_n(\lambda,t)$ is entire with respect to t (to λ);

2) $\dfrac{d\Phi_n(\lambda,t)}{dt} = \Phi_{n-1}(\lambda,t), n\in\mathbb{N}$;

$$\frac{d^k\Phi_n(\lambda,t)}{dt^k} = \lambda^k\Phi_n(\lambda,t) + \frac{\lambda^{k-1}t^n}{n!} + \cdots + \frac{t^{n-k+1}}{(n-k+1)!}, \quad k=0,1,\dots n;$$

3) $\left.\dfrac{d^k\Phi_n(\lambda,t)}{dt^k}\right|_{t=0} = 0, \quad k=0,1,\dots n;$

4) $\left.\dfrac{d^{n+1}\Phi_n(\lambda,t)}{dt^{n+1}}\right|_{t=0} = 1.$

The properties 2) - 4) can be verified directly. The fact that for a fixed t, the function $\Phi_n(\lambda,t)$ is entire in λ follows from the relation (see [5])

$$\sqrt[p]{\frac{|t|^p}{(n+2)\dots(n+p+1)}} = \frac{|t|}{\sqrt[p]{(n+2)\dots(n+p+1)}} < \frac{|t|}{\sqrt[p]{p!}} < \frac{|t|}{\sqrt[p]{(\sqrt{p})^p}},$$

which shows that

$$\overline{\lim_{p\to\infty}}\frac{|t|}{\sqrt[p]{(n+2)\dots(n+p+1)}} = 0,$$

so, the convergence radius of the series in (1) is infinite. It is also evident that for a fixed λ, $\Phi_n(\lambda,t)$ is entire in t as a product of two entire functions.

2 The conditions for solvability and well-posedness of the problem

In this section, the main attention is focussed on the case of normal A. So, let $X=\mathfrak{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$ and A be a normal operator in it. Starting from the properties 1)-4) of the function $\Phi_n(\lambda,t)$ and the operational calculus for normal operators (see [6, 7]) we arrive to the following assertion.

Theorem 1. *Suppose that the operator A is normal in $\mathfrak{H}$, $E(\lambda)$ and $\sigma(A)$ are its resolution of identity and spectrum respectively. The problem $C_{n+1}[\tau]$ has a solution if and only if*

$$\forall t\in[0,\tau]: \int\limits_{\sigma(A)} |\Phi_{n-1}(\lambda,t)|^2\, d(E(\lambda)x,x) < \infty. \tag{2}$$

Moreover, the solution may be represented in the form

$$v(t) = \int\limits_{\sigma(A)} \Phi_n(\lambda,t)\, dE(\lambda)x. \tag{3}$$

In order that this problem be well-posed, it is necessary and sufficient that

$$\sup_{\lambda\in\sigma(A)} |\Phi_{n-1}(\lambda,t)| < \infty, \; t\in[0,\tau]. \tag{4}$$

Proof. Assume that condition (2) is fulfilled. This condition stipulates the inclusion $x \in \mathcal{D}(\Phi_n(A,t))$.

Really, it follows from the property 2) of $\Phi_n(\lambda,t)$ that

$$\Phi_n(\lambda,t) = \int\limits_0^t \Phi_{n-1}(\lambda,s)\,ds,$$

so,

$$\int\limits_{\sigma(A)} |\Phi_n(\lambda,t)|^2\, d(E(\lambda)x,x) = \int\limits_{\sigma(A)} \left| \int\limits_0^t \Phi_{n-1}(\lambda,s)\,ds \right|^2 d(E(\lambda)x,x) \le$$

$$\le \int\limits_{\sigma(A)} \left(\int\limits_0^t |\Phi_{n-1}(\lambda,s)|\,ds \right)^2 d(E(\lambda)x,x) = \left(\int\limits_{\sigma(A)} \int\limits_0^t |\Phi_{n-1}(\lambda,s)|\,ds\, d(E(\lambda)x,x) \right)^2 =$$

$$= \left(\int\limits_0^t ds \int\limits_{\sigma(A)} |\Phi_{n-1}(\lambda,s)|\, d(E(\lambda)x,x) \right)^2 .$$

Since the function $\int\limits_{\sigma(A)} |\Phi_{n-1}(\lambda,s)|\, d(E(\lambda)x,x)$ is continuous on $[0,\tau]$ (see [6]), we have

$$\int\limits_0^t ds \int\limits_{\sigma(A)} |\Phi_{n-1}(\lambda,s)|\, d(E(\lambda)x,x) < \infty.$$

