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AVERAGING IN MULTIFREQUENCY SYSTEMS WITH LINEARLY TRANSFORMED ARGUMENTS AND INTEGRAL DELAY

The question of existence and uniqueness of the continuously differentiable solution for a multifrequency system of differential equations with variable linearly transformed and integral delay is investigated. The method of averaging by fast variables on a finite interval is substantiated. An estimate of the averaging method was obtained, which clearly depends on the small parameter and the number of fast variables and their delays.

Key words and phrases: multifrequency system, averaging method, resonance, delay, small parameter.

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Introduction

An oscillatory system of n oscillators with a delay and a small interaction force in a fairly general case takes the form

$$\frac{d^2u_\nu}{dt^2} + \omega_\nu^2(\tau)u_\nu = \varepsilon f_\nu(\tau, u, u_\Lambda, \frac{du}{dt}, \frac{du_\Lambda}{dt}),$$

$\nu = \overline{1,n}$, $\tau = \varepsilon t$, $0 < \varepsilon$ – small parameter, $u_\Lambda(t) = \big(u_1(t-\lambda_1), \dots, u_n(t-\lambda_n)\big)$, $\lambda_\nu > 0$ and characterize the delay, $\omega_\nu(\tau) > 0$ – slowly changing natural frequencies. After making the replacement

$$u_\nu = a_\nu \cos\varphi_\nu, \quad \frac{du_\nu}{dt} = -a_\nu\omega_\nu \sin\varphi_\nu, \tag{1}$$

we obtain a system of equations with n slow variables a_ν and fast variables φ_ν of the form

$$\frac{da_\nu}{d\tau} = A_\nu(\tau, a_\Lambda, \varphi_\Lambda)\sin\varphi_\nu + \frac{a_\nu(1-2\cos 2\varphi_\nu)}{2\omega_\nu(\tau)}\frac{d\omega_\nu(\tau)}{d\tau},$$

$$\frac{d\varphi_\nu}{d\tau} = \frac{\omega_\nu(\tau)}{\varepsilon} + A_\nu(\tau, a_\Lambda, \varphi_\Lambda) - \frac{\sin 2\varphi_\nu}{2\omega_\nu(\tau)}\frac{d\omega_\nu(\tau)}{d\tau},$$

where functions $A_\nu = f$ according to (1), depend on $\tau, a, \varphi, a_\Lambda, \varphi_\Lambda$, $\nu = \overline{1,n}$.

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This type of systems, the general form of which is

$$\frac{da}{d\tau} = X(\tau, a, \varphi), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau, a)}{\varepsilon} + Y(\tau, a, \varphi), \tag{2}$$

is difficult to study and solve, so the averaging procedure for fast variables on the cube of periods is used [1, 2, 3]. Multifrequency systems of ordinary differential equations were studied in the works of V. I. Arnold [1], E. O. Grebenikov, Yu. A. Mitropolsky and Yu. A. Ryabov [2] and others. A. M. Samoilenko and R. I. Petryshyn obtained significant results when researching multifrequency systems by the averaging method [3].

Multifrequency systems with a delay were studied in the works of [4, 5, 6] and others. A well-studied case is where the delay is set by linearly transformed arguments for systems with initial, multipoint, and integral conditions [7, 8].

The solution of the averaged system in the general case may deviate from the solution of the exact system by $O(1)$ on time intervals of the length $O(\varepsilon^{-1})$ or by $(0, \infty)$. The reason is the resonance phenomena, the condition of which is at the point τ for multifrequency systems (2)

$$(k, \omega) := k_1\omega_1 + \cdots + k_m\omega_m \cong 0, \quad k \neq 0.$$

For the systems with the delay, the condition takes the form

$$\gamma_k(\tau) := \sum_{\nu=1}^{q} \big(k_\nu, \theta_\nu \omega(\theta_\nu \tau)\big) = 0, \quad k_\nu \in \mathbb{Z}^m, \quad k \neq 0, \tag{3}$$

$0 < \theta_1 < \theta_2 < \cdots < \theta_q \leq 1$, $\varphi_{\theta_\nu}(\tau) = \varphi(\theta_\nu \tau)$, as shown in [4].

