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ASYMPTOTIC BEHAVIOR OF $P_\omega(Y_0, Y_1, \pm\infty)$-SOLUTIONS OF THE SECOND ORDER DIFFERENTIAL EQUATIONS WITH THE PRODUCT OF DIFFERENT TYPES OF NONLINEARITIES FROM AN UNKNOWN FUNCTION AND ITS FIRST DERIVATIVE

The task of establishing the conditions of existence, as well as finding asymptotic images of solutions of differential equations, which contain nonlinearities of various types in the right-hand side, is one of the most important tasks of the qualitative theory of differential equations. In this work, second-order differential equations, which contain in the right part the product of a regularly varying nonlinearity from an unknown function and a rapidly varying nonlinearity from the derivative of an unknown function when the corresponding arguments are directed to zero or infinity, are considered. Necessary and sufficient conditions for the existence of slowly varying $P_\omega(Y_0, Y_1, \pm\infty)$ solutions of such equations have been obtained. Asymptotic representations of such solutions and their first-order derivatives have also been obtained. When additional conditions are imposed on the coefficients of the characteristic equation of the corresponding equivalent system of quasi-linear differential equations, it is established that there is a one-parameter family of such $P_\omega(Y_0, Y_1, \pm\infty)$-solutions to the equation. Similar results were obtained earlier when considering second-order equations, which contain on the right-hand side the product of a rapidly varying function from an unknown function and a properly varying function from the derivative of an unknown function when the arguments go to zero or infinity. Results for the equation, considered in this paper, are new.

Key words and phrases: nonlinear second-order differential equations, asymptotic representations of solutions, $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, rapidly varying functions, regularly varying functions, slowly varying first-order derivative.

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Introduction

Let's consider the following differential equation

$$y'' = \alpha_0 p(t) \varphi_0(y') \varphi_1(y). \tag{1}$$

УДК 517.925
2010 *Mathematics Subject Classification:* 34A34, 34C41, 34E99.

In this equation the constant $\alpha_0 \in \{-1; 1\}$ is responsible for the sign of the equation, functions $p : [a, \omega[\to]0, +\infty[$, $(-\infty < a < \omega \leq +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, where Δ_{Y_i} is the some one-sided neighborhood of $Y_i \in \{0, \pm\infty\}$.

We also suppose that function $\varphi_1 \colon \Delta_{Y_0} \to]0, +\infty[$ is a regularly varying as $y \to Y_1$ function of the index σ_1 ([7], p.10-15), function $\varphi_0 \colon \Delta_{Y_1} \to]0, +\infty[$ is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y') \neq 0 \;\text{ as }\; y' \in \Delta_{Y_0}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \varphi_0(y') \in \{0, +\infty\}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \frac{\varphi_0(y')\varphi_0''(y')}{(\varphi_0'(y'))^2} = 1. \tag{2}$$

It follows from the above conditions(2) that the function φ_0 and its derivative of the first order are rapidly varying functions as the argument tends to Y_0 ([1]). So we have the second order differential equation that contains the product of a regularly varying function of unknown function and a rapidly varying function of its first derivative correspondingly.

From the conditions (2) it also follows that the function φ_0 and its first-order derivative belong to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the works of V. M. Evtukhov and A. G. Chernikova [3] as a generalization of the class Γ (L. Khan, see, for example, [1], p. 75). The properties of the class $\Gamma_{Y_0}(Z_0)$ were used to get our results.

For the equations of type (1) let's consider the following class of solutions.

