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ON THE BOUNDED CONTROL SYNTHESIS FOR THREE-DIMENSIONAL HIGH-ORDER NONLINEAR SYSTEMS

The paper deals with three-dimensional high-order nonlinear systems. A class of bounded finite-time stabilizing controls is presented. Korobov's controllability function is constructed to ensure global finite-time convergence. A simulation example is given to demonstrate the effectiveness of the proposed approach.

Key words and phrases: finite-time stabilization, bounded control synthesis, controllability function method, nonlinear systems, critical case.

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Introduction

High-order nonlinear systems sparked a great deal of interest in recent decades. These systems have uncontrollable first approximation and can not be mapped to linear systems. That is why the stabilization and finite-time stabilization of high-order nonlinear systems is considered to be one of the most challenging issues in nonlinear control theory.

The objective of finite-time stabilization is to find a controller so that the trajectories of the corresponding closed-loop system converge to the origin in finite time while ensuring stability. The controllability function method for finite-time stabilization was proposed by V.I. Korobov in [5]. The controllability function method was developed in many works, including [1, 6, 7, 8, 9].

Many stabilization and finite-time stabilization results for higher-order nonlinear systems were obtained using adding a power integrator technique, see, for example, [3, 4, 11, 12, 13, 14]. This recursive procedure produces feedback controllers of rather complicated structure and is often difficult to implement for systems of high dimension.

Simple classes of stabilizing controls for high-order nonlinear systems were proposed in [2, 10]. We develop the results presented in these works to achieve global finite-time convergence of the trajectories to the origin. To this end, we use the controllability function

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method. This allows us to construct bounded controls that satisfy preassigned constraint on their absolute values and can be easily implemented numerically.

The remainder of this paper is organized as follows. In Section 1, we give the problem formulation, introduce the control construction method, and prove the finite-time convergence. In Section 2, the boundedness of control is proved and the main result is stated. Additionally, within Section 2, we provide a simulation example demonstrating the effectiveness of the proposed method.

1 Problem formulation and control construction

We address the bounded control synthesis problem for the following nonlinear system

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^{2k_1+1}, \\ \dot{x}_3 = x_2^{2k_2+1}, \end{cases} \tag{1}$$

where $k_i = \frac{p_i}{q_i} > 1$ ($p_i > 0$ is an integer number, $q_i > 0$ is an odd numbers), $i = 1, 2$, $d > 0$ is a given number, and $k_1 < k_2$.

Our objective is to construct a class of bounded controls $u = u(x)$ ($|u(x)| \leq d$) such that for any initial point $x_0 \in U(0) \setminus \{0\} \subset \mathbb{R}^n$, the solution $x(t, x_0)$ of the corresponding closed-loop system is well-defined and reaches zero in a finite time $T(x_0) < +\infty$. In other words, $\lim_{t \to T(x_0)} x(t, x_0) = 0$. If the zero equilibrium point of the closed-loop system is stable, this problem is also known as the finite-time stabilization problem.

We use the controllability function method [5] to construct a class of finite-time stabilizing controls $u(x)$. The main idea of the controllability function method is to construct a positive definite function $\Theta(x)$ ($\Theta(0) = 0$, $\Theta(x) > 0$ for $x \neq 0$) so that

$$\dot{\Theta}(x) \leq -\beta \Theta^{1-\frac{1}{\alpha}} \tag{2}$$

for some $\beta > 0$, $\alpha \geq 1$, where $\dot{\Theta}(x)$ denotes the derivative along the trajectories of system (1) with $u = u(x)$. This inequality ensures finite-time convergence of the trajectories of the closed-loop system (1).

We introduce the following notation:

$$m_1 = 1,\ m_2 = (2k_1 + 1)m_1 + 1,\ m_3 = (2k_2 + 1)m_2 + 1 \equiv m.$$

Assume that $F = \{f_{ij}\}_{i,j=1}^n$ is a symmetric positive definite matrix such that the matrix $F^1 = 2mF - FH - HF$ is positive definite, where the matrix H has the form

$$H = \begin{pmatrix} m - m_1 & 0 & 0 \\ 0 & m - m_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose $a_0 > 0$ is a given number. Define the controllability function $\Theta(x)$, for $x \neq 0$, as a positive definite solution of the equation

$$2a_0 \Theta^{2m} = (FD(\Theta)x, D(\Theta)x), \tag{3}$$

where

$$D(\Theta) = \begin{pmatrix} \Theta^{m-m_1} & 0 & 0 \\ 0 & \Theta^{m-m_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We put $\Theta(0) = 0$.

Note that the equation (3) has a unique positive solution for every fixed $x \neq 0$ if the matrix F^1 is positive definite, which can be proved similarly to [5]. Moreover, $\Theta(x)$ is continuously differentiable at every point $x \neq 0$.