In this work, we investigated by averaging method a multifrequency system with delays, which are set by linearly transformed arguments and also contains variable integral delays of the form

$$v_\nu(a) = \int\limits_{\Delta_\nu \tau}^{\tau} g_\nu(s) a(s) ds, \quad \nu = \overline{1, r}, \quad 0 < \Delta_\nu < 1. \tag{4}$$

Parabolic systems with an integral delay of a similar form were studied in [9].

1 Formulation of the problem

We investigate a system of differential equations of the form

$$\frac{da}{d\tau} = X(\tau, a_\Lambda, v_\Delta(a), \varphi_\Theta), \tag{5}$$

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_\Lambda, v_\Delta(a), \varphi_\Theta), \tag{6}$$

where $\tau = \varepsilon t \in [0, L]$, $a \in D$ – limited closed area in $\mathbb{R}^n$, $\varphi \in \mathbb{R}^m$, small parameter $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, $a_\Lambda = (a_{\lambda_1}, \ldots, a_{\lambda_p})$, $\varphi_\Theta = (\varphi_{\theta_1}, \ldots, \varphi_{\theta_q})$, $0 < \lambda_1 < \cdots < \lambda_p \leq 1$, $0 < \theta_1 < \cdots < \theta_q \leq 1$, $a_{\lambda_i}(\tau) = a(\lambda_i \tau)$, $\varphi_{\theta_j}(\tau) = \varphi(\theta_j \tau)$. Vector-functions X and Y are 2π-periodic by components of the vector φ_Θ. The integral delay of the form (4) is set by the components of the vector $v_\Delta(a)$.

The corresponding (5), (6) system averaged over fast variables takes the form

$$\frac{d\overline{a}}{d\tau} = X_0(\tau, \overline{a}_\Lambda, \overline{v}_\Delta), \tag{7}$$

$$\frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y_0(\tau, \overline{a}_\Lambda, \overline{v}_\Delta). \tag{8}$$

To prove the existence and uniqueness of the classical solution of the system of equations (5), (6), the initial conditions of which coincide with the initial conditions of the solution of the averaged system (7), (8) and to justiy the averaging method we apply the estimate of the oscillatory integral corresponding to the system (5), (6)

$$I_k(\tau, \varepsilon) = \int_0^\tau f_k(s, \varepsilon) \exp\Big(\frac{i}{\varepsilon}\int_0^s \gamma_k(z)dz\Big)ds, \tag{9}$$

where the function f_k is determined through the Fourier coefficients of the vector functions X and Y. For a sufficiently small $\varepsilon^* \in (0, \varepsilon_0]$, sufficient conditions for performing the estimation are specified

$$u(\tau; \overline{y}, \overline{\psi}, \varepsilon) := \|a(\tau; \overline{y}, \overline{\psi}, \varepsilon) - \overline{a}(\tau; \overline{y})\| + \|\varphi(\tau; \overline{y}, \overline{\psi}, \varepsilon) - \overline{\varphi}(\tau; \overline{y}, \overline{\psi}, \varepsilon)\| \leq c_1\varepsilon^\alpha, \tag{10}$$

for $0 < \varepsilon \leq \varepsilon^*$ and for all $0 \leq \tau \leq L$, $\alpha = 1/mq$.

2 Technical lemma

Let $d = const > 0$, $J = [\tau_0, L]$, $\tau_0 \geq 0$.

