Bukovinian Math. Journal. 2023, **11** (2) , 51–58 Буковинський матем. журнал 2023, Т.11, №2, 51–58

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APPLICATION OF GENERATIVE FUNCTIONS TO THE PROBLEMS OF MAXIMUM CHESS ARRANGEMENTS OF N FIGURES.

A generating function is a formal structure that is closely related to a numerical sequence, but allows us to manipulate the sequence as a single entity, with the goal of understanding it better. Roughly speaking, generating functions transform problems about sequences into problems about functions. They provide a systematic way to encode sequences of numbers or other combinatorial objects, allowing for elegant solutions to complex problems across diverse mathematical domains.

In this article, we will approach a range of problems, involving placing n chess pieces on an $n \times n$ chessboard so that no two pieces attack each other, using the generating functions approach.

Key words and phrases: Exact enumeration problems, generating functions, n queens problem.

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Generating functions.

Given sequence $\{a_n\} \in \mathbb{R}$ the generative function for it is defined as: $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Generative functions provide a great way to operate over sequences, solve recurrent series, understand different enumeration problems etc. [2]

Lets take a look at a simple example of a famous Euler's problem, and how it can be solved using generating functions.

УДК 519.115.1
2010 *Mathematics Subject Classification:* 05A15.

Example 1. *Which weights can be weighted using given weights* 1, 2, 2^2,..., 2^n, ... *and in how many ways?*

Consider the function $f(x) = (1+x)(1+x^2)(1+x^4)... = \prod_{n=0}^{\infty}(1+x^{2^n}) = 1 + a_1x + a_2x^2 + ...$ Here, the a_n is exactly the amount of different ways one can add numbers $1, 2, 2^2, 2^3, ..., 2^k, ...$ in order to get n. Now we can calculate these a_n, by first calculating the function $f(x)$. For that, lets multiply it by $(1-x)$:

$$(1-x)f(x) = (1-x)\prod_{n=0}^{\infty}(1+x^{2^n}) = (1-x^2)\prod_{n=1}^{\infty}(1+x^{2^n}) = (1-x^4)\prod_{n=2}^{\infty}(1+x^{2^n}) = ... = 1$$

By dividing by $(1-x)$:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So we have, that $a_n = 1$ for every $n \in \mathbb{N} \cup \{0\}$. That implies, that there is always exactly one way to weight the n using weights 2^k, $k \in \mathbb{N} \cup \{0\}$.

Generative functions can be used to solve the recurrent series. Take, for example, famous Fibonacci sequence:

Example 2. *Given sequence:* $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$, *find its generating function.*

For it we can write generative function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty}(a_{n-1} + a_{n-2})x^n$$

$$= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 1 + x + \sum_{n=0}^{\infty} a_n x^{n+1} - x + \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= 1 + x\sum_{n=0}^{\infty} a_n x^n + x^2\sum_{n=0}^{\infty} a_n x^n = 1 + xf(x) + x^2 f(x).$$

So, we got the following equality: $f(x) = 1 + xf(x) + x^2f(x)$. From this, we get the generating function for Fibonacci sequence:

$$f(x) = \frac{1}{1-x-x^2}.$$

Now, we can write partial fraction expansion for this function:

$$f(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1}{2}(1+\sqrt{5})x} - \frac{1}{1-\frac{1}{2}(1-\sqrt{5})x}\right)$$

n queens problem and some approaches.

The n queens problem is the problem of placing n chess queens on an $n \times n$ chessboard so that no two queens attack each other. Solutions exist for all natural numbers n with the exception of $n = 2$ and $n = 3$. Although the exact number of solutions is only known for $n = 27$, the asymptotic growth rate of the number of solutions is approximately $(0.143n)^n$.

Although the n-queens problem is often commonly used as a benchmark for algorithms that solve combinatorial optimization problems; it has found several real-world applications, including practical task scheduling and assignment, computer resource management, and VLSI testing. [1])

Counting the number of solutions for the n queen problem is not easy. Since no known exact mathematical formula was found, just approximation, the solution's number became too large to enumerate one by one except for small n.

