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ON THE GROWTH OF THE MAXIMUM MODULUS OF DIRICHLET SERIES

For an entire Dirichlet series $F(s)=\sum_{n=0}^{\infty} f_n \exp\{s\lambda_n\}$ with $0\le \lambda_n\uparrow+\infty$, a connection between the growth of the maximum modulus $M(\sigma,F)=\sup\{|F(\sigma+it)|:\,t\in\mathbb{R}\}$ and the decrease of the coefficients is studied. For example, it is proved that if $\varlimsup\limits_{k\to\infty}\alpha(\lambda_k)/\beta\left(\frac{1}{\lambda_k}\ln\frac{1}{|f_k|}\right)=$ $=Q>0$, where α, β are positive continuous functions on $[x_0,+\infty)$ increasing to $+\infty$, then $\varlimsup\limits_{\sigma\to+\infty}\big(\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma)+\ln q),F))\}-p\exp\{\alpha(\ln M(\sigma,F))\}\big)=+\infty$ for any $q>1$ and $p>1$ such that $\ln p/\ln q<Q$. Similar results are obtained for Dirichlet series with zero abscissa of absolute convergence .

Key words and phrases: Dirichlet series, maximum modulus, generalized order.

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Introduction

For an entire transcendental function $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ let $M_f(r)=\max\{|f(z)|:\,|z|=r\}$. It is known [5] that $M_f(ar)/M_f(r)\nearrow+\infty$ as $r\to+\infty$ for every $a>1$. S. Singh [9] gave a more simple (on his opinion) proof of this fact, using the relation between $M_f(r)$ and the maximal term of the power expansion of f. A much simpler proof is given in [6] for entire functions given by both power series and Dirichlet series.

Let's remark that Dirichlet series

$$F(s)=\sum_{n=1}^{\infty} f_n\exp\{s\lambda_n\},\quad s=\sigma+it, \tag{1}$$

with non-negative increasing to $+\infty$ exponents λ_n are direct generalization of power series.

We suppose that series (1) has the abscissa of absolute convergence $\sigma_a\in(-\infty,+\infty]$, and for $\sigma<\sigma_a$ we put $M(\sigma,F)=\sup\{|F(\sigma+it)|:\,t\in\mathbb{R}\}$. If $\sigma_a=+\infty$ and series (1) is not reduced to an exponential polynomial then [6] $M(\sigma+h,F)/M(\sigma,F)\nearrow+\infty$ as $\sigma\to+\infty$ for every $h>0$.

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By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$. We remak that for every $\alpha \in L_{si}$ there exists $\alpha_1 \in L$ such that $\alpha_1(x) = (1+o(1))\alpha(x)$ and $\dfrac{x\alpha_1'(x)}{\alpha_1(x)} \to 0$ as $x \to +\infty$. Therefore, in the future we will assume that the function $\alpha \in L_{si}$ satisfies the condition $\dfrac{x\alpha'(x)}{\alpha(x)} \to 0$ as $x \to +\infty$.

In [6] the following theorems are proved.

Theorem A. *Let $\sigma_a = +\infty$, $\gamma \in L$, $p > 1$. If* $\overline{\lim\limits_{n\to\infty}} \dfrac{\ln\ln\gamma(\lambda_n)}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} = \eta^* > 0$ *then*

$$\overline{\lim_{\sigma\to+\infty}} \frac{\gamma(\ln M(q\sigma, F))}{\gamma^p(\ln M(\sigma, F))} = +\infty \tag{2}$$

for each $q > p^{1/\eta^*}$.

If the function γ is continuously differentiable, $\ln\gamma(e^x) \in L_{si}$, $\ln\ln\gamma(\lambda_{n+1}) = (1+o(1))\ln\ln\gamma(\lambda_n)$, $\ln n = O(\lambda_n)$ as $n \to \infty$, $\varkappa_n[F] := \dfrac{\ln|f_n| - \ln|f_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$ as $n_0 \leq n \to \infty$ and $\underline{\lim\limits_{n\to\infty}} \dfrac{\ln\ln\gamma(\lambda_n)}{\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)} = \eta_* < +\infty$, *then*

$$\underline{\lim_{\sigma\to+\infty}} \frac{\gamma(\ln M(q\sigma, F))}{\gamma^p(\ln M(\sigma, F))} = 0 \tag{3}$$

for each $q < p^{1/\eta_*}$.

Theorem B. *Let $\sigma_a = 0$, $p > 1$, $\gamma(e^x) \in L^0$ and $\ln\ln\gamma(x) = o(\ln x)$ as $\to +\infty$. If* $\overline{\lim\limits_{n\to\infty}} \dfrac{\ln\ln\gamma(\lambda_n)}{\ln(\lambda_n/\ln^+|f_n|)} = \eta^* > 0$ *then*

$$\overline{\lim_{\sigma\uparrow 0}} \frac{\gamma(\ln M(\sigma/q, F))}{\gamma^p(\ln M(\sigma, F))} = +\infty$$

for each $q > p^{1/\eta^*}$.