Thus, $x \in \mathcal{D}(\Phi_n(A,t))$ for any $t \in [0,\tau]$. Then the property 2) implies the relation

$$|\lambda\Phi_n(\lambda,t)|^2 = \left| \frac{d\Phi_n(\lambda,t)}{dt} - \frac{t^n}{n!} \right|^2 = \left| \Phi_{n-1}(\lambda,t) - \frac{t^n}{n!} \right|^2 \le$$

$$\le \left(|\Phi_{n-1}(\lambda,t)| + \frac{t^n}{n!} \right)^2 \le 2\left(|\Phi_{n-1}(\lambda,t)|^2 + \frac{t^{2n}}{n!^2} \right),$$

whence

$$\int\limits_{\sigma(A)} |\lambda\Phi_n(\lambda,t)|^2\, d(E(\lambda)x,x) \le 2\int\limits_{\sigma(A)} |\Phi_{n-1}(\lambda,t)|^2\, d(E(\lambda)x,x) + 2\left(\frac{\tau^n}{n!}\right)^2 \|x\|^2 < \infty,$$

that is, $\Phi_n(A,t)x \in \mathcal{D}(A)$.

By the direct verification one can ascertain that vector-valued function (3) is a solution of the problem $C_{n+1}[\tau]$. Indeed,

$$\frac{dv(t)}{dt} = \int\limits_{\sigma(A)} \Phi_{n-1}(\lambda,t)\, d(E(\lambda)x,x) = \int\limits_{\sigma(A)} \lambda\Phi_n(\lambda,t)\, d(E(\lambda)x,x) + \frac{t^n}{n!}x = Av(t) + \frac{t^n}{n!}x.$$

It is also obvious that, in view of (1), $v(0)=0$.

Now prove the necessity of the condition. Suppose that the problem $C_{n+1}[\tau]$ is solvable and $v(t)$ is its solution. Then

$$v'(t)-Av(t)=\frac{t^n}{n!}x=\Phi_{n-1}(\lambda,t)-\lambda\Phi_n(\lambda,t)x,$$

which implies the inclusion $x\in\mathcal{D}(\Phi_{n-1}(A,t))\bigcap\mathcal{D}(A\Phi_n(A,t))$. Therefore condition (2) is valid.

The well-posedness of $C_{n+1}[\tau]$ in the case of normal A is equivalent to the definability of the closed operator $\Phi_{n-1}(A,t)$, $t\in[0,\tau]$, on the whole space $\mathfrak{H}$, so, by the Banach closed graph theorem, to the boundedness of this operator, i.e. inequality (4). □

Theorem 2. *Let A be a normal operator in $\mathfrak{H}$. The problem $C_{n+1}[\tau]$ is well-posed if and only if there exist constants $R>0$ and $c>0$ such that*

$$\sigma(A)\subseteq K_R\bigcup E_c(n,\tau),$$

where

$$K_R=\{\lambda\in\mathbb{C}:|\lambda|\le R\},$$
$$E_c(n,\tau)=\left\{\lambda\in\mathbb{C}:Re\lambda>0,\ |Im\lambda|\ge ce^{\frac{\tau Re\lambda}{n}}\right\}.$$

Proof. Suppose that the problem $C_{n+1}[\tau]$ is well-posed. Then, in accordance with Theorem 1, there exists $c>0$ such that

$$\forall\lambda\in\sigma(A):|\Phi_{n-1}(\lambda,\tau)|\le c,$$

that is,

$$c\ge\frac{1}{|\lambda|^n}\left|e^{\lambda\tau}-\sum_{k=0}^{n-1}\frac{(\tau\lambda)^k}{k!}\right|\ge$$
$$\ge\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\sum_{k=0}^{n-1}\frac{\tau^k|\lambda|^k}{|\lambda|^n k!}=\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\sum_{k=0}^{n-1}\frac{\tau^k}{|\lambda|^{n-k}k!}.$$

For this reason, for $\lambda\in\sigma(A)$ with $\mathrm{Re}\lambda\ge\beta$, where $\beta>0$ is arbitrary fixed, we obtain

$$c\ge\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\sum_{k=0}^{n-1}\frac{\tau^k}{(\mathrm{Re}\lambda)^{n-k}k!}\ge\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\sum_{k=0}^{n-1}\frac{\tau^k}{\beta^{n-k}k!}=$$

$$=\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\frac{1}{\beta^n}\sum_{k=0}^{n-1}\frac{(\tau\beta)^k}{k!}\ge\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\frac{e^{\tau\beta}}{\beta^n}.$$

Set now $\beta=\dfrac{n}{\tau}$. Then for $\mathrm{Re}\lambda>0$ we have

$$c\ge\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\frac{e^n}{\left(\frac{n}{\tau}\right)^n}=\frac{e^{\tau\mathrm{Re}\lambda}}{|\lambda|^n}-\left(\frac{e\tau}{n}\right)^n,$$