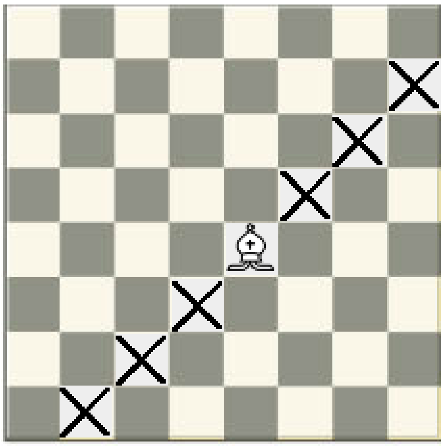
Interestingly, there is no guarantee to increase the number of solutions as n increases. For instance, the number of solutions for the six-Queen chessboard is less than that for the five-Queen. The 27×27 board is the highest-order board that has been wholly enumerated in approximately one year for more information see project Q27 on the web.

Other figures are also often considered as problems for non-attacking chess arrangements [3].

So far, there still remains open problem on the maximum arrangements of n queens on the board $n \times n$.

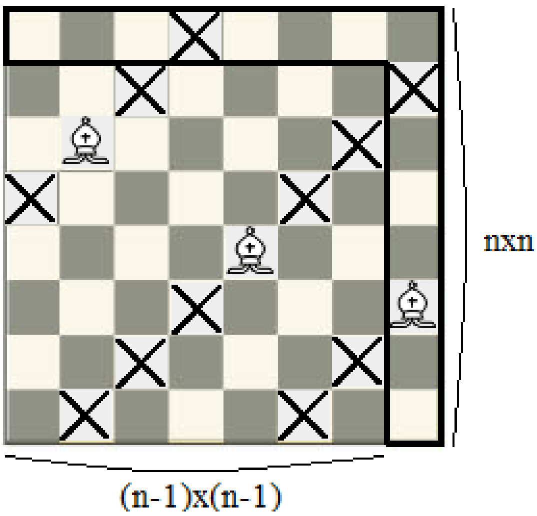
Partial solution for some figures.

Lets consider the following figure: partial bishop — the figure, that will only be attacking on one line. We will denote the amount of different ways we can place k such figures on a board $n \times n$ by b_n^k.



In case of board 1×1, we get the amount of possible allocations of one such bishop is 1; so $b_1^1 = 1$ and $b_1^0 = 1$.

Let's assume, that we have the amount of possible allocations of k figures for the board $(n-1) \times (n-1)$. The board $n \times n$ will contain board $(n-1) \times (n-1)$ as it's part (consider lower part to the left, see pic.). If we place $(k-i)$ figures in the part $(n-1) \times (n-1)$, in that case in the remaining $2n-1$ squares we place the rest i figures. But the remaining $2n-1$ slice has only $2n-1-(k-i)$ squares available (as the rest $(k-i)$ squares are under the attack from the already placed $(k-i)$ bishops). That implies the total amount of placements for the remaining slice as $C^i_{2n-1-(k-i)}$ (where $C_n^k = \frac{n!}{(n-k)!k!}$ — amount of combinations from n to k). From that we get, that there are in total $b_{n-1}^{k-i} \cdot C^i_{2n-1-(k-i)}$ ways to place $k-i$ figures on $(n-1) \times (n-1)$ part of the board and i figures on the rest of the board.



By adding all these products for $i = 0, 1, ..., k$ we get the total amount of placements of k figures on the board $n \times n$: $b_n^k = \sum\limits_{i=0}^{k} b_{n-1}^{k-i} \cdot C_{2n-1-(k-i)}^{i}$.

This way, we have proven the following theorem:

Theorem 1. *The amount of possible arrangements of k partial bishops on the board $n \times n$ is given by the following recursive formula:*

$$b_n^k = \sum_{i=0}^{k} b_{n-1}^{k-i} \cdot C_{2n-1-(k-i)}^{i} \tag{1}$$

Lets calculate b_n^k for the case of $k = 2$.