If the function γ is continuously differentiable, $\ln\gamma(x) \in L_{si}$, $\varkappa_n[F] \nearrow 0$ and $\ln\ln\gamma(\lambda_{n+1}) = (1+o(1))\ln\ln\gamma(\lambda_n)$ as $n_0 \leq n \to \infty$, $\overline{\lim\limits_{n\to\infty}} \dfrac{\ln\ln n}{\ln\lambda_n} < 1$ *and* $\underline{\lim\limits_{n\to\infty}} \dfrac{\ln\ln\gamma(\lambda_n)}{\ln(\lambda_n/\ln^+|f_n|)} = \eta_* < +\infty$ *then*

$$\underline{\lim_{\sigma\uparrow 0}} \frac{\gamma(\ln M(\sigma/q, F))}{\gamma^p(\ln M(\sigma, F))} = 0$$

for each $q < p^{1/\eta_*}$.

For the proof of these theorems in [6] the following theorem was used.

Theorem C. *Let $q > 1$, $p > 1$, $\gamma \in L$ and Φ be a positive function continuous on $(x_0, +\infty)$ and increasing to $+\infty$. If* $\overline{\lim\limits_{\sigma\to+\infty}} \dfrac{\gamma(\Phi(q\sigma))}{\gamma^p(\Phi(\sigma))} \leq \xi_1 < +\infty$ *then* $\overline{\lim\limits_{\sigma\to+\infty}} \dfrac{\ln\ln\gamma(\Phi(\sigma))}{\ln\sigma} \leq \dfrac{\ln p}{\ln q}$ *and if* $\underline{\lim\limits_{\sigma\to+\infty}} \dfrac{\gamma(\Phi(q\sigma))}{\gamma^p(\Phi(\sigma))} \geq \xi_2 > 0$ *then* $\underline{\lim\limits_{\sigma\to+\infty}} \dfrac{\ln\ln\gamma(\Phi(\sigma))}{\ln\sigma} \geq \dfrac{\ln p}{\ln q}$.

Using the generalized growth scale, here we will continue the studies started in [6].

1 Entire Dirichlet series

If $\alpha\in L$, $\beta\in L$ and F is an entire function then the quantities

$$\varrho_{\alpha,\beta}[F]:=\varlimsup_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma,F))}{\beta(\sigma)},\quad \lambda_{\alpha,\beta}[F]:=\varliminf_{\sigma\to+\infty}\frac{\alpha(\ln M(\sigma,F))}{\beta(\sigma)}$$

are called [7], [8] the generalized (α,β)-order and the generalized lower (α,β)-order of F respectively. We put

$$Q_{\alpha,\beta}[F]=\varlimsup_{k\to\infty}\frac{\alpha(\lambda_k)}{\beta\left(\frac{1}{\lambda_k}\ln\frac{1}{|f_k|}\right)},\quad q_{\alpha,\beta}[F]=\varliminf_{k\to\infty}\frac{\alpha(\lambda_k)}{\beta\left(\frac{1}{\lambda_k}\ln\frac{1}{|f_k|}\right)}.$$

Then the following lemma is true [7], [8].

Lemma 1. *Let $\alpha\in L_{si}$, $\beta\in L^0$ and $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. If $\ln k=o(\lambda_k\beta^{-1}(c\alpha(\lambda_k)))$ as $k\to\infty$ for each $c\in(0,+\infty)$ then $\varrho_{\alpha,\beta}[F]=Q_{\alpha,\beta}[F]$. If, moreover, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$ and $\kappa_k[F]\nearrow+\infty$ as $k_0\le k\to\infty$ then $\lambda_{\alpha,\beta}[F]=q_{\alpha,\beta}[F]$.*

If we choose $\alpha(x)=\ln\ln\gamma(x)$ and $\beta(x)=\ln^+ x$ then $\eta^*=Q_{\alpha,\beta}[F]$, $\eta_*=q_{\alpha,\beta}[F]$ and equalities (2) and (3) can be written, respectively, in the form

$$\varlimsup_{\sigma\to+\infty}(\exp\{\alpha(\ln M(q\sigma,F))\}-p\exp\{\alpha(\ln M(\sigma,F))\})=+\infty \tag{4}$$

and

$$\varliminf_{\sigma\to+\infty}(\exp\{\alpha(\ln M(q\sigma,F))\}-p\exp\{\alpha(\ln M(\sigma,F))\})=-\infty. \tag{5}$$

We remark also that Theorem C implies the following statement.

Lemma 2. *Let $q>1$, $p>1$, $\alpha\in L$ and Φ be a positive function continuous on $(x_0,+\infty)$ and increasing to $+\infty$. If*

$$\varlimsup_{\sigma\to+\infty}\frac{\alpha(\Phi(\sigma))}{\ln\sigma}>\frac{\ln p}{\ln q}$$

then

$$\varlimsup_{\sigma\to+\infty}(\exp\{\alpha(\Phi(q\sigma))\}-p\exp\{\alpha(\Phi(\sigma))\})=+\infty,$$

and if

$$\varliminf_{\sigma\to+\infty}\frac{\alpha(\Phi(\sigma))}{\ln\sigma}<\frac{\ln p}{\ln q}$$

then

$$\varliminf_{\sigma\to+\infty}(\exp\{\alpha(\Phi(q\sigma))\}-p\exp\{\alpha(\Phi(\sigma))\})=-\infty.$$

Using Lemmas 1 and 2 we prove the following theorem.

Theorem 1. *Let* $\sigma_a = +\infty$, $\alpha \in L$ *and* $\beta \in L^0$. *If* $Q_{\alpha,\beta}[F] > 0$ *then*

$$\overline{\lim_{\sigma\to+\infty}} \big(\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma)+\ln q), F))\} - p\exp\{\alpha(\ln M(\sigma, F))\}\big) = +\infty. \tag{6}$$

for each $p > 1$ *and* $q > 1$ *such that* $Q_{\alpha,\beta}[F] > \ln p/\ln q$.

