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Zaporizhzhya National University

STABILITY OF RICH SUBSPACES

В роботі розглядається відомий результат: якщо E – симетричний банахів простір на $[0,1]$, відмінний від L_∞ з точністю до еквівалентної норми, K – компактний оператор на E та X – багатий підпростір простору E, то підпростір $(I+K)(X)$ – також багатий. Пропонується доведення, яке не використовує слабку збіжність системи Радемахера до нуля, що дає можливість узагальнити результат на довільний F-простір Кете з абсолютно неперервною нормою на просторі зі скінченною безатомною мірою (Ω, Σ, μ).

We consider the well-known result: if E is an r.i. Banach space on $[0,1]$ not equal to L_∞, up to an equivalent norm, K is a compact operator on E and X is a rich subspace of E, then the subspace $(I+K)(X)$ is also rich. We provide a proof without using the Rademacher system's weak convergence to zero. This permits us to generalize the proposition to any Köthe F-space with an absolutely continuous norm on a finite atomless measure space.

It is well known [3, p. 72], [4, Proposition 5.7] that if E is an r.i. Banach space on $[0,1]$, not equal to L_∞, up to an equivalent norm, K is a compact operator on E and X is a rich subspace of E then the subspace $(I+K)(X)$ is also rich. Note that the image $(I+K)(X)$ is closed because K is a compact operator on X [1, p. 474]). Proofs of this proposition in [3] and [4] use the fact that the Rademacher system is weakly null in such a space E. Using the technique of [3, Section 1.3], we provide another proof which does not use the Rademacher system's weak convergence to zero. It enables us to generalize (extend) the proposition to Köthe spaces.

Recall some definitions. An $F-space$ is a complete metric linear space X over the real or complex scalar field $\mathbb{K}$ with an invariant metric ρ (i.e. $\rho(x,y) = \rho(x+z, y+z)$ for all $x, y, z \in X$). As in a normed space, the distance between x and zero is denoted by $\|x\|$ and called the $F-norm$.

Let (Ω, Σ, μ) be a finite atomless measure space. An F-space E of equivalence classes of measurable functions $x : \Omega \to \mathbb{K}$ is called a *Köthe F-space* if the following conditions hold:

(1) if $y \in E$ and $|x(\omega)| \leqslant |y(\omega)|$ for almost all $\omega \in \Omega$ then $x \in E$ and $\|x\| \leqslant \|y\|$;

(2) $\mathbf{1}_A \in E$ for every set $A \in \Sigma$.

The symbol $\mathbf{1}_A$ denotes the characteristic function of a set $A \in \Sigma$. let E be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ). A linear continuous operator $T : E \to Y$ (Y is any F-space) is called *narrow*, if $\forall\, \varepsilon > 0$ and $\forall\, A \in \Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$ there exists an element $x \in E$ such that $x^2 = \mathbf{1}_A$, $\int_\Omega x\, d\mu = 0$ (such an element is called a mean zero sign on A) and $\|Tx\| < \varepsilon$. A subspace X of E is called *rich* if the quotient map from E onto E/X is narrow. In other wording, X is a rich subspace if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist elements $x \in X$ and $y \in E$, such that $y^2 = \mathbf{1}_A$, $\int_\Omega y\, d\mu = 0$ and $\|x - y\| < \varepsilon$.

For Banach or F-space E the symbol $\mathcal{L}(X,Y)$, stands for the linear space of all continuous linear operators from E to E.

Theorem 1. *Let E be a Köthe F-space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ). Let $K \in$*

$\mathcal{L}(E)$ *be a compact operator. If X is a rich subspace of E then the subspace $(I+K)(X)$ is also rich.*

Proof. Fix $\varepsilon > 0$ and $A \in \Sigma^+$. Since X is a rich subspace, there exist a sequence $x_n \in X$, $n \in \mathbb{N}$ and a Rademacher system (r_n) on A such that $\|x_n - r_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

By a Rademacher system on A we mean any sequence (r_n) such that $r_n^2 = \mathbf{1}_A$, $\displaystyle\int_\Omega r_n\, d\mu = 0$ and $\mu\{r_n(\omega) = 1\} = \mu\{r_n(\omega) = -1\} = 2^{-n}\mu(A)$.