We introduce the following feedback law $u(x)$:

$$\begin{aligned} u(x) &= a_1 \frac{x_1}{\Theta(x)} + a_2 \frac{x_2}{\Theta(x)^{m_2}} + a_3 \frac{x_3}{\Theta(x)^{m_3}} \\ &+ a_4 \frac{x_1^{2k_1+1}}{\Theta(x)^{m_2-1}} + a_5 \frac{x_2^{2k_2+1}}{\Theta(x)^{m_3-1}}, \end{aligned} \tag{4}$$

where $a_i < 0$ are some real numbers, $i = 1, \ldots, 5$. Additional conditions on a_i, $i = 1, \ldots, 5$, and F will be obtained later.

Apply the feedback $u = u(x)$ given by (4) to system (1). To find the derivative of the function $\Theta(x)$ along the trajectories of the system (1) with $u = u(x)$, we take the derivative of both sides of equation (3):

$$4a_0 m \theta(x)^{2m-1} \dot{\Theta}(x) = (F\dot{y}(x), y(x)) + (Fy(x), \dot{y}(x)), \tag{5}$$

where $y(x) = D(\Theta(x))x = (x_1 \Theta^{m-m_1}(x), x_2 \Theta^{m-m_2}(x), x_3)$.

Computing $\dot{y}$, we get

$$\dot{y} = \dot{D}(\Theta)x + D(\Theta)\dot{x}.$$

It is clear, that

$$\dot{D}(\Theta) = HD(\Theta)\frac{\dot{\Theta}}{\Theta}.$$

We proceed with $\dot{y}$ by rewriting system (1) in the form

$$\dot{x} = A(\Theta(x))x + h_1(\Theta(x))x_1^{2k_1+1} + h_2(\Theta(x))x_2^{2k_2+1},$$

where the matrix $A(\Theta)$ is given by

$$A(\Theta) = \begin{pmatrix} \frac{a_1}{\Theta} & \frac{a_2}{\Theta^{m_2}} & \frac{a_3}{\Theta^{m_3}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the vectors $h_1(\Theta)$ and $h_2(\Theta)$ are defined by

$$h_1(\Theta) = \begin{pmatrix} \frac{a_4}{\Theta^{m_2-m_1}} \\ 1 \\ 0 \end{pmatrix}, \quad h_2(\Theta) = \begin{pmatrix} \frac{a_5}{\Theta^{m-m_1}} \\ 0 \\ 1 \end{pmatrix}.$$

It is straightforward to check that

$$D(\Theta)A(\Theta) = AD(\Theta)\Theta^{-1}, \quad D(\Theta)h_1(\Theta) = h_1\Theta^{m-m_2}, \quad D(\Theta)h_2(\Theta) = h_2,$$

where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} a_4 \\ 1 \\ 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} a_5 \\ 0 \\ 1 \end{pmatrix}. \tag{6}$$

Thus,

$$\begin{aligned} \dot{y} &= HD(\Theta)x\frac{\dot{\Theta}}{\Theta} + AD(\Theta)x\frac{1}{\Theta} + h_1\Theta^{m-m_2}x_1^{2k_1+1} + h_2x_2^{2k_2+1} \\ &= Hy\frac{\dot{\Theta}}{\Theta} + Ay\frac{1}{\Theta} + h_1\Theta^{m-m_2}x_1^{2k_1+1} + h_2x_2^{2k_2+1}. \end{aligned} \tag{7}$$

Substituting (7) into (5), we obtain

$$\begin{aligned} 4a_0m\Theta^{2m-1}(x)\dot{\Theta}(x) =& (FAy(x), y(x))\Theta(x)^{-1} + (Fy(x), Ay(x))\Theta(x)^{-1} \\ &+ (Fh_1, y(x))x_1^{2k_1+1}\Theta(x)^{m-m_2} + (y(x), Fh_1)x_1^{2k_1+1}\Theta^{m-m_2} \\ &+ (Fh_2, y(x))x_2^{2k_2+1} + (Fy(x), h_2)x_2^{2k_2+1} + (FHy(x), y(x))\frac{\dot{\Theta}(x)}{\Theta(x)} \\ &+ (Fy(x), Hy(x))\frac{\dot{\Theta}(x)}{\Theta(x)}. \end{aligned}$$

Multiplying both sides by $\Theta(x)$ and substituting $2a_0\Theta^{2m} = (Fy, y)$, we deduce that

$$\dot{\Theta} = \frac{((A^*F + FA)y, y) + 2(Fh_1, y)x_1^{2k_1+1}\Theta^{m-m_2+1} + 2(Fh_2, y)x_2^{2k_2+1}\Theta}{((2mF - FH - HF)y, y)} \tag{8}$$

Denote the matrix $2mF - FH - HF$ by F^1.

Note that since the matrix A is singular, it is impossible to choose a positive definite matrix F in such a way that the matrix $A^*F + FA$ is negative definite. Therefore, we choose a positive definite matrix F so that the matrix $A^*F + FA$ is positive semi-definite. To do this, we consider the following matrix Lyapunov equation

$$A^*F + FA = -W, \tag{9}$$

where $W = \{w_{i,j}\}_{i,j=1}^n$ is a positive semi-definite matrix.