Definition 1. *The solution y of the equation* (1)*, that is defined on the interval* $[t_0, \omega[\subset [a, \omega[$*, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution ($-\infty \leq \lambda_0 \leq +\infty$), if the following conditions take place*

$$y^{(i)} : [t_0, \omega[\longrightarrow \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \quad (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t) y(t)} = \lambda_0.$$

This class of solutions was defined in the work of V. M. Evtukhov and A .M. Samoilenko [4] for the n-th order differential equations of Emden-Fowler type and was concretized for the second-order equation. Due to the asymptotic properties of functions in the class of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions [4], every such solution belongs to one of four non-intersecting sets according to the value of λ_0: $\lambda_0 \in \mathbb{R}\backslash\{0, 1\}$, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty$. In this article we consider the case $\lambda_0 = \pm\infty$ of such solutions, every $P_\omega(Y_0, Y_1, \pm\infty)$-solution and its derivative satisfy the following limit relations

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) y'(t)}{y(t)} = 1, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) y''(t)}{y'(t)} = 0. \tag{3}$$

The class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions for equations of the form (1) is one of the most difficult to study due to the fact that the second-order derivative is not explicitly expressed through the first-order derivative. From (3) it follows that the the first order derivative of each such $P_\omega(Y_0, Y_1, \pm\infty)$-solution is a slowly varying function as $t \uparrow \omega$.

The conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions in equation (1) were established in the case $\lambda_0 \in \mathbb{R}\backslash\{0, 1\}$ in the work [2].

The purpose of this work is establishing the necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of the equation (1), as well as asymptotic images for these solutions and their first-order derivatives as $t \uparrow \omega$ in case of the existence of some infinite limit. We also indicate the number of such solutions.

1 Section with results

To formulate the main results, we introduce the following definitions

Definition 2. *Let $Y \in \{0, \infty\}$, Δ_Y is some one-sided neighborhood of Y. Continuous-differentiable function $L : \Delta_Y \to]0; +\infty[$ is called ([6], p.2-3) a normalized slowly varying function as $z \to Y \quad (z \in \Delta_Y)$ if the next statement is valid*

$$\lim_{\substack{y\to Y\\ y\in \Delta_Y}} \frac{yL'(y)}{L(y)} = 0.$$

Definition 3. *We say that a slowly varying as $z \to Y \quad (z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0; +\infty[$ satisfies the condition S as $z \to Y$, if for any continuous differentiable normalized slowly varying as $z \to Y \quad (z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ the next relation is valid*

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \quad as \quad z \to Y \quad (z \in \Delta_Y).$$

Condition S is satisfied, for example, for such functions as $\ln|y|$, $|\ln|y||^\mu$ $(\mu \in R)$, $\ln\ln|y|$.
The following theorem is obtained.

Theorem 1. *Let's $\sigma_1 \neq 1$, функція $\varphi_1(y')|y'|^{-\sigma_1}$satisfies the condition S as $y' \to Y_1 \quad (y' \in \Delta_{Y_1})$. Then each $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of the equation (1) can be represented as*

$$y(t) = \pi_\omega(t)L(t), \tag{4}$$

in which $L : [t_0, \omega[\to R$ is twice continuously differentiable and satisfies the next conditions

$$y_0^0\pi_\omega(t)L(t) > 0, \quad L'(t) \neq 0 \quad при \quad t \in [t_1, \omega[\ \ (t_0 \leq t_1 < \omega), \tag{5}$$

$$\lim_{t\uparrow\omega} L(t) \in \{0; \pm\infty\}, \quad \lim_{t\uparrow\omega} \pi_\omega(t)L(t) = Y_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)} = 0. \tag{6}$$

Thus, in the case of the existence of a finite or infinite limit

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L''(t)}{L'(t)}, \tag{7}$$

the following relations take place

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L''(t)}{L'(t)} = -1, \quad \alpha_0L'(t) > 0 \quad при \quad t \in [t_1, \omega[(t_0 \leq t_1 < \omega), \tag{8}$$

$$p(t) = \frac{\alpha_0L'(t)}{|\pi_\omega(t)L(t)|^{\sigma_1}\theta_1(\pi_\omega(t)) \cdot \varphi_0\left(L(t)\left[1 + \frac{\pi_\omega(t)L'(t)}{L(t)}\right]\right)}[1 + o(1)] \quad при \quad t \uparrow \omega. \tag{9}$$

Proof. Let the function $y : [t_0, \omega[\to \Delta_{Y_0}$ $P_\omega(Y_0, Y_1, \pm\infty)$ is the solution of equation (1). Then this solution and its first- and second-order derivatives retain their sign on some interval $[t_1, \omega[(t_0 \leq t_1 < \omega)$ and conditions (3) are fulfilled. Due to the first of the conditions (3), there exists ([7], p.15) the normalized slowly variable as $t \uparrow \omega$ function $L(t) : [t_0, \omega[\longrightarrow R$, which satisfies the first of conditions (5) and the last of conditions (6), as well as for which equality (4) holds.