Since K is compact, the set $\{Kr_n : n \in \mathbb{N}\}$ is relatively compact. Hence we can choose a convergent subsequence (Kr_{n_i}) of (Kr_n). Consider two numbers n_1, n_2 for which

$$\Big\|K\frac{r_{n_1} - r_{n_2}}{2}\Big\| < \frac{\varepsilon}{2},$$

$$\Big\|\frac{x_{n_1} - r_{n_1}}{2}\Big\| < \frac{\varepsilon}{4(1+\|K\|)},$$

$$\Big\|\frac{x_{n_2} - r_{n_2}}{2}\Big\| < \frac{\varepsilon}{4(1+\|K\|)}.$$

We denote

$$h_1 = \frac{r_{n_1} - r_{n_2}}{2}.$$

It is clear that the function h_1 is a mean zero sign on $A_1 \subset A$ such that $\mu(A_1) = \mu(A)/2$. Put

$$\widetilde{x}_1 = \frac{x_{n_1} - x_{n_2}}{2} \in X$$

and

$$y_1 = \widetilde{x}_1 + K\widetilde{x}_1 \in (I+K)(X).$$

Then

$$\begin{aligned}\|y_1 - h_1\| &= \|\widetilde{x}_1 + K\widetilde{x}_1 - h_1\| = \\ &= \|(I+K)(\widetilde{x}_1 - h_1) + Kh_1\| \leqslant \\ &\leqslant (1+\|K\|)\Big(\Big\|\frac{x_{n_1} - r_{n_1}}{2}\Big\| + \\ &+ \Big\|\frac{x_{n_2} - r_{n_2}}{2}\Big\|\Big) + \|Kh_1\| < \varepsilon.\end{aligned}$$

Now consider the set $A \setminus A_1$. Analogously, there exist a sequence $x_n^1 \in X$, $n \in \mathbb{N}$ and a Rademacher system (r_n^1) on $A \setminus A_1$, for which $\|x_n^1 - r_n^1\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since the set $\{Kr_n^1 : n \in \mathbb{N}\}$ is relatively compact, we can choose two numbers n_1, n_2, for which

$$\Big\|K\frac{r_{n_1}^1 - r_{n_2}^1}{2}\Big\| < \frac{\varepsilon}{2^2},$$

$$\Big\|\frac{x_{n_1}^1 - r_{n_1}^1}{2}\Big\| < \frac{\varepsilon}{8(1+\|K\|)},$$

$$\Big\|\frac{x_{n_2}^1 - r_{n_2}^1}{2}\Big\| < \frac{\varepsilon}{8(1+\|K\|)}.$$

Denote by

$$h_2 = \frac{r_{n_1}^1 - r_{n_2}^1}{2}$$

the function, which is a mean zero sign on $A_2 \subset A \setminus A_1$ and $\mu(A_2) = \mu(A)/2^2$. Put

$$\widetilde{x}_2 = \frac{x_{n_1}^1 - x_{n_2}^1}{2} \in X$$

и

$$y_2 = \widetilde{x}_2 + K\widetilde{x}_2 \in (I+K)(X).$$

Then

$$\begin{aligned}\|y_2 - h_2\| &= \|\widetilde{x}_2 + K\widetilde{x}_2 - h_2\| \leqslant \\ &\leqslant (1+\|K\|)\Big(\Big\|\frac{x_{n_1}^1 - r_{n_1}^1}{2}\Big\| + \\ &+ \Big\|\frac{x_{n_2}^1 - r_{n_2}^1}{2}\Big\|\Big) + \|Kh_2\| < \frac{\varepsilon}{2}.\end{aligned}$$

Suppose for $i = 1, 2, ..., m$ the functions $\widetilde{x}_i$ and the signs h_i on $A_i \subset A \setminus A_1 \cup A_2 \cup ... \cup A_{i-1}$ with $\mu(A_i) = \mu(A)/2^i$ have been chosen. Then there are a sequence $(x_n^m) \subset X$ and a Rademacher system (r_n^m) on $A \setminus A_1 \cup A_2 \cup ... \cup A_m$, for which $\|x_n^m - r_n^m\| \longrightarrow 0$ as $n \longrightarrow \infty$. Again, using the relative compactness of $\{Kr_n^m : n \in \mathbb{N}\}$, we choose two numbers n_1, n_2, for which

$$\Big\|K\frac{r_{n_1}^m - r_{n_2}^m}{2}\Big\| < \frac{\varepsilon}{2^{m+1}},$$

$$\Big\|\frac{x_{n_1}^m - r_{n_1}^m}{2}\Big\| < \frac{\varepsilon}{2^{m+2}(1+\|K\|)},$$

$$\left\| \frac{x^m_{n_2} - r^m_{n_2}}{2} \right\| < \frac{\varepsilon}{2^{m+2}(1+\|K\|)}.$$