Theorem 2. *The amount of arrangements of two partial bishops on board $n \times n$ is:*

$$b_n^{(2)} = 1-10(n+1)+\frac{29}{2}(n+2)(n+1)+\frac{16}{3}(n+3)(n+2)(n+1)+\frac{1}{2}(n+4)(n+3)(n+2)(n+1).$$

Proof. By the recursive formula (1) we have: $b_n^{(2)} = b_{n-1}^{(2)} + b_n^{(1)}(2n-2) + \frac{1}{2}b_n^{(0)}(2n-1)(2n-2)$. Considering, that $b_{n-1}^{(0)} = 1$ and $b_{n-1}^{(1)} = (n-1)^2$ (there are always 1 way to place 0 figures on any board and n^2 ways to place 1 figure on the board $n \times n$) we get $b_n^{(2)} = b_{n-1}^{(2)} + 2n^3 - 4n^2 + 3n - 1$.

Lets find the generating function for this recursive sequence, and after that, we will be able to find the solution to this sequence. First, we will write down the generative function:

$$f(x) = \sum_{n=1}^{\infty} b_n^{(2)} x^n = \sum_{n=1}^{\infty} (b_{n-1}^{(2)} + 2n^3 - 4n^2 + 3n - 1)x^n$$

$$= \sum_{n=1}^{\infty} b_{n-1}^{(2)} x^n + \sum_{n=1}^{\infty} 2n^3 x^n - \sum_{n=1}^{\infty} 4n^2 x^n + \sum_{n=1}^{\infty} 3n x^n - \sum_{n=1}^{\infty} x^n.$$

Therefore, after taking all sums to the left side, we get:

$$\sum_{n=1}^{\infty} b_n^{(2)} x^n - \sum_{n=1}^{\infty} b_{n-1}^{(2)} x^n - 2\sum_{n=1}^{\infty} n^3 x^n + 4\sum_{n=1}^{\infty} n^2 x^n - 3\sum_{n=1}^{\infty} n x^n + \sum_{n=1}^{\infty} x^n = 0 \tag{2}$$

In order to get the value of the generating function, we first have to find the generating functions for sequences

$$\sum_{n=1}^{\infty} n^{\alpha} x^n.$$

First, we already know, that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(and thus, $\sum\limits_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1$). Then,

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\sum_{n=1}^{\infty}(x^n)' = x(\frac{1}{1-x} - 1)' = \frac{x}{(1-x)^2}.$$

Similarly, we get

$$\sum_{n=1}^{\infty} n^2x^n = x\sum_{n=1}^{\infty} n^2x^{n-1} = x\sum_{n=1}^{\infty} n(x^n)' = x(\frac{x}{(1-x)^2})' = \frac{x^2+x}{(1-x)^3}$$

and

$$\sum_{n=1}^{\infty} n^3x^n = x\sum_{n=1}^{\infty} n^3x^{n-1} = x\sum_{n=1}^{\infty} n^2(x^n)' = x(\frac{x^2+x}{(1-x)^3})' = \frac{x(x^2+4x+1)}{(1-x)^4}.$$

Now we can use these generating functions (and the fact, that $f(x) = \sum\limits_{n=1}^{\infty} b_n^{(2)} x^n$) in order to calculate (2):

$$f(x) - xf(x) - 2\frac{x(x^2+4x+1)}{(1-x)^4} + 4\frac{x^2+x}{(1-x)^3} - 3\frac{x}{(1-x)^2} + \frac{1}{1-x} - 1 = 0.$$

From that result we get that:

$$f(x) = \frac{x^4+6x^3+5x^2}{(1-x)^5}.$$

Using partial fraction expansion, we get

$$f(x) = \frac{1}{1-x} - \frac{10}{(1-x)^2} + \frac{29}{(1-x)^3} - \frac{32}{(1-x)^4} + \frac{12}{(1-x)^5}.$$