Lemma 1. *Let the following conditions be satisfied:*

1) *nondecreasing on J differentiable functions $\alpha_\nu : J \to J$, $\alpha_\nu(\tau) \leq \tau$, $\nu = \overline{1, p}$;*
2) *nondecreasing continuous on J functions $\beta_\nu : J \to J$, $\beta_\nu(\tau) \leq \tau$, $\nu = \overline{1, r}$;*
3) *$f_\nu \in \mathbb{C}(J)$, $f_\nu : J \to \mathbb{R}_+$, $\nu = \overline{1, p}$;*
4) *$h_\nu \in \mathbb{C}(J)$, $h_\nu : J \to \mathbb{R}_+$, $\nu = \overline{1, r}$;*
5) *the inequality is true*

$$u(t) \leq d + \sum_{\nu=1}^{p} \int_{\alpha_\nu(\tau_0)}^{\alpha_\nu(\tau)} f_\nu(s)u(s)ds + \int_{\tau_0}^{\tau} \sum_{\nu=1}^{r} \Big(\int_{\beta_\nu(s)}^{s} h_\nu(z)u(z)dz\Big)ds, \quad \tau \in J. \tag{11}$$

Then for $\tau \in J$

$$u(t) \leq d \cdot exp\Big(\int_{\tau_0}^{\tau} \sum_{\nu=1}^{p} f_\nu\big(\alpha_\nu(s)\big)\alpha_\nu'(s)ds + \int_{\tau_0}^{\tau} \sum_{\nu=1}^{r} (\int_{\beta_\nu(s)}^{s} h_\nu(z)dz)ds\Big). \tag{12}$$

Proof. We denote by $v(\tau)$ the right-hand side of the inequality (11). Then

$$v(\tau_0) = d; \quad u(\tau) \leq v(\tau), \quad \tau \in J.$$

Since

$$v^{'}(\tau) \geq 0, \quad u(\alpha_\nu(\tau)) \leq v(\alpha_\nu(\tau)) \leq v(\tau)$$

and similarly for $\beta_\nu(\tau)$, then calculating the derivative $v^{'}(\tau)$, we obtain

$$v^{'}(\tau) = \sum_{\nu=1}^{p} f_\nu(\alpha_\nu(\tau))\alpha_\nu^{'}(\tau)u(\alpha_\nu(\tau)) + \sum_{\nu=1}^{r} \int\limits_{\beta_\nu(\tau)}^{\tau} h_\nu(z)u(z)dz$$

$$\leq \Big(\sum_{\nu=1}^{p} f_\nu(\alpha_\nu(\tau))\alpha_\nu^{'}(\tau) + \sum_{\nu=1}^{r} \int\limits_{\beta_\nu(\tau)}^{\tau} h_\nu(z)dz\Big)v(\tau).$$

Dividing by $v(\tau)$ and integrating, we obtain the inequality (12). □

Corollary 1. *Let $\tau \in [0, L]$, f_ν and h_ν – non-negative numbers, $\alpha_i(\tau) = \lambda_i\tau$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_p \leq 1$; $\beta_j = \Delta_j\tau$, $0 < \Delta_1 < \Delta_2 < \cdots < \Delta_r \leq 1$, and the inequality* (11) *holds true. Then*

$$u(t) \leq d \cdot exp\big(\sum_{\nu=1}^{p} f_\nu\lambda_\nu + 0.5\sum_{\nu=1}^{r} h_\nu(1-\Delta_\nu)\tau\big)\tau, \tau \in [0, L]. \tag{13}$$

Corollary 2. *Let $h_\nu = 0$, $\nu = \overline{0, r}$, $p = 1$. Then the solution of the integral inequality* (11) *takes the form[10]*

$$u(\tau) \leq d \cdot \exp\Big(\int\limits_{\tau_0}^{\tau} f_\nu(\alpha(s))\alpha^{'}(s)ds\Big) = d \cdot \exp\Big(\int\limits_{\alpha(\tau_0)}^{\alpha(\tau)} f_\nu(s)ds\Big).$$

For $\alpha(\tau) = \tau$, $\tau \in [0, L]$ the Bellman inequality is obtained[10]

$$u(\tau) \leq d \cdot \exp\Big(\int\limits_{0}^{\tau} f(s)ds\Big).$$

3 Justification Of The Averaging Method

Theorem 1. *Let the conditions be satisfied:*

1) *vector functions X and Y are defined and continuously differentiable by all variables at $(\tau, a, \varphi) \in G = [0, L] \times \mathbb{D} \times \mathbb{R}^m$ and are bounded together with the first derivatives by the constant σ_1;*