From (3) та (4) it follows that

$$y'(t) = L(t)\left[1 + \frac{\pi_\omega(t)L'(t)}{L(t)}\right] = L(t)[1 + o(1)] \quad \text{при} \quad t \uparrow \omega, \tag{10}$$

from where, considering (2), the first and second conditions of (6) theorem are fulfilled.

From (4), (6) and from the condition that y is a solution of the equation (1) we have the following relation

$$2L'(t) + \pi_\omega(t)(t)L''(t) = \alpha_0 p(t)\varphi_0(\pi_\omega(t)L(t))\varphi_1(y'(t)). \tag{11}$$

In the case of the existence of a finite or infinite limit (7), using Lopital's rule in the form of Stolz, using conditions (5) and (6), we have

$$0 = \lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)} = 1 + \lim_{t\uparrow\omega} \frac{\pi_\omega L''(t)}{L'(t)}, \tag{12}$$

whence follows the first of the conditions (8). From (10), (11) and (12), we have as $t \uparrow \omega$

$$\alpha_0 p(t)\varphi_0\left(L(t)\left[1 + \frac{\pi_\omega(t)L'(t)}{L(t)}\right]\right)\varphi_1(\pi_\omega(t)L(t)) = L'(t)\left[2 + \frac{\pi_\omega(t)L''(t)}{L'(t)}\right] = L'(t)[1 + o(1)].$$

Since the function $\theta_1(y') = \varphi_1(y')|y'|^{-\sigma_1}$ satisfies the condition S and (10) is fulfilled, then

$$\alpha_0 p(t)\varphi_0\left(L(t)\left[1 + \frac{\pi_\omega(t)L'(t)}{L(t)}\right]\right)|\pi_\omega(t)L(t)|^{\sigma_1}\theta_1(\pi_\omega(t)) = L'(t)[1 + o(1)] \quad \text{as} \quad t \uparrow \omega.$$

Therefore, the second of the conditions (8) and the asymptotic representation (9) are valid. The theorem is proved.

Definition 4. *We will say that condition N is fulfilled if for some continuously differentiable function $L(t) : [t_0, \omega[\longrightarrow R(t_0 \in [a, \omega[)$, which satisfies conditions (4)–(6) and (8), the following image takes place*

$$p(t) = \frac{\alpha_0 L'(t)}{|\pi_\omega(t)L(t)|^{\sigma_1}\theta_1(\pi_\omega(t)) \cdot \varphi_0\left(L(t)\left[1 + \frac{\pi_\omega(t)L'(t)}{L(t)}\right]\right)}[1 + r(t)],$$

where $r(t) : [t_0, \omega[\longrightarrow] - 1; +\infty[$ is a continuous function that tends to zero as $t \uparrow \omega$.

Let's introduce the following notations

$$\mu_0 = \operatorname{sign}\varphi_0'(y'), \quad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1}, \quad X(t) = L(t) \cdot e_1(t),$$

$$H(t) = \frac{L^2(t)\varphi_0'(X(t))}{\pi_\omega(t)L'(t)\varphi_0(X(t))}, \quad q_1(t) = \left.\frac{\left(\frac{\varphi_0'(y)}{\varphi_0(y)}\right)'}{\left(\frac{\varphi_0'(y)}{\varphi_0(y)}\right)^2}\right|_{y=X(t)},$$

$$e_1(t) = 1 + \frac{\pi_\omega(t)L'(t)}{L(t)}, \quad e_2(t) = 2 + \frac{\pi_\omega(t)L''(t)}{L'(t)}.$$

For these functions, from (2) and (6), the following statements hold:

1)

$$\lim_{t\uparrow\omega} e_1(t) = \lim_{t\uparrow\omega} e_2(t) = 1 \tag{13}$$

$$\lim_{t\uparrow\omega} H(t) = \pm\infty, \quad \lim_{t\uparrow\omega} q_1(t) = 0, \tag{14}$$

2) if there is exists the limit

$$\lim_{t\uparrow\omega} \frac{L(t)}{L'(t)} \cdot \frac{H'(t)}{|H(t)|^{\frac{3}{2}}},$$

then

$$\lim_{t\uparrow\omega} \frac{L(t)}{L'(t)} \cdot \frac{H'(t)}{|H(t)|^{\frac{3}{2}}} = 0. \tag{15}$$

Really, the statement (1) directly follows from conditions (6) and (8). Statements (14) follow from the validity of the statements:

$$H(t) = \frac{L(t)}{\pi_\omega(t)L'(t)} \cdot \frac{X(t)\varphi_0'\left(X(t)\right)}{\varphi_0\left(X(t)\right)} \cdot \frac{1}{e_1(t)},$$

$$\frac{\varphi_0(X(t))\varphi_0''(X(t))}{\left(\varphi_0'(X(t))\right)^2} = 1 + \frac{\left(\frac{\varphi_0'(X(t))}{\varphi_0'(X(t))}\right)'}{\frac{\varphi_0'(X(t))}{\varphi_0'(X(t))}}.$$

Statement (15) is proved analogously to the corresponding statement given in the work of V.M. Evtukhov. and A.G. Chernikova [3].

Theorem 2. *Let's $\sigma_1 \neq 1$, the function θ_1 satisfies the condition S, the condition N is true and the following statement takes place*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)} |H(t)|^{\frac{1}{2}} = \pm\infty. \tag{16}$$

Then if the following condition is true

$$\alpha_0\mu_0 > 0 \tag{17}$$

the differential equation (1) has a one-parameter family of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions, for each of which the following asymptotic representations take place as $t \uparrow \omega$:

$$y(t) = \pi_\omega(t) \cdot L(t)(1 + o(1)), \tag{18}$$

$$y'(t) = X(t) + \frac{\varphi_0(X(t))}{\varphi_0'(X(t))} \cdot |H(t)|^{\frac{1}{2}} \cdot o(1). \tag{19}$$

Proof. Let's $\sigma_1 \neq 1$, the function θ_1 satisfies the condition S, the conditions N and (16) are true. Then we apply the transformation to the equation (1):

$$y(t) = \pi_\omega(t) \cdot L(t) \cdot [1 + z_1(t)],$$
$$y'(t) = X(t) + \frac{\varphi_0(X(t))}{\varphi_0'(X(t))} \cdot z_2(t).$$

We obtain a system of differential equations

$$z_1' = \frac{1}{\pi_\omega(t)} \cdot [-e_1 z_1 + 1 + K(t) e_1(t) z_2], \tag{20}$$

$$z_2' = L'(t) \cdot e_2(t) \cdot \frac{\varphi_0'(X(t))}{\varphi_0(X(t))} \times$$
$$\times \left[\frac{\alpha_0 p(t) |\pi_\omega(t) \cdot L(t)|^{\sigma_1} \theta_1(\pi_\omega(t)) \varphi_0(Y_2(t, z_2)) \cdot N(t, z_1)}{L'(t)} \cdot [1 + z_1]^{\sigma_1} - 1 + z_2 \cdot q_1(t)\right], \tag{21}$$

where

$$K(t) = \frac{\varphi_0(X(t))}{X(t)\varphi_0'(X(t))}, \quad N(t, z_1) = \frac{\theta_1(Y_1(t, z_1))}{\theta_1(\pi_\omega(t)},$$

$$Y_1(t, z_1) = \pi_\omega(t) \cdot L(t) \cdot [1 + z_1(t)], \quad Y_2(t, z_2) = X(t) + \frac{\varphi_0(X(t))}{\varphi_0'(X(t))} \cdot z_2(t).$$