If $\alpha \in L_{si}$, $\beta \in L^0$ *and* $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ *as* $x \to +\infty$, $\ln k = o(\lambda_k\beta^{-1}(c\alpha(\lambda_k)))$ *as* $k \to \infty$ *for each* $c \in (0, +\infty)$, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$, $\kappa_k[F] \nearrow +\infty$ *as* $k_0 \le k \to \infty$ *and* $q_{\alpha,\beta}[F] < +\infty$ *then*

$$\underline{\lim}_{\sigma\to+\infty} \big(\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma)+\ln q), F))\} - p\exp\{\alpha(\ln M(\sigma, F))\}\big) = -\infty \tag{7}$$

for each $p > 1$ *and* $q > 1$ *such that* $q_{\alpha,\beta}[F] < \ln p/\ln q$.

Proof. If we put $\Phi(\sigma) = \Phi_1(\beta^{-1}(\ln\sigma))$ then

$$\begin{aligned}&\overline{\lim_{\sigma\to+\infty}}(\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma)+\ln q)))\} - p\exp\{\alpha(\Phi_1(\sigma))\}) =\\ &= \overline{\lim_{\sigma\to+\infty}}(\exp\{\alpha(\Phi_1(\beta^{-1}(\ln(qe^{\beta(\sigma)}))))\} - p\exp\{\alpha(\Phi_1(\beta^{-1}(\ln(e^{\beta(\sigma)}))))\}) =\\ &= \overline{\lim_{x\to+\infty}}(\exp\{\alpha(\Phi_1(\beta^{-1}(\ln(qx))))\} - p\exp\{\alpha(\Phi_1(\beta^{-1}(\ln x)))\}) =\\ &= \overline{\lim_{\sigma\to+\infty}}(\exp\{\alpha(\Phi(q\sigma))\} - p\exp\{\alpha(\Phi(\sigma))\}),\end{aligned}$$

$$\begin{aligned}&\underline{\lim}_{\sigma\to+\infty}(\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma)+\ln q)))\} - p\exp\{\alpha(\Phi_1(\sigma))\}) =\\ &= \underline{\lim}_{\sigma\to+\infty}(\exp\{\alpha(\Phi(q\sigma))\} - p\exp\{\alpha(\Phi(\sigma))\}),\end{aligned}$$

$$\overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\Phi(\sigma))}{\ln\sigma} = \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\Phi_1(\beta^{-1}(\ln\sigma)))}{\ln\sigma} = \overline{\lim_{x\to+\infty}} \frac{\alpha(\Phi_1(\beta^{-1}(x)))}{x} = \overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)}$$

and

$$\underline{\lim}_{\sigma\to+\infty} \frac{\alpha(\Phi(\sigma))}{\ln\sigma} = \underline{\lim}_{\sigma\to+\infty} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)}.$$

Therefore, in view of Lemma 2 if Φ_1 is a positive function continuous on $(x_0, +\infty)$ and increasing to $+\infty$ and

$$\overline{\lim_{\sigma\to+\infty}} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)} > \frac{\ln p}{\ln q}$$

then

$$\overline{\lim_{\sigma\to+\infty}}(\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma)+\ln q)))\} - p\exp\{\alpha(\Phi_1(\sigma))\}) = +\infty,$$

and if

$$\underline{\lim}_{\sigma\to+\infty} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)} < \frac{\ln p}{\ln q}$$

then

$$\underline{\lim}_{\sigma\to+\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma)+\ln q)))\} - p\exp\{\alpha(\Phi_1(\sigma))\}) = -\infty$$

Finally, we choose $\Phi_1(\sigma) = \ln\, M(\sigma, F)$. Then from hence it follows that if

$$\varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln\, M(\sigma, F))}{\beta(\sigma)} > \frac{\ln\, p}{\ln\, q}, \tag{8}$$

then (6) holds and if

$$\varliminf_{\sigma\to+\infty} \frac{\alpha(\ln\, M(\sigma, F))}{\beta(\sigma)} < \frac{\ln\, p}{\ln\, q} \tag{9}$$

then (7) holds.

If $Q_{\alpha,\beta}[F] > 0$ then for every $Q \in (0,\, Q_{\alpha,\beta}[F])$ there exists an increasing to ∞ sequence (k_n) such that $\ln\,|f_{k_n}| \geq -\lambda_{k_n}\beta^{-1}(\alpha(\lambda_{k_n})/Q)$ for all n. We choose $\sigma_n = \beta^{-1}(\alpha(\lambda_{k_n})/Q) + 1$. Then by the Cauchy inequality

$$\ln\, M(\sigma_n, F) \geq \ln\,|f_{k_n}| + \sigma_n\lambda_{k_n} \geq -\lambda_{k_n}\beta^{-1}(\alpha(\lambda_{k_n})/Q) + \lambda_{k_n}(\beta^{-1}(\alpha(\lambda_{k_n})/Q) + 1) =$$

$$= \lambda_{k_n} = \alpha^{-1}(Q\beta(\sigma_n - 1)),$$

whence in view of the condition $\beta \in L^0$ and of the arbitrariness of Q we get

$$\varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln\, M(\sigma, F))}{\beta(\sigma)} \geq \varlimsup_{n\to\infty} \frac{\alpha(\ln\, M(\sigma_n, F))}{\beta(\sigma_n)} \geq Q_{\alpha,\beta}[F].$$

Therefore, if $Q_{\alpha,\beta}[F] > \ln\, p/\ln\, q$ then (8) and, thus, (6) hold. The first part of Theorem 1 is proved.