Singular Lyapunov matrix equation (9) is studied in [2]. In [2], it is proved that for a positive semi-definite matrix W, the matrix equation (9) is solvable within the class of positive definite matrices F if and only if the matrix W has the form

$$W = \begin{pmatrix} w_{11} & w_{11}\frac{a_2}{a_1} & w_{11}\frac{a_3}{a_1} \\ w_{11}\frac{a_2}{a_1} & w_{11}\frac{a_2^2}{a_1^2} & w_{11}\frac{a_2a_3}{a_1^2} \\ w_{11}\frac{a_3}{a_1} & w_{11}\frac{a_2a_3}{a_1^2} & w_{11}\frac{a_3^2}{a_1^2} \end{pmatrix} \tag{10}$$

It is clear that the matrix W of the form (10) is positive semi-definite if and only if $w_{11} \geq 0$. We chose $w_{11} > 0$.

The following theorem describes the class of positive definite solutions of the matrix equation (9).

Theorem 1 ([2])**.** *Suppose that the matrices A and W are defined by* (6) *and* (10) *respectively. Assume that $w_{11} > 0$, $a_1 < 0$. Then the Lyapunov matrix equation* (9) *is solvable and its solutions have the form*

$$F = \begin{pmatrix} f_{11} & \frac{a_2}{a_1} f_{11} & \frac{a_3}{a_1} f_{11} \\ \frac{a_2}{a_1} f_{11} & f_{22} & f_{23} \\ \frac{a_3}{a_1} f_{11} & f_{23} & f_{33} \end{pmatrix}, \tag{11}$$

where $f_{11} = -\dfrac{w_{11}}{2a_1}$, and the elements $\{f_{ij}\}_{i,j=1}^{2}$ are some real numbers. Moreover, matrix (10) *is positive definite for sufficiently large $f_{22} > 0$ and $f_{33} > 0$.*

Let us chose f_{23}, a_4, a_5 as follows:

$$f_{23} = \frac{a_3}{a_2} f_{22}, \quad a_4 = -\frac{a_1}{a_2}\frac{f_{22}}{f_{11}}, \quad a_5 = -\frac{a_1}{a_3}\frac{f_{33}}{f_{11}}. \tag{12}$$

Then (8) takes the form

$$\begin{aligned} \dot{\Theta} = & \frac{((A^*F + FA)y, y) + 2b_1^1 x_1^{2k_1+2}\Theta^{2m-m_2} + 2b_1^2 x_2^{2k_2+1} x_1 \Theta^m}{(F^1 y, y)} \\ & + \frac{2b_2^2 x_2^{2k_2+2}\Theta^{m-m_2+1}}{(F^1 y, y)}, \end{aligned} \tag{13}$$

where

$$b_1^1 = -\frac{a_1}{a_2}\left(f_{22} - f_{11}\frac{a_2^2}{a_1^2}\right), \quad b_1^2 = -\frac{a_1}{a_3}\left(f_{33} - f_{11}\frac{a_3^2}{a_1^2}\right), \quad b_2^2 = -\frac{a_2}{a_3}\left(f_{33} - f_{22}\frac{a_3^2}{a_2^2}\right). \tag{14}$$

Note that since $f_{23} = \frac{a_3}{a_2} f_{22}$ by (12), the matrix F, given by (11), takes the form

$$F = \begin{pmatrix} f_{11} & \frac{a_2}{a_1} f_{11} & \frac{a_3}{a_1} f_{11} \\ \frac{a_2}{a_1} f_{11} & f_{22} & \frac{a_3}{a_2} f_{22} \\ \frac{a_3}{a_1} f_{11} & \frac{a_3}{a_2} f_{22} & f_{33} \end{pmatrix}. \tag{15}$$

It is straightforward to show that the matrix F of the form (15) is positive definite if and only if the following inequalities hold:

$$f_{11} > 0, \quad f_{22} > f_{11}\frac{a_2^2}{a_1^2}, \quad f_{33} > f_{22}\frac{a_3^2}{a_2^2}. \tag{16}$$

Inequalities (16) guaranty that $b_1^1 < 0$, $b_1^2 < 0$, $b_2^2 < 0$ for $a_i < 0$, $i = 1, \ldots, 3$. Then $b_1^1 = -|b_1^1|$, $b_1^2 = -|b_1^2|$, $b_2^2 = -|b_2^2|$.

Let us recall that according to Young's inequality, it is true that

$$ab \le \frac{1}{1+c} a^{1+c} + \frac{c}{1+c} b^{1+\frac{1}{c}}$$

for every $a > 0$, $b > 0$, $c > 0$.