2) *vector function $\omega \in \mathbb{C}^{mq-1}[0, L]$ and according to the system $\big\{\omega(\theta_1\tau), \ldots, \omega(\theta_q\tau)\big\}$ Vronsky determinant*

$$W(\tau) \neq 0, \quad \tau \in [0, L];$$

3) in the area $G_1 = [0, L] \times \Delta$

$$\sum_{k\neq 0}\Big(\sup_{G_1}\|X_k\| + \frac{1}{\|k\|_\Theta}\big(\sup_{G_1}\Big\|\frac{\partial X_k}{\partial\tau}\Big\| + \sum_{\nu=1}^{p}\lambda_\nu\sup_{G_1}\Big\|\frac{\partial X_k}{\partial a_{\lambda_\nu}}\Big\|$$
$$+\sum_{\nu=1}^{r}(1-\Delta_\nu)\sup_{G_1}\Big\|\frac{\partial X_k}{\partial v_{\Delta\nu}}\frac{\partial v_{\Delta\nu}}{\partial\tau}\Big\|\big)\Big) \leq \sigma_2$$

4) functions $g_\nu \in \mathbb{C}([0, L])$*;*

5) there is a unique solution $\big(\overline{a}(\tau;\overline{y}), \overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\big)$ *of the averaged problem, which lies in* $\mathbb{D}$ *for* $\tau \in [0, L]$ *together with some* ρ*-circumference.*

Then for a sufficiently small $\varepsilon^* > 0$ *there is a unique solution of the system of equations* (5) – (6) *with initial conditions* $(\overline{y}, \overline{\psi})$ *and for each* $\varepsilon \in (0, \varepsilon^*]$ *and for* $\tau \in [0, L]$ *the estimation* (10) *is performed.*

Proof. From the differentiability of the right parts of the system of equations (5), (6) it follows the existence of a solution on the interval $(0, T)$. From the equation (5), (7) for $\tau \in (0, T)$ we have

$$\|a(\tau; y, \psi, \varepsilon) - \overline{a}(\tau; y)\| = \int_0^\tau \Big(X(s, a_\Lambda, v_\Delta, \varphi_\Theta) - X_0(s, \overline{a}_\Lambda, \overline{v}_\Delta)\Big)ds$$
$$= \int_0^\tau \Big(X(s, a_\Lambda, v_\Delta, \varphi_\Theta) - X(s, \overline{a}_\Lambda, \overline{v}_\Delta, \overline{\varphi}_\Theta)\Big)ds + \int_0^\tau \tilde{X}(s, a_\Lambda, v_\Delta, \varphi_\Theta)ds = I_1 + I_2,$$

where

$$I_2(\tau, \varepsilon) = \sum_{k\neq 0}\int_0^\tau X_k(s, \overline{a}_\Lambda, \overline{v}_\Delta)e^{i(k,\varphi_\Theta)}ds, \quad (k, \varphi_\Theta) = \sum_{\nu=1}^{q}(k_\nu, \varphi_{\theta_\nu}).$$

Let's construct estimates for the norms I_1 and I_2. We have

$$I_1 = \int_0^\tau \Big(X(s, a_\Lambda, v_\Delta, \varphi_\Theta) - X(s, \overline{a}_\Lambda, v_\Delta, \varphi_\Theta)\Big)ds$$
$$+ \int_0^\tau \Big(X(s, \overline{a}_\Lambda, v_\Delta, \varphi_\Theta) - X(s, \overline{a}_\Lambda, \overline{v}_\Delta, \varphi_\Theta)\Big)ds$$
$$+ \int_0^\tau \Big(X(s, \overline{a}_\Lambda, \overline{v}_\Delta, \varphi_\Theta) - X(s, \overline{a}_\Lambda, \overline{v}_\Delta, \overline{\varphi}_\Theta)\Big)ds = I_{11} + I_{12} + I_{13}.$$