If $q_{\alpha,\beta}[F] < +\infty$ then by Lemma 1

$$\varliminf_{\sigma\to+\infty} \frac{\alpha(\ln\, M(\sigma, F))}{\beta(\sigma)} = \lambda_{\alpha,\beta}[F] = q_{\alpha,\beta}[F].$$

Therefore, if $q_{\alpha,\beta}[F] < \ln\, p/\ln\, q$ then (9) and, thus, (7) hold. The proof of Theorem 1 is complete. □

We remark that if $\beta(x) = \ln^+ x$ then (6) and (7) imply (4) and (5).

If we choose $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ then we get the following result in the R-order scale.

Corollary 1. *Let $\sigma_a = +\infty$. If $Q_R[F] := \varlimsup_{k\to\infty} \dfrac{\lambda_k \ln\, \lambda_k}{-\ln\,|f_k|} > 0$ then*

$$\varlimsup_{\sigma\to+\infty} (\ln\, M(\sigma + \ln\, q, F) - p \ln\, M(\sigma, F)) = +\infty$$

for each $p > 1$ and $q > 1$ such that $Q_R[F] > \ln\, p/\ln\, q$.

If $\ln\, k = o(\lambda_k \ln\, \lambda_k)$, $\ln\, \lambda_{k+1} \sim \ln\, \lambda_k$, $\kappa_k[F] \nearrow +\infty$ as $k_0 \leq k \to \infty$ and $q_R[F] := \varliminf_{k\to\infty} \dfrac{\lambda_k \ln\, \lambda_k}{-\ln\,|f_k|} < +\infty$ then

$$\varliminf_{\sigma\to+\infty} (\ln\, M(\sigma + \ln\, q, F) - p \ln\, M(\sigma, F)) = -\infty$$

for each $p > 1$ and $q > 1$ such that $q_R[F] < \ln\, p/\ln\, q$.

2 Dirichlet series absolutely convergent in a half-plane

If $\sigma_a[F]=0$, $\alpha\in L$ and $\beta\in L$ then the quantities

$$\varrho^{(0)}_{\alpha,\beta}[F]:=\overline{\lim_{\sigma\uparrow 0}}\frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)},\quad \lambda^{(0)}_{\alpha,\beta}[F]:=\underline{\lim_{\sigma\uparrow 0}}\frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)}$$

are called [3], [4] the generalized (α,β)-order and the generalized lower (α,β)-order of F accordingly. We put

$$Q^{(0)}_{\alpha,\beta}[F]=\overline{\lim_{k\to\infty}}\frac{\alpha(\lambda_k)}{\beta(\lambda_k/\ln^+|f_k|)},\quad q^{(0)}_{\alpha,\beta}[F]=\underline{\lim_{k\to\infty}}\frac{\alpha(\lambda_k)}{\beta(\lambda_k/\ln^+|f_k|)}.$$

The following lemma is correct [3], [4].

Lemma 3. *Let $\alpha\in L_{si}$, $\beta\in L_{si}$ and*

$$\frac{x}{\beta^{-1}(c\alpha(x))}\uparrow+\infty,\quad \alpha\left(\frac{x}{\beta^{-1}(c\alpha(x))}\right)=(1+o(1))\alpha(x) \tag{10}$$

as $x_0(c)\le x\to+\infty$ for each $c\in(0,+\infty)$. Suppose that $\sigma_a[F]=0$ and $\ln k= =o\left(\lambda_k/\beta^{-1}(c\alpha(\lambda_k))\right)$ as $k\to\infty$ for each $c\in(0,+\infty)$. Then $\varrho^{(0)}_{\alpha,\beta}[F]=Q^{(0)}_{\alpha,\beta}[F]$. If, moreover, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$ and $\kappa_k[F]\nearrow 0$ as $k_0\le k\to\infty$ then $\lambda^{(0)}_{\alpha,\beta}[F]=q^{(0)}_{\alpha,\beta}[F]$.

Using Lemmas 2 and 3 we prove the following theorem.

Theorem 2. *Let $\sigma_a=0$, $\beta\in L_{si}$, $\alpha(e^x)\in L_{si}$ and $\alpha(x)=o(\beta(x))$ as $x\to+\infty$. If $Q^{(0)}_{\alpha,\beta}[F]>0$ then*

$$\overline{\lim_{\sigma\uparrow 0}}\left(\exp\left\{\alpha\left(\ln M\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|)+\ln q)},F\right)\right)\right\}-p\exp\{\alpha(\ln M(\sigma,F))\}\right)=+\infty. \tag{11}$$

for each $p>1$ and $q>1$ such that $Q^{(0)}_{\alpha,\beta}[F]>\ln p/\ln q$.