Since the function $Y_1(t, z_1)$ is properly a variable of order 1, the function θ_1 satisfies the condition S, then

$$\lim_{t \uparrow \omega} N(t, z_1) = 1 \text{ рівномірно за } \quad z_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{22}$$

From the condition N it follows that

$$\frac{\alpha_0 p(t) |\pi_\omega(t) L(t)|^{\sigma_1} \theta_1(\pi_\omega(t)) \varphi_0(Y_2(t, z_2))}{L'(t)} = \frac{\varphi_0(Y_2(t, z_2))}{\varphi_0(X(t))} [1 + r(t)]. \tag{23}$$

Expanding the right-hand side of (23) at a fixed $t \in [t_1; \omega[$ according to Maclauren's formula with a remainder in Lagrange form, we have

$$\frac{\varphi_0(Y_1(t, z_1))}{\varphi_0(X(t))} \cdot [1 + r(t)] = [1 + r(t)] \cdot (1 + z_2) + R(t, z_2),$$

where

$$R(t, z_2) = [1 + r(t)] \cdot \frac{\varphi_0''\left(X(t) + \frac{\varphi_0(X(t))}{\varphi_0'(X(t))} \cdot \xi\right) \varphi_0(X(t))}{\varphi_0'(X(t))} \cdot z_2^2,$$

$$|\xi| < |z_2|.$$

As

$$Y(t, z_1) = X(t) \left[1 + \frac{1}{\frac{X(t))\varphi_0(X(t))}{\varphi_0'(X(t))}} \xi\right],$$

then from the conditions (2) and (6) it follows that

$$\varphi_0''\left(X(t)+\frac{\varphi_0(X(t))}{\varphi_0'(X(t))}\cdot\xi\right)=\frac{\varphi_0'^2\left(X(t)+\frac{\varphi_0(X(t))}{\varphi_0'(X(t))}\cdot\xi\right)}{\varphi_0\left(X(t)+\frac{\varphi_0(X(t))}{\varphi_0'(X(t))}\cdot\xi\right)}\cdot[1+d_1(t,z_2)],$$

where

$$\lim_{t\uparrow\omega} d_1(t,z_2)=0 \text{ evenly for } \quad z_2\in\left[-\frac{1}{2},\frac{1}{2}\right].$$

From the lema 1.2. in [3] and φ_0,$\varphi_0' \in \Gamma_{Y_1}(Z_1)$ with the additional function $g=\frac{\varphi_0}{\varphi_0'}$, then the equality is true

$$\varphi_0''\left(X(t)+\frac{\varphi_0(X(t))}{\varphi_0'(X(t))}\cdot\xi\right)=\frac{\varphi_0'^2(X(t))}{\varphi_0(X(t))}e^{\xi}[1+d_1(t,z_2)],$$

where

$$\lim_{t\uparrow\omega} d_1(t,z_2)=0 \text{ evenly for } \quad z_2\in\left[-\frac{1}{2},\frac{1}{2}\right]. \tag{24}$$

Therefore, for any $\varepsilon>0$ there exist such $t_1\in[t_0;\omega[$ та $0<\delta\le\frac{1}{2}$, then

$$|R(t,z_2)|\le(1+\varepsilon)|z_2|^2 \quad \text{при} \quad t\in[t_1;\omega[,|z_1|\le\delta. \tag{25}$$

We arbitrarily choose the number $\varepsilon>0$ and consider the system (20)–(21) on the set

$$\Omega=[t_1;\omega[\times D, \quad \text{where } D=\{(z_1;z_2)\in R^2, \quad |z_1|\le\delta, \quad |z_2|\le\frac{1}{2}\}. \tag{26}$$