Assume that the matrix F is a positive definite solution of the Lyapunov matrix equation (9) with the matrix W of the form (10) ($w_{11} > 0$). So $f_{11} = -\frac{w_{11}}{2a_1} > 0$, and F has the form (15). In this case

$$((A^*F + FA)y, y) = -w_{11} \left(y_1 + \frac{a_2}{a_1} y_2 + \frac{a_3}{a_1} y_3 \right)^2 .$$

First we establish conditions under which $\dot{\Theta}(x) < 0$ along the trajectories of system (1) with $u = u(x)$, given by (4). To estimate the numerator of $\dot{\Theta}(x)$, given by (13), we will derive some useful inequalities. Using Young's inequality, we obtain

$$\begin{aligned} &2b_1^2 x_1 x_2^{2k_2+1} \Theta^m = 2b_1^2 \left(\varepsilon^{-\frac{2k_2+1}{2k_2+2}} x_1 \Theta^{-1} \right) \left(\varepsilon^{\frac{1}{2k_2+2}} x_2 \Theta^{-m_2} \right)^{2k_2+1} \Theta^{2m} \\ &\le 2|b_1^2| \left(\frac{1}{2k_2+2} \varepsilon^{-(2k_2+1)} \left(x_1 \Theta^{-1} \right)^{2k_2+2} + \frac{2k_2+1}{2k_2+2} \varepsilon \left(x_2 \Theta^{-m_2} \right)^{2k_2+2} \right) \Theta^{2m} \\ &= 2|b_1^2| \left(\frac{1}{2k_2+2} \varepsilon^{-(2k_2+1)} \left(x_1 \Theta^{-1} \right)^{2(k_2-k_1)} x_1^{2k_1+2} \Theta^{2m-m_2} + \frac{2k_2+1}{2k_2+2} \varepsilon x_2^{2k_2+2} \Theta^{m-m_2+1} \right) \end{aligned} \tag{17}$$

for every $\varepsilon > 0$.

Let us chose ε by the condition

$$0 < \varepsilon < \frac{2k_2+2}{2k_2+1} \frac{|b_2^2|}{|b_1^2|}.$$

Thus,

$$g_2 = |b_2^2| - \frac{2k_2+1}{2k_2+2} \varepsilon |b_1^2| > 0.$$

From (3) it follows that

$$2a_0 \Theta(x)^{2m} \ge \lambda_{min}(F) (x_1^2 \Theta^{2(m-m_1)}(x) + x_2^2 \Theta^{2(m-m_2)}(x) + x_3^2),$$

where $\lambda_{min}(F) > 0$ is the smallest eigenvalue of the matrix F. Therefore

$$\frac{x_1^2}{\Theta^{2m_1}(x)} \le \frac{2a_0}{\lambda_{min}(F)}, \quad \frac{x_2^2}{\Theta^{2m_2}(x)} \le \frac{2a_0}{\lambda_{min}(F)}, \quad \frac{x_3^2}{\Theta^{2m}(x)} \le \frac{2a_0}{\lambda_{min}(F)}. \tag{18}$$

Suppose that a_0 satisfy the condition

$$0 < a_0 < \frac{1}{2} \lambda_{min}(F) \sqrt[k_2-k_1]{(2k_2+2) \varepsilon^{2k_2+1} \frac{|b_1^1|}{|b_1^2|}}. \tag{19}$$

Then

$$|b_1^1| - |b_1^2| \frac{\varepsilon^{-(2k_2+1)}}{2k_2+2} \left(x_1 \Theta^{-1} \right)^{2(k_2-k_1)} \ge g_1 > 0,$$

where

$$g_1 = |b_1^1| - |b_1^2| \frac{\varepsilon^{-(2k_2+1)}}{2k_2+2} \left(\frac{2a_0}{\lambda_{min}(F)} \right)^{k_2-k_1} > 0.$$

We now estimate the denominator of $\dot{\Theta}(x)$, given by (13). Calculating FH, we get

$$FH = \begin{pmatrix} (m-m_1)f_{11} & (m-m_2)f_{11}\frac{a_2}{a_1} & 0 \\ (m-m_1)f_{11}\frac{a_2}{a_1} & (m-m_2)f_{22} & 0 \\ (m-m_1)f_{11}\frac{a_3}{a_1} & (m-m_2)\frac{a_3}{a_2}f_{22} & 0 \end{pmatrix}$$

We find that

$$F^1 = 2mF - FH - HF = \begin{pmatrix} 2m_1 f_{11} & (m_1+m_2)\frac{a_2}{a_1}f_{11} & (m+m_1)\frac{a_3}{a_1}f_{11} \\ (m_1+m_2)\frac{a_2}{a_1}f_{11} & 2m_2 f_{22} & (m+m_2)\frac{a_3}{a_2}f_{22} \\ (m+m_1)\frac{a_3}{a_1}f_{11} & (m+m_2)\frac{a_3}{a_2}f_{22} & 2mf_{33} \end{pmatrix}$$