If the functions $\alpha\in L_{si}$ and $\beta\in L_{si}$ satisfy conditions (10), *$\ln k=o\left(\lambda_k/\beta^{-1}(c\alpha(\lambda_k))\right)$ as $k\to\infty$ for each $c\in(0,+\infty)$, $\alpha(\lambda_{k+1})\sim\alpha(\lambda_k)$, $\kappa_k[F]\nearrow 0$ as $k_0\le k\to\infty$ and $q^{(0)}_{\alpha,\beta}[F]<+\infty$ then*

$$\underline{\lim_{\sigma\uparrow 0}}\left(\exp\left\{\alpha\left(\ln M\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|)+\ln q)},F\right)\right)\right\}-p\exp\{\alpha(\ln M(\sigma,F))\}\right)=-\infty \tag{12}$$

for each $p>1$ and $q>1$ such that $q^{(0)}_{\alpha,\beta}[F]<\ln p/\ln q$.

Proof. If we put $\Phi(x)=\Phi_1\left(-1/\beta^{-1}(\ln x)\right)$ then

$$\overline{\lim_{\sigma\uparrow 0}}\left(\exp\left\{\alpha\left(\Phi_1\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|)+\ln q)}\right)\right)\right\}-p\exp\{\alpha(\Phi_1(\sigma))\}\right)=$$

$$=\overline{\lim_{\sigma\uparrow 0}}\left(\exp\left\{\alpha\left(\Phi_1\left(\frac{-1}{\beta^{-1}(\ln(qe^{\beta(1/|\sigma|)}))}\right)\right)\right\}-p\exp\left\{\alpha\left(\Phi_1\left(\frac{-1}{\beta^{-1}(\ln e^{\beta(1/|\sigma|)})}\right)\right)\right\}\right)=$$

$$=\overline{\lim_{x\to+\infty}}\left(\exp\left\{\alpha\left(\Phi_1\left(-\frac{1}{\beta^{-1}(\ln(qx))}\right)\right)\right\}-p\exp\left\{\alpha\left(\Phi_1\left(-\frac{1}{\beta^{-1}(\ln x)}\right)\right)\right\}\right)=$$

$$=\overline{\lim_{x\to+\infty}}(\exp\{\alpha(\Phi(qx))\}-p\exp\{\alpha(\Phi(x))\}),$$

$$\underline{\lim}_{\sigma\uparrow 0}\left(\exp\left\{\alpha\left(\Phi_1\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|)+\ln q)}\right)\right)\right\}-p\exp\{\alpha(\Phi_1(\sigma))\}\right)=$$
$$=\underline{\lim}_{x\to+\infty}\left(\exp\{\alpha(\Phi(qx))\}-p\exp\{\alpha(\Phi(x))\}\right),$$

$$\overline{\lim}_{x\to+\infty}\frac{\alpha(\Phi(x))}{\ln x}=\overline{\lim}_{x\to+\infty}\frac{\alpha(\Phi_1(-1/\beta^{-1}(\ln x)))}{\ln x}=\overline{\lim}_{\sigma\uparrow 0}\frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)}$$

and

$$\underline{\lim}_{x\to+\infty}\frac{\alpha(\Phi(x))}{\ln x}=\underline{\lim}_{\sigma\uparrow 0}\frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)}$$

Therefore, in view of Lemma 2 if Φ_1 is a positive function continuous on $(x_0,0)$ and increasing to $+\infty$ and

$$\overline{\lim}_{\sigma\uparrow 0}\frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)}>\frac{\ln p}{\ln q}$$

then

$$\overline{\lim}_{\sigma\uparrow 0}\left(\exp\left\{\alpha\left(\Phi_1\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|)+\ln q)}\right)\right)\right\}-p\exp\{\alpha(\Phi_1(\sigma))\}\right)=+\infty$$

and if

$$\underline{\lim}_{\sigma\uparrow 0}\frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)}<\frac{\ln p}{\ln q}$$

then

$$\underline{\lim}_{\sigma\uparrow 0}\left(\exp\left\{\alpha\left(\Phi_1\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|)+\ln q)}\right)\right)\right\}-p\exp\{\alpha(\Phi_1(\sigma))\}\right)=-\infty$$

Finally, we choose $\Phi_1(\sigma)=\ln M(\sigma,F)$. Then from hence it follows that if

$$\overline{\lim}_{\sigma\uparrow 0}\frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)}>\frac{\ln p}{\ln q} \tag{13}$$

then (11) holds, and if

$$\underline{\lim}_{\sigma\uparrow 0}\frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)}<\frac{\ln p}{\ln q} \tag{14}$$

then (12) holds.

If $Q^{(0)}_{\alpha,\beta}[F]>0$ then for every $Q\in(0,\,Q_{\alpha,\beta}[F])$ there exists an increasing to ∞ sequence (k_n) such that $\ln^+|f_{k_n}|\ge\dfrac{\lambda_{k_n}}{\beta^{-1}(\alpha(\lambda_{k_n})/Q)}$ for all n. We choose $\sigma_n=-\dfrac{1}{2\beta^{-1}(\alpha(\lambda_{k_n})/Q)}$. Then by the Cauchy inequality

$$\ln M(\sigma_n,F)\ge\ln|f_{k_n}|+\sigma_n\lambda_{k_n}\ge\lambda_{k_n}\left(\frac{1}{\beta^{-1}(\alpha(\lambda_{k_n})/Q)}-\frac{1}{2\beta^{-1}(\alpha(\lambda_{k_n})/Q)}\right)=$$
$$=\frac{\lambda_{k_n}}{2\beta^{-1}(\alpha(\lambda_{k_n})/Q)}=\lambda_{k_n}|\sigma_n|=|\sigma_n|\alpha^{-1}\left(Q\beta\left(\frac{1}{2|\sigma_n|}\right)\right),$$