The system (20)–(21) on Ω has the form

$$z_1'=\frac{1}{\pi_\omega(t)}\cdot[A_{11}z_1+A_{12}z_2+1], \tag{27}$$

$$z_2'=L'(t)e_2(t)\cdot\frac{\varphi_0'(X(t))}{\varphi_0(X(t))}\cdot[A_{21}(t)z_1+A_{22}(t)z_2+R_1(t,z_1,z_2)+R_2(t,z_1,z_2)], \tag{28}$$

where

$$A_{11}(t)=-e_1(t), \quad A_{12}=K(t)e_1(t), \quad A_{21}(t)=\sigma_1, \quad A_{22}(t)=1,$$

$$R_1(t,z_1,z_2)=\left(\frac{(1+r_1)N(t,z_1)}{e_2(t)}-1\right)(1+\sigma_1z_1+z_2)+q_1z_2,$$

$$R_2(t,z_1,z_2)=\frac{(1+r_1)N(t,z_1)}{e_2(t)}(\sigma_1z_1z_2+(1+z_2)((1+z_1)^{\sigma_1}-1-\sigma_1z_1))+$$
$$+R(t,z_2)\cdot\frac{(1+z_2)(1+z_1)^{\sigma_1}N(t,z_1)}{e_2(t)}.$$

Note that from (2), (14) and (22) we have

$$\lim_{t\uparrow\omega}A_{11}=-1, \quad \lim_{t\uparrow\omega}A_{12}=0.$$

In addition, we have

$$\lim_{t\to+\infty} R_1(t;z_1;z_2) = 0 \text{ рівномірно за } \quad z_1, z_2 : \ |z_i| < \frac{1}{2}, \ i = 1, 2.$$

$$\lim_{t\to+\infty} \frac{R_2(t;z_1;z_2)}{|z_1| + |z_2|} = 0 \ .$$

We apply an additional transformation to the system (27)–(28)

$$z_1(t) = v_1(t),$$

$$z_2(t) = |H(t)|^{-\frac{1}{2}} v_2(t).$$

As a result, we get

$$v_1' = h(t) \cdot [c_{11}(t)v_1 + c_{12}v_2 + 1], \tag{29}$$

$$v_2' = h(t)\Bigg[\frac{1}{2}\frac{H'(t)\mathrm{sign}H(t)}{|H(t)|^{\frac{3}{2}}}v_2 + \frac{e_2(t)}{e_1^2(t)}A_{21}v_1 +$$

$$+\frac{e_2(t)}{e_1^2(t)}\frac{A_{22}}{|H(t)|^{\frac{1}{2}}}v_2 + \frac{e_2(t)}{e_1^2(t)}R_1(t, v_1, |H(t)|^{-\frac{1}{2}}v_2(t)) + \frac{e_2(t)}{e_1^2(t)}R_2(t, v_1, |H(t)|^{-\frac{1}{2}}v_2(t))\Bigg], \tag{30}$$

where

$$h(t) = \frac{L'(t)e_1(t)}{L(t)}|H(t)|^{\frac{1}{2}}, \quad c_{11} = \alpha_0\mu_0 q_1(t)|H(t)|^{\frac{1}{2}}, \quad c_{12} = \alpha_0\mu_0 \tag{31}$$

From (5), (6),we have

$$\int_{t_1}^{t} h(\tau)d\tau = \pm\infty.$$

From (12)–(15) we have

$$\lim_{t\uparrow\omega} c_{12}(t) = \alpha_0\mu_0$$

$$\lim_{t\uparrow\omega} \frac{e_2(t)}{e_1^2(t)}\frac{A_{22}}{|H(t)|^{\frac{1}{2}}} = 0$$

$$\lim_{t\uparrow\omega} \frac{1}{2}\frac{H'(t)\mathrm{sign}H(t)}{|H(t)|^{\frac{3}{2}}} = 0$$

Because of

$$H'(t) = \left(\frac{L^2(t)}{L'(t)}\right)' \cdot \frac{\varphi_0'(\pi_\omega(t)L(t))}{\varphi_0(\pi_\omega(t)L(t))} + \frac{L^2(t)}{L'(t)} \cdot (L(t) + \pi_\omega(t)L(t)) \cdot \left(\frac{\varphi_0'(y)}{\varphi_0(y)}\right)'\Bigg|_{y=\pi_\omega(t)L(t)},$$

we have

$$\left(\frac{\varphi_0'(y)}{\varphi_0(y)}\right)'\Bigg|_{y=\pi_\omega(t)L(t)} = \frac{H'(t)}{\frac{L^2(t)}{L'(t)}\cdot(L(t)+\pi_\omega(t)L(t))} - \frac{\varphi_0'(\pi_\omega(t)L(t))}{\varphi_0(\pi_\omega(t)L(t))}\times$$

$$\times\frac{\frac{L^2(t)}{L'(t)}}{\frac{L^2(t)}{L'(t)}\cdot(L(t)+\pi_\omega(t)L(t))}.$$