Since minimum of the function $\dfrac{e^{\tau\beta}}{\beta^n}$, $\beta > 0$, is attained at the point $\dfrac{n}{\tau}$, it follows for any $\mathrm{Re}\lambda > 0$ the inequality

$$|\lambda|^n\left(c+\left(\frac{e\tau}{n}\right)^n\right) \geq e^{\tau\mathrm{Re}\lambda},$$

whence

$$|\lambda|^n \geq \frac{e^{\tau\mathrm{Re}\lambda}}{c+\left(\frac{e\tau}{n}\right)^n} \geq \frac{e^{\tau\mathrm{Re}\lambda}}{\left(\sqrt[n]{c}+\frac{e\tau}{n}\right)^n} \geq \frac{e^{\tau\mathrm{Re}\lambda}}{(\sqrt[n]{c}+e\tau)^n},$$

and, consequently,

$$|\lambda| \geq c_1 e^{\frac{\tau\mathrm{Re}\lambda}{n}},$$

where $c_1 = (\sqrt[n]{c}+e\tau)^{-1}$ does not depend on $\mathrm{Re}\lambda$.

For a a fixed $\mathrm{Re}\lambda = \beta > 0$, we have

$$|\mathrm{Im}\lambda| = |\lambda|\sin\left(\arccos\frac{\beta}{|\lambda|}\right) \geq c_2 e^{\frac{\tau\beta}{n}}$$

with $c_2 = c_1\sin\left(\arccos\frac{\beta}{|\lambda|}\right)$.

Assume now that $\mathrm{Re}\lambda \leq 0$. As has been shown in [8],

$$\sigma(A) \subseteq \left\{\lambda \in \mathbb{C}\middle|\lambda = tz : t \in [1,\infty), |z| \leq R\right\},$$

where R is the norm of the bounded normal operator $A(I+A^\star)^{-1}$. Since $\mathrm{Re}\lambda = t\mathrm{Re}z \leq 0$, we have $\mathrm{Re}z \leq 0$, namely, $\mathrm{Re}z \in [-R,0]$. Taking into account correlation (1), we conclude that

$$c \geq |\Phi_{n-1}(\lambda,\tau)| = \frac{\tau^n}{n!}\left|\sum_{k=0}^{\infty}\frac{\tau^k\lambda^k}{(n+1)\dots(n+k)}\right| = \frac{\tau^n}{n!}\left|\sum_{k=0}^{\infty}\frac{\tau^k t^k z^k}{(n+1)\dots(n+k)}\right|. \tag{5}$$

Let now $\lambda_0 = z_0 t_0 \in \sigma(A)$. Then the power series

$$\sum_{k=0}^{\infty}\frac{\tau^k{\lambda_0}^k}{(n+1)\dots(n+k)}$$

converges at the point λ_0. By the former Abel theorem (see, for instance, [9]), the circle $|\lambda| = |\lambda_0|$ divides the whole plane $\mathbb{C}$ into the convergence region $\{\lambda : |\lambda| < |\lambda_0|\}$ of series (5) and its divergence one $\{\lambda : |\lambda| > |\lambda_0|$. Thus, there are no poits of spectrum of the operator A outside the circle $|\lambda| \leq |\lambda_0|$. As $|z_0| \leq R$ and the minimal value of t_0 is equal to 1, $K_R = \{\lambda \in \mathbb{C} : |\lambda|\} \leq R$ is the minimal circle containing all the points $\lambda \in \sigma(A) : \mathrm{Re}\lambda \leq 0$. □

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Як відомо, класична теорія C_0-півгруп лінійних операторів є важливим інструментом для вивчення багатьох питань теорії диференціальних рівнянь у банаховому просторі, зокрема задачі Коші $C_0[\tau]$ відшукання розв'язку $u(t), t \in [0,\tau]$ рівняння $u'(t) = Au(t)$, що задовольняє умову $u(0) = x \in X$, де A - замкнений лінійний оператор у банаховому просторі X. Виявляється, що одним із самих плідних методів дослідження $(n+1)$-раз ($n \in N$) проінтегрованої задачі Коші $C_{n+1}[\tau] : v'(t) = Av(t) + \frac{t^n}{n!}x, v(0) = 0$, є вивчення введених Арендтом так званих $(n+1)$-раз проінтегрованих півгруп, теорію яких у подальшому розробляли Келлерман і Гебер, Танака і Міядера, деЛаубенфелс та ін.

У цій статті основна увага сконцентрована на випадку, коли A є нормальним оператором у гільбертовому просторі. Виходячи з властивостей функції

$\Phi_n(\lambda,t) = \frac{1}{\lambda^{n+1}}\left(e^{\lambda t} - \sum\limits_{k=0}^{n}\frac{(t\lambda)^k}{k!}\right), \lambda \in \mathbb{C}$, пов'язаної певним чином з відповідною $(n+1)$-раз проінтегрованою півгрупою, та операційного числення для нормальних операторів,з допомогою зазначеної функції описано всі розв'язки задачі $C_{n+1}[\tau]$ і знайдено умови, необхідні й достатні для її коректної постановки. Більше того, установлено критерій коректності цієї задачі в термінах локалізації спектра операторп A.