whence in view of the conditions $\beta \in L_{si}$, $\alpha(e^x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \to +\infty$ we get

$$\begin{gathered}Q(1+o(1))\beta(1/|\sigma_n|) \le \alpha\left(\ln M(\sigma_n,F)/|\sigma_n|\right) = \alpha\left(\exp\{\ln\ln M(\sigma_n,F) + \ln(1/|\sigma_n|)\}\right) \le \\ \le \alpha\left(\exp\{2\max\{\ln\ln M(\sigma_n,F), \ln(1/|\sigma_n|)\}\}\right) = \\ = (1+o(1))\alpha\left(\exp\{\max\{\ln\ln M(\sigma_n,F), \ln(1/|\sigma_n|)\}\}\right) = \\ = (1+o(1))\max\{\alpha(\ln M(\sigma_n,F)), \alpha(1/|\sigma_n|)\} \le \\ \le (1+o(1))\left(\alpha(\ln M(\sigma_n,F)) + \alpha((1/|\sigma_n|))\right) = \\ = (1+o(1))\alpha(\ln M(\sigma_n,F)) + o\left(\beta((1/|\sigma_n|))\right), \quad \sigma \uparrow 0.\end{gathered}$$

Therefore, in view of the arbitrariness of Q we have

$$\varlimsup_{\sigma\uparrow 0} \frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)} \ge \varlimsup_{n\to\infty} \frac{\alpha(\ln M(\sigma_n,F))}{\beta(1/|\sigma_n|)} \ge Q^{(0)}_{\alpha,\beta}[F], \tag{15}$$

and if $Q^{(0)}_{\alpha,\beta}[F] > \ln p/\ln q$ then (13) and, thus, (11) hold. The first part of Theorem 2 is proved.

If $q^{(0)}_{\alpha,\beta}[F] < +\infty$ then by Lemma 3

$$\varliminf_{\sigma\uparrow 0} \frac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)} = \lambda^{(0)}_{\alpha,\beta}[F] = q^{(0)}_{\alpha,\beta}[F].$$

Therefore, if $q^{(0)}_{\alpha,\beta}[F] < \ln p/\ln q$ then (14) and, thus, (12) hold. The proof of Theorem 2 is complete. □

From conditions of Theorem 2 it follows that the function α grows more slowly than the function β. In the case if the function β grows more slowly than the function α, the following theorem is true.

Theorem 3. *Let $\sigma_a = 0$, $\beta \in L$ and $\alpha \in L^0$. If $P^{(0)}_{\alpha,\beta}[F] := \varlimsup\limits_{n\to\infty} \dfrac{\alpha(\ln|f_n|)}{\beta(\lambda_n)} > 0$ then (11) holds for each $p > 1$ and $q > 1$ such that $P^{(0)}_{\alpha,\beta}[F] > \ln p/\ln q$.*

If the functions $\alpha \in L_{si}$ and $\beta \in L_{si}$ satisfy conditions

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\gamma^{-1}(c\beta(x))}\right) = (1+o(1))\beta(x), \quad x \to +\infty. \tag{16}$$

for each $c \in (0,+\infty)$, $\gamma(\ln n) = o(\beta(\lambda_n))$, $\beta(\lambda_{n+1}) \sim \beta(\lambda_n)$ and $\kappa_n[F] \nearrow 0$ as $n_0 \le n \to \infty$ and $p^{(0)}_{\alpha,\beta}[F] := \varliminf\limits_{n\to\infty} \dfrac{\alpha(\ln|f_n|)}{\beta(\lambda_n)} < +\infty$ then (12) holds for each $p > 1$ and $q > 1$ such that $p^{(0)}_{\alpha,\beta}[F] < \ln p/\ln q$.

Proof. If $P^{(0)}_{\alpha,\beta}[F] > 0$ then for every $P \in (0,\, P_{\alpha,\beta}[F])$ there exists an increasing to ∞ sequence (k_n) such that $\ln|f_{k_n}| \ge \alpha^{-1}(P\beta(\lambda_{k_n}))$ for all n. We choose $\sigma_n = -1/\lambda_{k_n}$. Then by the Cauchy inequality

$$\ln M(\sigma_n,F) \ge \alpha^{-1}(P\beta(\lambda_{k_n})) + \sigma_n\lambda_{k_n} = \alpha^{-1}(P\beta(\lambda_{k_n})) - 1 = \alpha^{-1}(P\beta(1/|\sigma_n|)) - 1,$$

whence we obtain (15) with $P^{(0)}_{\alpha,\beta}[F]$ instead $Q^{(0)}_{\alpha,\beta}[F]$, and thus, the first part of Theorem 3 is proved.

If the functions $\alpha\in L_{si}$ and $\beta\in L_{si}$ satisfy conditions (16), $\gamma(\ln n)=o(\beta(\lambda_n))$, $\beta(\lambda_{n+1})\sim\beta(\lambda_n)$ and $\varkappa_n[F]\nearrow 0$ as $n_0\le n\to\infty$ then [4] $\underline{\lim}_{\sigma\uparrow 0}\dfrac{\alpha(\ln M(\sigma,F))}{\beta(1/|\sigma|)}=p^{(0)}_{\alpha,\beta}[F]$ and as above we get the correctness of the second part of Theorem 3. □

As a conclusion, we present two statements corresponding to scales of finite order and finite R-order.