Then the function

$$h_{m+1} = \frac{r^m_{n_1} - r^m_{n_2}}{2}$$

is a mean zero sign on

$A_{m+1} \subset A \setminus A_1 \cup A_2 \cup ... \cup A_m$,

such that $\mu(A_{m+1}) = \mu(A)/2^{m+1}$.

Putting

$$\widetilde{x}_{m+1} = \frac{x^m_{n_1} - x^m_{n_2}}{2} \in X$$

and

$$y_{m+1} = \widetilde{x}_{m+1} + K\widetilde{x}_{m+1} \in (I+K)(X),$$

we obtain the estimate

$$\|y_{m+1} - h_{m+1}\| < \frac{\varepsilon}{2^m}.$$

Now let

$$\widetilde{x} = \sum_{i=1}^{\infty} \widetilde{x}_i \in X, \quad h = \sum_{i=1}^{\infty} h_i.$$

The series converges because E has an absolutely continuous norm. It is clear that the function h is a mean zero sign on A. Putting $y = \widetilde{x} + K\widetilde{x} = \sum\limits_{i=1}^{\infty} y_i \in (I+K)(X)$, we obtain the estimate

$$\|y - h\| \leqslant \sum_{i=1}^{\infty} \|y_i - h_i\| < 2\varepsilon.$$

This means that the subspace is rich. □

Corollary 1. *Let E be a Köthe F-space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ). If X is a rich subspace of E and Y is a subspace of X of finite co-dimension then Y is also rich.*

Theorem 2. *Let E be a Banach Köthe space with an absolutely continuous norm on a finite atomless measure space (Ω, Σ, μ) and let X be a rich subspace of E with a normalized basis $(x_n)_{n=1}^{\infty}$. If a sequence $(y_n)_{n=1}^{\infty}$ in E satisfies $\sum\limits_{n=1}^{\infty} \|x_n - y_n\| < \infty$ then the subspace $Y = [y_n]_{n=1}^{\infty}$ is also rich.*

The symbol $[y_n]_{n=1}^{\infty}$ (or simply $[y_n]$) denotes the closed linear span of a sequence $(y_n)_{n=1}^{\infty}$. If we consider the closed linear span of a sequence $(y_n)_{n=n_0}^{\infty}$, we denote it by the symbol $[y_n]_{n=n_0}^{\infty}$.

Proof. According to [4,p.60], fix any $A \in \Sigma^+$ and $\varepsilon \in (0, 2\mu(\Omega))$. Let M be the basis constant of (x_n). Then choose $n_0 \in \mathbb{N}$ so that $\sum\limits_{n=n_0}^{\infty} \|x_n - y_n\| < \varepsilon/8M\mu(\Omega)$. By [2, p. 5], there exists a continuous linear operator $T : [x_n]_{n=n_0}^{\infty} \to [y_n]_{n=n_0}^{\infty}$ with $Tx_n = y_n$ for each $n \geqslant n_0$ and

$$\|Tx - x\| \leqslant \varepsilon\|x\|/4\mu(\Omega)$$

for each $x \in [x_n]_{n=n_0}^{\infty}$.

By Corollary 1, $[x_n]_{n=n_0}^{\infty}$ is a rich space of E. So, there exist $x = [x_n]_{n=n_0}^{\infty}$ and $y \in E$, such that $y^2 = \mathbf{1}_A$, $\int\limits_{\Omega} y\, d\mu = 0$ and $\|x - y\| < \varepsilon/2$. Then $Tx \in Y$ and

$$\|x\| \leqslant \|x - y\| + \|y\| \leqslant \varepsilon/2 + \mu(\Omega) < 2\mu(\Omega),$$

$$\|Tx - y\| \leqslant \|Tx - x\| + \|x - y\| \leqslant \varepsilon\|x\|/4 + \varepsilon/2 < \varepsilon.$$

So the subspace Y is rich. □

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