We need to choose f_{11}, f_{22}, and f_{33} in such a way that the matrix F^1 is positive definite. To achieve this we employ the Sylvester criterion, according to which all determinants of the principal minors of F^1 must be positive. This leads to the following conditions:

$$\begin{aligned} &f_{11} > 0, \quad f_{22} > \frac{(m_1+m_2)^2}{4m_1m_2}\frac{a_2^2}{a_1^2}f_{11}, \\ &f_{33} > \frac{(m+m_2)\frac{a_3}{a_2}f_{22}\Delta_2 - (m+m_1)\frac{a_3}{a_1}f_{11}\Delta_3}{2m\Delta_1}, \end{aligned} \tag{20}$$

where $\Delta_1, \Delta_2, \Delta_3$ are given by

$$\Delta_1 = \det \begin{pmatrix} 2m_1 f_{11} & (m_1+m_2)\frac{a_2}{a_1}f_{11} \\ (m_1+m_2)\frac{a_2}{a_1}f_{11} & 2m_2 f_{22} \end{pmatrix} > 0,$$

$$\Delta_2 = \det \begin{pmatrix} 2m_1 f_{11} & (m_1+m_2)\frac{a_2}{a_1}f_{11} \\ (m+m_1)\frac{a_3}{a_1}f_{11} & (m+m_2)\frac{a_3}{a_2}f_{22} \end{pmatrix},$$

$$\Delta_3 = \det \begin{pmatrix} (m_1+m_2)\frac{a_2}{a_1}f_{11} & 2m_2 f_{22} \\ (m+m_1)\frac{a_3}{a_1}f_{11} & (m+m_2)\frac{a_3}{a_2}f_{22} \end{pmatrix}.$$

It is obvious that (16) and (20) hold simultaneously for sufficiently large $f_{22} > 0$ and $f_{33} > 0$. So assume that the positive definite matrix F of the form (15) is such that (16) and (20) hold. Then the matrix $F^1 = 2mF - FH - HF$ is positive definite. Therefore

$$(F^1 y, y) \le \lambda_{max}(F^1)\|y\|^2 = \lambda_{max}(F^1)(x_1^2\Theta^{2(m-m_1)}(x) + x_2^2\Theta^{2(m-m_2)}(x) + x_3^2),$$

where $\lambda_{max}(F^1) > 0$ is the largest eigenvalue of the matrix F^1.

Combining the last inequality with the inequality (17), we derive the estimation

$$\dot{\Theta} \le -\frac{w_{11}(y_1 + \frac{a_2}{a_1}y_2 + \frac{a_3}{a_1}y_3)^2 + 2g_1x_1^{2k_1+2}\Theta^{2m-m_2} + 2g_2x_2^{2k_2+2}\Theta^{m-m_2+1}}{\lambda_{max}(F^1)\|y\|^2} < 0, \tag{21}$$

for $\|x\| \neq 0$, since $w_{11} > 0$, $g_1 > 0$, $g_2 > 0$.

From (21), using Lyapunov function method, we deduce the asymptotic stability of the origin for system (1) with $u = u(x)$ of the form (4). To prove the finite-time convergence of the trajectories of system (1) with $u = u(x)$, given by (4), we show that the inequality (2) holds for $\alpha = 1$.

Consider a family of curves defined by

$$\begin{cases} x_1 = x_1^0|x_3^0|^{-\frac{m_1}{m}} sign(x_3^0)|x_3|^{\frac{m_1}{m}} sign(x_3), \\ x_2 = x_2^0|x_3^0|^{-\frac{m_2}{m}} sign(x_3^0)|x_3|^{\frac{m_2}{m}} sign(x_3), \\ x_3 = x_3. \end{cases} \tag{22}$$

Note that, for every fixed point $x^0 = (x_1^0, x_2^0, x_3^0) \in \mathbb{R}^n \setminus \{0\}$ such that $x_3^0 \neq 0$, the system (22) defines a continuous curve passing through x_0.

Assume that $x \in \mathbb{R}^n$ lies on the curve defined by (22) for some fixed $x^0 \in \mathbb{R}^n \setminus \{0\}$ with $x_3^0 \neq 0$. It is straightforward to verify that

$$\Theta(x) = \Theta(x_0)|x_3^0|^{-\frac{1}{m}}|x_3|^{\frac{1}{m}}. \tag{23}$$

Then

$$y = (\frac{x_1^0}{x_3^0}\Theta(x_0)^{m-m_1}, \frac{x_2^0}{x_3^0}\Theta(x_0)^{m-m_2}, 1)x_3.$$

Denote

$$z = (z_1, z_2, z_3) = (\frac{x_1^0}{x_3^0}\Theta(x_0)^{m-m_1}, \frac{x_2^0}{x_3^0}\Theta(x_0)^{m-m_2}, 1).$$

For every point $x \in \mathbb{R}^3$ that lies on the curve (22) with some fixed $x^0 \in \mathbb{R}^3$ ($x_3^0 \neq 0$), using (22) and (23), we rewrite (21) as follows:

$$\dot{\Theta} \le -\frac{w_{11}(z_1 + \frac{a_2}{a_1}z_2 + \frac{a_3}{a_1})^2 + 2g_1 z_1^{2k_1+2}\left(\frac{2a_0}{(Fz,z)}\right)^{k_1} + 2g_2 z_2^{2k_2+2}\left(\frac{2a_0}{(Fz,z)}\right)^{k_2}}{\lambda_{max}(F^1)\|z\|^2} \equiv G(z), \tag{24}$$