Proposition 1. *Let* $\sigma_a=0$. *If* $\overline{\lim}_{n\to\infty}\dfrac{\ln^+\ln|f_n|}{\ln\lambda_n}=\tau\in(0,\,1)$ *then*

$$\overline{\lim_{\sigma\uparrow 0}}(\ln M(\sigma/q,F)-p\ln M(\sigma,F))=+\infty.$$

for each $p>1$ *and* $q>1$ *such that* $\tau/(1-\tau)>\ln p/\ln q$.

If $\ln\ln n=o(\ln\lambda_n)$, $\ln\lambda_{n+1}\sim\ln\lambda_n$, $\varkappa_n[F]\nearrow 0$ *as* $n_0\le n\to\infty$ *and* $\overline{\lim}_{n\to\infty}\dfrac{\ln^+\ln|f_n|}{\ln\lambda_n}=$ $=\eta<1$ *then*

$$\underline{\lim}_{\sigma\uparrow 0}(\ln M(\sigma/q,F)-p\ln M(\sigma,F))=-\infty.$$

for each $p>1$ *and* $q>1$ *such that* $\eta/(1-\eta)<\ln p/\ln q$.

Proof. If $\tau>0$ then for every $\tau^0\in(0,\,\tau)$ there exists an increasing to ∞ sequence (k_n) such that $\ln|f_{k_n}|\ge\lambda_{k_n}^{\tau^0}$ for all n. We choose $\sigma_n=-(1/2)\lambda_{k_n}^{\tau^0-1}$. Then by the Cauchy inequality

$$\ln M(\sigma_n,F)\ge\lambda_{k_n}^{\tau^0}-\frac{1}{2}\lambda_{k_n}^{\tau^0}=\frac{1}{2}\lambda_{k_n}^{\tau^0}=\frac{1}{2}\left(\frac{1}{2|\sigma|}\right)^{\tau^0/(1-\tau^0)},$$

i. e. in view the arbitrariness of τ^0 we have

$$\overline{\lim_{\sigma\uparrow 0}}\frac{\ln\ln M(\sigma,F)}{\ln(1/|\sigma|)}\ge\frac{\tau^0}{1-\tau^0}>0$$

whence we obtain (15) with $\alpha(x)=\beta(x)=\ln^+x$ and $\tau^0/(1-\tau^0)$ instead $Q^{(0)}_{\alpha,\beta}[F]$, and thus, the first part of Proposition 1 is proved.

To prove the second part, it is enough to note that if $\ln\ln n=o(\ln\lambda_n)$, $\ln\lambda_{n+1}\sim\ln\lambda_n$, $\varkappa_n[F]\nearrow 0$ as $n_0\le n\to\infty$ and $\eta<1$ then [1] $\underline{\lim}_{\sigma\uparrow 0}\dfrac{\ln\ln M(\sigma,F)}{\ln(1/|\sigma|)}\ge\dfrac{\eta}{1-\eta}<+\infty$. □

Proposition 2. *Let* $\sigma_a=0$. *If* $\overline{\lim}_{k\to\infty}\dfrac{\ln\lambda_k}{\lambda_k}\ln^+|f_k|=Q_R>0$ *then*

$$\overline{\lim_{\sigma\uparrow 0}}(\ln M(\sigma+\ln q,F)-p\ln M(\sigma,F))=+\infty.$$

for each $p>1$ *and* $q>1$ *such that* $Q_R>\ln p/\ln q$.

If $\overline{\lim\limits_{k\to\infty}}\dfrac{\ln\ln k}{\ln\lambda_k} < 1$, $\ln\lambda_{k+1}\sim\ln\lambda_k$, $\varkappa_k[F]\nearrow 0$ *as* $k_0\le k\to\infty$ *and* $\underline{\lim\limits_{k\to\infty}}\dfrac{\ln\lambda_k}{\lambda_k}\ln^+|f_k| = q_R < +\infty$ *then*

$$\underline{\lim_{\sigma\uparrow 0}}(\ln M(\sigma+\ln q, F) - p\ln M(\sigma, F)) = -\infty.$$

for each $p > 1$ *and* $q > 1$ *such that* $q_R < \ln p/\ln q$.

Proof. If $Q_R > 0$ then for every $Q\in(0, Q_R)$ there exists an increasing to ∞ sequence (k_n) such that $\ln|f_{k_n}|\ge Q\lambda_{k_n}/\ln\lambda_{k_n}$ for all n. We choose $\sigma_n = -\xi/\ln\lambda_{k_n}$, where $0<\xi<Q$. Then by the Cauchy inequality

$$\ln M(\sigma_n, F)\ge\lambda_{k_n}\left(\frac{Q}{\ln\lambda_{k_n}}+\sigma\right) = \frac{(Q-\xi)\lambda_{k_n}}{\ln\lambda_{k_n}} = \frac{Q-\xi}{\xi}|\sigma|\exp\left\{\frac{\xi}{|\sigma|}\right\}$$

i. e. in view the arbitrariness of Q and ξ we have

$$\overline{\lim_{\sigma\uparrow 0}}|\sigma|\ln\ln M(\sigma, F)\ge Q_R > 0$$

whence we obtain (15) with $\alpha(x)=\ln^+ x$, $\beta(x)=x^+$ and Q_R instead $Q^{(0)}_{\alpha,\beta}[F]$, and thus, the first part of Proposition 2 is proved.