It follows from condition 1 of the theorem, that

$$\|I_{11}\| \leq \sigma_1\sum_{\nu=1}^{p}\int_0^\tau \|a_{\lambda_\nu} - \overline{a}_{\lambda_\nu}\|ds = \sigma_1\sum_{\nu=1}^{p}\lambda_\nu\int_0^{\lambda_\nu\tau}\|a - \overline{a}\|ds. \tag{14}$$

Since

$$v_{\Delta_\nu} - \overline{v}_{\Delta_\nu} = \int\limits_{\lambda_\nu \tau}^{\tau} g_\nu(s)(a - \overline{a})ds,$$

then for estimation of the norms I_{12} we have

$$\|I_{12}\| \le \sigma_1 \sum_{\nu=1}^{r} \int\limits_0^\tau \Big(\int\limits_{\Delta_s s}^{s} |g_\nu(z)| \|a - \overline{a}\| dz \Big) ds. \tag{15}$$

For the norm I_{13} we obtain

$$\|I_{13}\| \le \sqrt{2} \sum_{k \ne 0} \|X_k\| \int\limits_0^\tau \sum_{\nu=1}^{q} \|\varphi_{\theta_\nu} - \overline{\varphi}_{\theta_\nu}\| ds = c_2 \sum_\nu \theta_\nu \int\limits_0^{\theta_\nu \tau} \|\varphi - \overline{\varphi}\| ds,$$

where $c_2 = \sqrt{2} \sum\limits_{k \ne 0} \|k\| \max\limits_{\tau \in [0,L]} \|X_k(\tau, \overline{a}(\tau), \overline{v}_\Delta)\|$.

We estimate the norm I_2 based on the estimation of the oscillatory integral (9)

$$\|I_k(\tau, \varepsilon)\| \le \sigma_3 \varepsilon^\alpha \Big(\sup_{G_1} \|f_k(\tau, \varepsilon)\| + \frac{1}{\|k\|_\Theta} \Big\| \frac{df_k(\tau, \varepsilon)}{d\tau} \Big\| \Big),$$

where $k \ne 0$, $G_1 = [0, L] \times (0, \varepsilon_0]$, $\|k\|_\Theta = \sum\limits_{\nu=1}^{q} \theta_\nu \|k_\nu\|$, $\sigma_3 > 0$ and does not depend on ε.

Let's represent I_2 in the form (9), where

$$f_k(\tau, \varepsilon) = X_k(\tau, \overline{a}_\Lambda, \overline{v}_\Delta) \exp\Big(i \sum_{\nu=1}^{q} (k_\nu, \varphi_0)\Big) \exp\Big(i \sum_{\nu=1}^{q} \int\limits_0^{\theta_\nu \tau} \big(k_\nu, Y_0(x, \overline{a}_\Lambda, \overline{v}_\Delta)\big)\Big) dx \equiv$$

$$\equiv X_k(\tau, \overline{a}_\Lambda, \overline{v}_\Delta) \cdot I_3 I_4.$$

It follows from the form of the vector function f_k, that

$$\|f_k(\tau, \varepsilon)\| = \|X_k(\tau, \overline{a}_\Lambda, \overline{v}_\Delta)\|.$$

Since

$$\Big\| \frac{dv_{\Delta_\nu}}{d\tau} \Big\| \le (1 - \Delta_\nu) \sigma_4 \max_{\tau \in I} \|\overline{a}(\tau; \overline{y})\| = (1 - \Delta_\nu) c_2,$$

then we will get an estimate

$$\Big\| \frac{df_k}{d\tau} \Big\| \le \Big\| \frac{\partial X_k}{\partial \tau} \Big\| + \sigma_1 \sum_{\nu=1}^{p} \lambda_\nu \Big\| \frac{\partial X_k}{\partial \overline{a}_{\lambda_\nu}} \Big\| + c_2 \sum_{\nu=1}^{r} (1 - \Delta_\nu) \Big\| \frac{\partial X_k}{\partial \overline{v}_{\Delta \nu}} \frac{\partial \overline{v}_{\Delta \nu}}{\partial \tau} \Big\| + \sigma_1 \|k\|_\Theta \|X_k\|.$$