This way of writing down the generating function is very useful, as we can now, similarly to fibonacci sequence, write down the solution to $b_n^{(2)}$. For that, we consider the following:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

$$\frac{1}{(1-x)^2} = (\frac{1}{1-x})' = (\sum_{n=0}^{\infty} x^n)' = \sum_{n=0}^{\infty}(x^n)' = \sum_{n=0}^{\infty}(n+1)x^n,$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}(\frac{1}{1-x})'' = \frac{1}{2}(\sum_{n=0}^{\infty} x^n)'' = \frac{1}{2}\sum_{n=0}^{\infty}(x^n)'' = \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)x^n,$$

$$\frac{1}{(1-x)^4} = \frac{1}{6}(\frac{1}{1-x})''' = \frac{1}{6}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)x^n,$$

$$\frac{1}{(1-x)^5} = \frac{1}{24}(\frac{1}{1-x})'''' = \frac{1}{24}\sum_{n=0}^{\infty}(n+4)(n+3)(n+2)(n+1)x^n.$$

Using that expansion of these functions, we get the following way to write down the generative function for $b_n^{(2)}$:

$$\sum_{n=0}^{\infty} b_n^{(2)} x^n = f(x)$$

$$= \sum_{n=0}^{\infty}(1-10(n+1)+\frac{29}{2}(n+2)(n+1)+\frac{16}{3}(n+3)(n+2)(n+1)+\frac{1}{2}(n+4)(n+3)(n+2)(n+1))x^n.$$

Form this we get the initial statement of the theorem.

□

In case of the $k > 2$ the problem of calculation of the generative function somewhat increases.

References

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Received 20.11.2023

Лазорик А.Б., Мельник Г.В., Мельник В.С. *Застосування твірних функцій в задачах про розміщення n шахматних фігур.* // Буковинський матем. журнал. — 2023. — Т.11, №2. — С. 51–58.

Твірна функція — це структура, яка тісно пов'язана з числовою послідовністю, і дозволяє нам маніпулювати послідовністю як єдиною сутністю з метою її кращого розуміння. Грубо кажучи, твірні функції перетворюють проблеми про послідовності в проблеми про функції. Вони забезпечують систематичний спосіб кодування послідовностей чисел або інших комбінаторних об'єктів, дозволяючи вишукано розв'язувати складні проблеми в різних математичних областях.

У цій статті ми розглянули ряд проблем, пов'язаних із розміщенням n шахових фігур на $n \times n$ шахівниці так, щоб дві фігури не атакували одна одну, використовуючи підхід твірних функцій. Задача про розміщення n королев на дошці $n \times n$ до сих пір є відкритою комбінаторною задачею, точної формули обчислення поки не відомо, лише приблизні апроксимації. Тим не менше, часто розглядаються часткові випадки, зокрема специфічні умови розміщень, інші фігури, які мають інші лінії атак і т.д.

Хоча проблема n-королев часто використовується як практичний приклад для алгоритмів, які вирішують задачі комбінаторної оптимізації; він знайшов кілька реальних застосувань, включаючи практичне планування та призначення завдань, керування комп'ютерними ресурсами та тестування схем надвеликого рівня інтеграції.

В цій статті було розглянуто спеціальний тип шахових фігур, та побудовано рекурентну формулу для обчислення кількості розміщень k цих фігур на дошці розміру $n \times n$. Нами було вказано спосіб обчислення кількості розміщень цих фігур на дошці $n \times n$ з використанням твірних функцій, та продемонстровано це обчислення при $k = 2$. Для цього було обчислено твірну функцію для цього часткового випадку. Потім цю твірну функцію було розкладено на прості дроби, для яких було обчислено їхні відповідні ряди. Таким чином, прирівнявши два ряди, було отримано значення кількості розміщень відповідних фігур на шахматній дошці $n \times n$.