Let us estimate the right-hand side of (24) to show that it is bounded from zero. Note that the function $G(z)$ is continuous, and $G(z) < 0$ at each $(z_1, z_2) \in \mathbb{R}^2$, $z = (z_1, z_2, 1)$. Choose $R \in \mathbb{R}$ so that

$$0 < R < \frac{1}{2}\frac{a_3}{a_1}\frac{1}{\sqrt{1 + \frac{a_2^2}{a_1^2}}}. \tag{25}$$

First we estimate $G(z)$ for every point (z_1, z_2) satisfying $z_1^2 + z_2^2 \le R^2$. Using (25), from (24) we obtain

$$\begin{aligned} G(z) = &-\frac{w_{11}(z_1 + \frac{a_2}{a_1}z_2)^2 + w_{11}\frac{a_3^2}{a_1^2} + 2w_{11}\frac{a_3}{a_1}z_1 + 2w_{11}\frac{a_2 a_3}{a_1^2}z_2}{\lambda_{max}(F^1)\|z\|^2} \\ &-\frac{2g_1 z_1^{2k_1+2}\left(\frac{2a_0}{(Fz,z)}\right)^{k_1} + 2g_2 z_2^{2k_2+2}\left(\frac{2a_0}{(Fz,z)}\right)^{k_2}}{\lambda_{max}(F^1)\|z\|^2} \\ &\le -\frac{w_{11}\frac{a_3^2}{a_1^2} - 2w_{11}\frac{a_3}{a_1}\sqrt{1 + \frac{a_2^2}{a_1^2}}\sqrt{z_1^2 + z_2^2}}{\lambda_{max}(F^1)\|z\|^2} \\ &\le -\frac{w_{11}\frac{a_3^2}{a_1^2} - 2w_{11}\frac{a_3}{a_1}\sqrt{1 + \frac{a_2^2}{a_1^2}}R}{\lambda_{max}(F^1)(R^2+1)} \equiv -M_1(R) < 0. \end{aligned} \tag{26}$$

We now estimate the function $G(z)$ for $(z_1, z_2) \in \mathbb{R}^2$ such that $z_1^2 + z_2^2 \geq R^2$. Using (25), from (24) we derive

$$
\begin{aligned}
G(z) = &-\frac{w_{11}(z_1 + \frac{a_2}{a_1} z_2 + \frac{a_3}{a_1})^2 (Fz, z)^{k_1+k_2} + 2g_1 z_1^{2k_1+2} (2a_0)^{k_1} (Fz, z)^{k_2}}{\lambda_{max}(F^1)\|z\|^2 (Fz, z)^{k_1+k_2}} \\
&- \frac{2g_2 z_2^{2k_2+2} (2a_0)^{k_2} (Fz, z)^{k_1}}{\lambda_{max}(F^1)\|z\|^2 (Fz, z)^{k_1+k_2}} \leq -\frac{2g_1 z_1^{2k_1+2} (2a_0)^{k_1} \lambda_{min}(F)^{k_2} \|z\|^{2k_2}}{\lambda_{max}(F^1)\lambda_{max}(F)^{k_1+k_2} \|z\|^{2+2k_1+2k_2}} \\
&- \frac{2g_2 z_2^{2k_2+2} (2a_0)^{k_2} \lambda_{min}(F)^{k_1} \|z\|^{2k_1}}{\lambda_{max}(F^1)\lambda_{max}(F)^{k_1+k_2} \|z\|^{2+2k_1+2k_2}} \\
&\leq -\frac{2\min\left\{g_1 (2a_0)^{k_1} \lambda_{min}(F)^{k_2}, g_2 (2a_0)^{k_2} \lambda_{min}(F)^{k_1}\right\}}{\lambda_{max}(F^1)\lambda_{max}(F)^{k_1+k_2}} \\
&\times \frac{z_1^{2k_1+2}\|z\|^{2k_2+2} + z_2^{2k_2+2}\|z\|^{2k_1+2}}{\|z\|^{2k_1+2k_2+4}} \\
&\leq -\frac{2\min\left\{g_1 (2a_0)^{k_1} \lambda_{min}(F)^{k_2}, g_2 (2a_0)^{k_2} \lambda_{min}(F)^{k_1}\right\}}{\lambda_{max}(F^1)\lambda_{max}(F)^{k_1+k_2}} \frac{z_1^{2k_1+2k_2+4} + z_2^{2k_1+2k_2+4}}{(z_1^2 + z_2^2 + 1)^{k_1+k_2+2}} \\
&\leq -\frac{2\min\left\{g_1 (2a_0)^{k_1} \lambda_{min}(F)^{k_2}, g_2 (2a_0)^{k_2} \lambda_{min}(F)^{k_1}\right\}}{\lambda_{max}(F^1)\lambda_{max}(F)^{k_1+k_2}} \frac{2^{-(k_1+k_2+1)}(z_1^2 + z_2^2)^{k_1+k_2+2}}{(z_1^2 + z_2^2 + 1)^{k_1+k_2+2}} \\
&\leq -\frac{2\min\left\{g_1 (2a_0)^{k_1} \lambda_{min}(F)^{k_2}, g_2 (2a_0)^{k_2} \lambda_{min}(F)^{k_1}\right\}}{\lambda_{max}(F^1)\lambda_{max}(F)^{k_1+k_2}} \frac{2^{-(k_1+k_2+1)} R^{2k_1+2k_2+2}}{(R^2 + 1)^{k_1+k_2+2}} \\
&\equiv -M_2(R) < 0.
\end{aligned} \tag{27}
$$