To prove the second part, it is enough to note that if $\overline{\lim\limits_{k\to+\infty}}\dfrac{\ln\ln k}{\ln\lambda_k} < 1$, $\ln\lambda_{k+1}\sim\ln\lambda_k$ and $\varkappa_k[F]\nearrow 0$ as $k_0\le k\to\infty$ then [2] $\underline{\lim\limits_{\sigma\uparrow 0}}|\sigma|\ln\ln M(\sigma, F) = q_R < +\infty$. □

References

[1] Boychuk V.S. *On the growth of Dirichlet series absolutely convergent in a half-plane.* Matem. sbornik. Naukova dumka, Kyiv, 1976, 238-240. (in Russian)

[2] Gaisin A. M. *A bound for the growth in a half-strip of a function represented by a Dirichlet series.* Math. sbornik. 1982, **117**(159):(3), 412-424. (in Russian)

[3] Gal' Yu.M., Sheremeta M.M. *On the growth of analytic fuctions in a half-plane given by Dirichlet series.* Doklady AN USSR, Ser. A. 1978, no. 12, 1064-1067. (in Russian)

[4] Gal' Yu.M. *On the growth of analytic functions given by Dirichlet series absolute convergent in a half-plane.* Drohobych, 1980. Dep. in VINITI, no. 4080-80 Dep. (in Russian)

[5] Goodstein R.L. Complex functions. New York, 1965.

[6] Mulyava O.M., Sheremeta M.M. *A remark to the growth of positive functions and its application to Dirichlet series.* Mat. Stud. 2015, **44**(2), 161-170. doi:10.15330/ms.44.2.161-170

[7] Pyanylo Ya.D., Sheremeta M.M. *On the growth of entire fuctions given by Dirichlet series.* Izv. Vyssh. Uchebn. Zaved. Mat. 1975, no. 10, 91-93. (in Russian)

[8] Sheremeta M.M. Entire Dirichlet series. ISDO, Kyiv, 1993. (in Ukrainian)

[9] Singh S.K. *On the maximum modulus and the means of an entire function.* Matem. Vesnik. 1976, **13**(28), 211-213.

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Шеремета М.М., Трухан Ю.С. *Про зростання максимуму модуля рядів Діріхле* // Буковинський матем. журнал — 2024. — Т.12, №1. — С. 32–42.

Для ряду Діріхле $F(s) = \sum_{n=0}^{\infty} f_n \exp\{s\lambda_n\}$ з невід'ємними зростаючими $+\infty$ показниками λ_n і абсцисою абсолютної збіжності $\sigma_a \in (-\infty, +\infty]$ вивчено зв'язок між зростанням на $(-\infty, \sigma_a)$ максимума модуля $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ і поводженням коефіцієнтів f_n. Для цього через L позначено клас неперервних зростаючих до $+\infty$ на $(x_0, +\infty)$ функцій α. Належність α до класу L^0 означає, що $\alpha \in L$ і $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ при $x \to +\infty$, а $\alpha \in L_{si}$, якщо $\alpha \in L$ і $\alpha(cx) = (1 + o(1))\alpha(x)$ при $x \to +\infty$.

Для цілих рядів Діріхле ($\sigma_a = +\infty$), наприклад, доведено, що якщо $\alpha \in L$, $\beta \in L^0$, то $\overline{\lim\limits_{\sigma\to+\infty}} \big(\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}\big) = +\infty$ для таких $p > 1$ і $q > 1$, що $\overline{\lim\limits_{n\to\infty}} \alpha(\lambda_n)/\beta\left(\lambda_n^{-1} \ln(1/|f_n|)\right) > \ln p/\ln q$. Якщо ж $\alpha \in L_{si}$, $\beta \in L^0$, $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ при $x \to +\infty$ і $\ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n)))$ при $n \to \infty$ для кожного $c \in (0, +\infty)$, $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ при $n \to \infty$ і $\dfrac{\ln|f_n| - \ln|f_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$ при $n_0 \le n \to \infty$, то $\underline{\lim\limits_{\sigma\to+\infty}} \big(\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}\big) = -\infty$ для таких $p > 1$ і $q > 1$, що $\underline{\lim\limits_{n\to\infty}} \alpha(\lambda_n)/\beta\left(\lambda_n^{-1} \ln(1/|f_n|)\right) < \ln p/\ln q$.

Подібні результати отримано для рядів Діріхле, абсолютно збіжних у півплощині $\{s : \text{Re}s < 0\}$. Наприклад, доведено, що якщо $\sigma_a = 0$, $\beta \in L_{si}$, $\alpha(e^x) \in L_{si}$ і $\alpha(x) = o(\beta(x))$ при $x \to +\infty$, то

$$\overline{\lim_{\sigma\uparrow 0}} \left(\exp\left\{\alpha\left(\ln M\left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)}, F\right)\right)\right\} - p \exp\{\alpha(\ln M(\sigma, F))\}\right) = +\infty$$

для таких $p > 1$ і $q > 1$, що $\overline{\lim\limits_{n\to\infty}} \dfrac{\alpha(\lambda_n)}{\beta(\lambda_n/\ln^+ |f_n|)} > \ln p/\ln q$.