Let $c3 = \max(1 + \sigma_1, c_2)$. Then the estimate for I_2 takes the form

$$\begin{aligned} \|I_2(\tau, \varepsilon)\| \le c_3 \sigma_3 \varepsilon^\alpha \sum_{k \ne 0} \Big(\sup_G \|X_k\| + \frac{1}{\|k\|_\Theta} \big(\sup_G \Big\| \frac{\partial X_k}{\partial \tau} \Big\| + \sum_{\nu=1}^{p} \lambda_\nu \sup_G \Big\| \frac{\partial X_k}{\partial \overline{a}_{\lambda_\nu}} \Big\| \\ + \sum_{\nu=1}^{r} (1 - \Delta_\nu) \sup_G \Big\| \frac{\partial X_k}{\partial \overline{v}_{\Delta \nu}} \frac{\partial \overline{v}_{\Delta \nu}}{\partial \tau} \Big\| \big) \Big) = c_4 \varepsilon^\alpha \end{aligned} \tag{16}$$

where $\alpha = 1/mq$, $\tau \in [0, L]$, $\varepsilon \in (0, \varepsilon_1]$, $\varepsilon_1 \le \varepsilon_0$.

Based on the estimates (14)–(16) we obtain

$$\|a(\tau;\overline{y},\overline{\psi},\varepsilon) - \overline{a}(\tau;\overline{y})\| \le \sigma_1 \sum_{\nu=1}^{p} \lambda_\nu \int_0^{\lambda_\nu \tau} \|a - \overline{a}\| ds$$

$$+\sigma_1 \sum_{\nu=1}^{r} \int_0^{\tau} \Big(\int_{\Delta_\nu s}^{s} |g_\nu(z)| \|a - \overline{a}\| dz \Big) ds + c_2 \sum_{\nu=1}^{q} \theta_\nu \int_0^{\theta_\nu \tau} \|\varphi - \overline{\varphi}\| ds + c_4 \varepsilon^\alpha.$$

A similar inequality is obtained in the estimation $\|\varphi(\tau;\overline{y},\overline{\psi},\varepsilon) - \overline{\varphi}(\tau;\overline{y})\|$. Therefore, in the end we have for $\tau \in (0, T)$ and $\varepsilon \in (0, \varepsilon_1]$

$$u(\tau;\overline{y},\overline{\psi},\varepsilon) \le 2c_4\varepsilon^\alpha + \sigma_1 \sum_{\nu=1}^{p} \lambda_\nu \int_0^{\lambda_\nu \tau} u(s;\overline{y},\overline{\psi},\varepsilon) ds$$

$$+c_2 \sum_{\nu=1}^{q} \theta_\nu \int_0^{\theta_\nu \tau} u(s;\overline{y},\overline{\psi},\varepsilon) ds + \sigma_1 \sum_{\nu=1}^{r} \int_0^{\tau} \Big(\int_{\Delta_\nu s}^{s} |g_\nu(z)| u(z;\overline{y},\overline{\psi},\varepsilon) dz \Big) ds.$$

Let us apply the integral inequality (13) from the corollary 1, where $d = 2c_4\varepsilon^\alpha$, $h_\nu = \max_{[0,L]} |g(\tau)|$.

We obtain

$$u(\tau;\overline{y},\overline{\psi},\varepsilon) \le 2c_4 \Big(\exp\big(\sigma_1 \sum_{\nu=1}^{p} \lambda_\nu + c_2 \sum_{\nu=1}^{q} \theta_\nu + 0.5\sigma_1 \tau \sum_{\nu=1}^{r} h_\nu (1 - \Delta_\nu)\big)\tau \Big) \varepsilon^\alpha \equiv c_5(\tau)\varepsilon^\alpha,$$

where $\tau \in [0, T)$, $0 < \varepsilon \le \varepsilon_1$.