Thus, we have proved that

$$
\dot{\Theta}(x) \leq -\min\{M_1(R), M_2(R)\} < 0, \tag{28}
$$

and the inequality (2) holds for $\alpha = 1$, $\beta = \min\{M_1(R), M_2(R)\} > 0$. This proves the global finite-time convergence of the trajectories of system (1) with $u = u(x)$, given by (4).

2 Boundedness of control

Let us rewrite the control $u(x)$, given by (4), in the form

$$
u(x) = (a, D(\Theta(x))x)\frac{1}{\Theta^m(x)} + a_4 \frac{x_1^{2k_1+1}}{\Theta^{m_2-1}(x)} + a_5 \frac{x_2^{2k_2+1}}{\Theta^{m-1}(x)},
$$

where $a = (a_1, a_2, a_3)$.

Using (18), we estimate $\frac{x_1^{2k_1+1}}{\Theta^{m_2-1}(x)}$ and $\frac{x_2^{2k_2+1}}{\Theta^{m-1}(x)}$ as follows:

$$
\frac{x_1^{2k_1+1}}{\Theta^{m_2-1}(x)} = \frac{x_1^{2k_1}}{\Theta^{2k_1}(x)} \frac{x_1}{\Theta(x)} \leq \left(\frac{2a_0}{\lambda_{min}(F)}\right)^{\frac{2k_1+1}{2}}, \tag{29}
$$

$$
\frac{x_2^{2k_2+1}}{\Theta^{m-1}(x)} = \frac{x_2^{2k_2+1}}{\Theta^{(2k_2+1)m_2}(x)} \leq \left(\frac{2a_0}{\lambda_{min}(F)}\right)^{\frac{2k_2+1}{2}}. \tag{30}
$$

From (3), using the estimate

$$2a_0\Theta^{2m} \geq \lambda_{min}(F)||D(\Theta)x||^2,$$

we get

$$\frac{||D(\Theta)x||}{\Theta^m(x)} \leq \sqrt{\frac{2a_0}{\lambda_{min}(F)}}. \tag{31}$$

The inequalities (29), (30), and (31) give the following estimate of the feedback

$$|u(x)| \leq \|a\|\|D(\Theta)x\|\frac{1}{\Theta^m(x)} + \left(\frac{2a_0}{\lambda_{min}(F)}\right)^{\frac{2k_1+1}{2}} + \left(\frac{2a_0}{\lambda_{min}(F)}\right)^{\frac{2k_2+1}{2}}$$
$$\leq \frac{\sqrt{2a_0}}{\sqrt{\lambda_{min}(F)}}\left(||a|| + \left(\frac{2a_0}{\lambda_{min}(F)}\right)^{k_1} + \left(\frac{2a_0}{\lambda_{min}(F)}\right)^{k_2}\right) \equiv Q(a_0).$$

Note that the function $Q(a_0)$ is continuous. Moreover, $Q(0) = 0$, and $Q \to +\infty$ as $a_0 \to +\infty$. This implies that the equation $Q(a_0) = d$ has a positive solution a_0 for every $d > 0$.

Let a_0^* be the smallest root of the equation

$$Q(a_0) = d.$$

Then for every a_0 such that $0 < a_0 \leq a_0^*$ the feedback control $u(x)$ satisfies the restriction $|u(x)| \leq d$.

We summarize the procedure for designing the solution of the global bounded control synthesis problem in the following theorem.

Theorem 2. *Suppose that $a_1 < 0$, $a_2 < 0$, and $a_3 < 0$ are arbitrary real numbers. Let the matrix W be of the form (10) with $w_{11} > 0$. Define the matrix F by (15). Suppose the numbers $f_{22} > 0$, $f_{33} > 0$ are sufficiently large so that the matrixes F and $F^1 = 2mF - HF - FH$ are both positive definite. Choose a_0 by (19) so that $0 < a_0 \leq a_0^*$, where a_0^* is the smallest root of the equation $Q(a_0) = d$, $d > 0$. Suppose that, for every $x \in \mathbb{R}^n \setminus \{0\}$, the controllability function $\Theta(x)$ is the positive solution of the equation (3). Let a_4 and a_5 be given by (12). Then the control defined by (4) solves the global bounded finite-time stabilizing control synthesis problem for system (1). Moreover, the time of motion $T(x_0)$ from x_0 to the origin satisfies the estimate*

$$T(x_0) \leq \frac{1}{\min\{M_1(R), M_2(R)\}}\Theta(x_0),$$

where $M_1(R)$ and $M_2(R)$ are given by (26) and (27), respectively.