From the last and from the conditionals (6) and (8) we have

$$q_1(t)|H(t)|^{\frac{1}{2}} = \frac{L(t)}{L'(t)e_1(t)}\cdot\frac{H'(t)}{|H(t)|^{\frac{3}{2}}} - \frac{1+o(1)}{\frac{\pi_\omega(t)L'(t)}{L(t)}\cdot e_1(t)|H(t)|^{\frac{1}{2}}\cdot \operatorname{sign}H'(t)} \quad \text{as} \quad t\uparrow\omega. \tag{32}$$

In (32), the first term on the right goes to zero due to (15), and the second also goes to zero due to condition (16).

So,

$$\lim_{t\uparrow\omega} c_{11}(t) = 0. \tag{33}$$

From (31), (32), (33) the characteristic equation of the limiting matrix of coefficients as v_1 та v_2

$$\begin{pmatrix} 0 & \alpha_0\mu_0 \\ 1 & 0 \end{pmatrix}$$

is

$$\rho^2 - \alpha_0\mu_0 = 0.$$

It follows from the conditions of the theorem that this equation has exactly two real roots of different signs.

We get that for the differential system of equations (29)–(30), all the conditions of Theorem 2.2 with [5] are fulfilled. According to this theorem, the system (29)–(30) has a one-parameter family of solutions $\{v_i\}_{i=1}^2 : [t*, +\infty[\longrightarrow \mathbb{R}^2$ $(t* \geq t_1)$, which go to zero at $t\uparrow\omega$. These solutions correspond to the solutions $y : [t*, +\infty[\longrightarrow \mathbb{R}$ $(t* \geq t_1)$ equation (1), which allow for $t\uparrow\omega$ asymptotic images (19).

By virtue of the appearance of these images, it is clear that they were obtained solutions are $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1). The theorem is completely proved.

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Received 25.11.2023

Чепок О.О. *Асимптотична поведінка $P_\omega(Y_0, Y_1, \pm\infty)$-розв'язків диференціальних рівнянь другого порядку з добутком різних типів нелінійностей від невідомої функції та її першої похідної* // Буковинський матем. журнал — 2023. — Т.11, №2. — С. 41–50.

Задача встановлення умов існування, а також знаходженння асимптотичних зображень розв'язків диференціальних рівняння, які містять у правій частині нелінійності різних типів є однією з найважливіших задач якісної теорії диференціальних рівнянь. У даній роботі розглянуті диференціальні рівняння другого порядку, які містять у правій частині добуток правильно змінної нелінійності від невідомої функції та швидко змінної нелінійності від похідної невідомої функції при прямуванні відповідних аргументів до нуля або нескінченності. Отримано необхідні та достатні умови існування повільно змінних $P_\omega(Y_0, Y_1, \pm\infty)$-розв'язків таких рівнянь. Також отримані асимптотичні зображення таких розв'язків та їх похідних першого порядку. При накладанні додаткових умов на коефіцієнти характеристичного рівняння відповідної еквівалентної системи квазілінійних диференціальних рівнянь встановлено, що таких $P_\omega(Y_0, Y_1, \pm\infty)$-розв'язків у рівняння існує однопараметрична сім'я. Подібні результати були отримані раніше при розгляді рівнянь другого порядку, які містять у правій частині добуток швидко змінної функції від невідомої функції та правильно змінної функції від похідної невідомої функції при прямуванні аргументів до нуля або нескінченності. Для рівнянь, які розглядаються у даній роботі, подібні результати є новими.