It follows from the satisfaction of condition 5 of the theorem that for $c_5(L)\varepsilon^\alpha \le \rho/2$, that is $\varepsilon \le \varepsilon_2 = (\rho/2c_5(L))^{mq}$ the component of the solution $a(\tau;\overline{y},\overline{\psi},\varepsilon)$ lies in the ρ-circumference of the solution $a(\tau;\overline{y})$. Therefore, for $\varepsilon^* = \min(\varepsilon_1, \varepsilon_2)$ we have $T = L$ and the estimate (10) is performed for all $\tau \in [0, L]$ with the constant $c_1 = c_5(L)$.

□

4 Model Example

Example 1. *Consider the single-frequency system of equations*

$$\frac{da}{d\tau} = b_0 a + b_1 a_\lambda + b_2 \int_{\lambda\tau}^{\tau} a(s) ds + b_3 \cos(k\varphi + l\varphi_\theta),$$

$$\frac{d\varphi}{d\tau} = \frac{1 + 2\tau}{\varepsilon}, \quad a(0) - 1 = \varphi(0) = 0,$$

where $b_\nu \in \mathbb{R}/\{0\}$, $\nu = \overline{0,3}$, $\lambda, \theta \in (0,1)$, $k, l \in \mathbb{Z}$, $k + l\theta = 0$.

The solution of the equation for the fast variable is $\varphi(\tau, \varepsilon) = \tau(1+\tau)/\varepsilon$, *therefore* $k\varphi + l\varphi_\theta = \kappa\tau^2/\varepsilon$, $\kappa = k + l\theta^2 \ne 0$.

The resonance condition is satisfied because $\gamma_{kl}(\tau) = 2\tau\kappa = 0$ *at* $\tau = 0$.
The averaged equation for the slow variable is

$$\frac{d\overline{a}}{d\tau} = b_0\overline{a} + b_1\overline{a}_\lambda + b_2 \int\limits_{\lambda\tau}^{\tau} \overline{a}(s)ds, \quad \overline{a}(0) = 1.$$

If $b_1 = b_2 = 1 - b_0$, *then the solution of the problem is* $\overline{a}(\tau) = e^\tau$.
Then we have

$$v(\tau,\varepsilon) := a(\tau,\varepsilon) - \overline{a}(\tau) = b_0 \int\limits_0^{\tau} \big(a(s,\varepsilon) - \overline{a}(s)\big)ds$$

$$+b_1\lambda^{-1} \int\limits_0^{\lambda\tau} \big(a(s,\varepsilon) - \overline{a}(s)\big)ds + b_2 \int\limits_0^{\tau}\int\limits_{\lambda s}^{s} \big(a(z,\varepsilon) - \overline{a}(z)\big)dzds + \frac{\sqrt{\varepsilon}}{\sqrt{\kappa}} \int\limits_0^{\sqrt{\pi}\tau/\sqrt{\kappa}} \cos x^2 dx.$$

Based on the inequality (13) *we obtain*

$$|v(\tau,\varepsilon)| \le d\exp(b_0 + b_1 + b_2\tau(1-\lambda)/2)\tau.$$

Let $\varepsilon \le 4\kappa/\pi^2$, *then* $\tau \le 1$. *It follows from the asymptotics of the Fresnel integral [11]*

$$d = \frac{\sqrt{\varepsilon}}{\sqrt{\kappa}}\frac{\sqrt{\pi}}{2\sqrt{2}} + O(\sqrt[4]{\varepsilon^3}) \le c_5\sqrt{\varepsilon}, c_5 = \frac{\sqrt{\pi}}{\sqrt{2\kappa}}.$$

Therefore, for $\tau \in [0,1]$ *we have*

$$|v(\tau,\varepsilon)| \le |v(1,\varepsilon)| \le c_6\sqrt{\varepsilon}, \quad c_6 = c_5\exp\big(b_0 + b_1 + b_2(1-\lambda)/2\big).$$

It should be noted that if $k + l\theta^2 = 0$, *then* $k + l\theta \ne 0$, *there is no resonance and the estimation of the error of the averaging method is of* ε *order.*

References

[1] Arnold V.I. Mathematical methods of classical mechanics. Springer, 1989, Vol. 60.

[2] Grebennikov E.A., Ryabov Yu.A. Constructive Methods in the Analysis of Nonlinear Systems. Mir, Moscow, 1983.

[3] Samoilenko A.M., Petryshyn R.I. Multifrequency Oscillations of Nonlinear Systems. Kluwer Academic Publishers, Dordrecht, 2004.

[4] Bihun Ya.Y. *Averaging of a multifrequency boundary-value problem with linearly transformed argument.* Ukrainian Mathematical Journal. 2000, **52** (3), 291–299.

[5] Samoilenko A.M., Bihun Ya.Y. *The averaging of nonlinear oscillation systems of the highest approximation with a delay.* Nonlinear Oscillations. 2002, **5** (1), 77–85.

[6] Petryshyn R.I, Bihun Ya.Y. *About Averagin in systems with linearly transformed agrument in resonance case.* Scientific Herald of Yuriy Fedkovych Chernivtsi national university. Series of math. 2008, 84–89.

[7] Bihun Y., Skutar I. *Averaging in Multifrequency Systems with Delay and Local-Integral Conditions.* Bukovynian Mathematical Journal. 2020, **8** (2), 14–23. doi:https://doi.org/10.31861/bmj2020.02.02

[8] Bihun Y., Petryshyn R., Krasnokutska I. *Averaging method in multifrequency systems with linearly transformed arguments and with point and integral condstions.* Acta et Coomentationes,Exact and Natural Sciences. 2018, **2** (6), 20–27.

[9] Bokalo M., Ilnytska O. *The classical solutions of the parabolic equations with variable integral delay.* Bukovinian Math. Journal. 2017, **5** (1–2), 18–36.

[10] Pachpatte B.G. *Explicit Bounds on Certain Integral Inequalities.* Journal of Mathematical Analysis and Applications. 2002, **267**, 48–61.

[11] Bateman H., Erdelyi A. Higher transcendental functions. McGraw-Hill, New York, 1953, Vol. II.

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Бігун Я.Й., Скутар І.Д., Бардан А.О. *Усереднення в багаточастотних системах із лінійним перетворенням й інтегральним запізненням* // Буковинський матем. журнал — 2023. — Т.11, №2. — С. 24–32.

Математичними моделями багатьох коливних систем є диференціальні рівняння з повільними $a(\tau)$ і швидкими $\varphi(\tau)$ змінними. Для дослідження і побудови наближеного розв'язку застосовується процедура усереднення за швидкими змінними.

У статті досліджено існування і єдиність диференціального розв'язку m-частотної системи вигляду

$$\frac{da}{d\tau} = X(\tau, a_\Lambda, v_\Delta(a), \varphi_\Theta), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_\Lambda, v_\Delta(a), \varphi_\Theta),$$

із початковими умовами в точці $\tau = 0$. Тут $\tau \in [0, L]$, ε–малий параметр. Компоненти векторів Λ, Δ, Θ задають лінійно перетворені аргументи, які характеризують запізнення. Змінною v_Δ задається розподілене запізнення.

Система в процесі еволюції може проходити через резонанси, умова яких

$$\sum_{\nu=1}^{q} \big(k_\nu, \theta_\nu \omega(\theta_\nu \tau)\big) = 0.$$

Вказано умови, при виконанні яких існує єдиний розв'язок й отримано оцінку похибки методу усереднення, порядок якої ε^α, $\alpha = 1/(mq)$.