Example 1. *Let us solve the global bounded control synthesis problem for the system (1) in the case $k_1 = 1, k_2 = 2$. Thus, (1) takes the form*

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3, \\ \dot{x}_3 = x_2^5. \end{cases} \tag{32}$$

We take $a_1 = -1$, $a_2 = -\frac{1}{3}, a_3 = -\frac{1}{5}$. *Set* $w_{11} = -2$. *According to* (10), *the matrix* W *has the form*

$$W = -\begin{pmatrix} 2 & \frac{2}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{2}{9} & \frac{2}{15} \\ \frac{2}{5} & \frac{2}{15} & \frac{2}{25} \end{pmatrix}.$$

By Theorem 1, a positive definite solution of the following Lyapunov matrix equation

$$A^*F + FA = -W,$$

for $f_{22} = 0.2$, $f_{33} = 0.3$, *and* $f_{23} = \frac{a_3}{a_2} f_{22}$, *has the form*

$$F = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} & \frac{3}{25} \\ \frac{1}{5} & \frac{3}{25} & \frac{3}{10} \end{pmatrix}.$$

Note that $F^1 = 2mF - HF - FH$ *is positive definite.*

Define the controllability function $\Theta(x)$ *as the positive definite solution of the equation*

$$2a_0\Theta^{2m} = (FD(\Theta)x, D(\Theta)x),$$

where $m = (2k_1 + 2)(2k_2 + 1) + 1 = 21$, $D(\Theta) = \begin{pmatrix} \Theta(t)^{20} & 0 & 0 \\ 0 & \Theta(t)^{17} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Take $a_0 = 0.6567802403$. *Theorem 2 gives the following finite-time stabilizing control:*

$$u(x) = -\frac{x_1(t)}{\Theta(t)} - \frac{1}{3}\frac{x_2(t)}{\Theta(t)^4} - \frac{1}{5}\frac{x_3(t)}{\Theta(t)^{21}} - \frac{3}{5}\frac{x_1(t)^3}{\Theta(t)^3} - \frac{3}{2}\frac{x_2^5}{\Theta(t)^{20}},$$

which solves the global bounded control synthesis problem for system (32).

Apply this control to system (32). *For example, we take the initial point* $x_0 = (1, -2, -1)$. *Numerical computation of the solution* $x(t)$ *of the closed-loop system* (32) *with* $x(0) = x_0$ *gives the following results:* $||x(0)|| = 6$, $||x(100)|| = 0.031\ldots$, $||x(150)|| = 0.00409\ldots$, $||x(180)|| = 0.6932\ldots \times 10^{-4}$, $||x(184.5|| = 0.185\ldots \times 10^{-16}$.

3 Conclusion

In the paper, we solve the finite-time stabilization problem for the high-order nonlinear three-dimensional system (1). These systems are difficult to control in view of the fact that they have uncontrollable first approximation and cannot be mapped to linear systems. We develop the method for stabilization proposed in [2] to present a class of finite-time stabilizing controls. Employing the controllability function method [5], we achieve the finite-time stabilization. We establish the conditions under which the control satisfies preassigned constraint on its absolute value.

Our approach produces controls that can be easily implemented numerically. This approach seems to be suitable for systems of high dimension. The simulation example is provided to demonstrate the effectiveness of the proposed approach.

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Бебія М.О. *Про синтез обмежених керувань для тривимірних нелінійних систем високого порядку* // Буковинський матем. журнал — 2023. — Т.11, №2. — С. 11–23.

У статті розглянуто задачу побудови обмежених керувань, що забезпечують потрапляння траєкторій відповідної замкнутої системи у початок координат за скінченний час. Досліджено клас нелінійних некерованих за першим наближенням тривимірних систем,

які не можна відобразити на лінійні. Складність вивчення таких систем полягає у неможливості їх дослідження за першим наближенням, тому такі системи називають суттєво нелінійними. Крім того, оскільки нескінченна кількість траєкторій замкнутої системи має проходити через початок координат, то з теореми єдиності розв'язку випливає, що шукане керування не задовольняє умову Ліпшиця та не є гладким в нулі. У випадку стійкості нульової точки спокою замкнутої системи, цю задачу називають задачею стабілізації за скінченний час.

Запропонований метод побудови керувань ґрунтується на методі функції керованості В.І. Коробова. Функцію керованості задано неявно як єдиний додатний корінь відповідного рівняння. Керування вибрано таким чином, щоб досягти виконання спеціальної нерівності для похідної функції керованості. Ця нерівність гарантує потрапляння траєкторій у початок координат за скінченний час. При побудові керувань використано сингулярне матричне рівняння Ляпунова, що було досліджено у більш ранніх роботах автора. Знайдене керування забезпечує прямування розв'язків системи до нуля за скінченний час для будь-якої початкової точки, такий синтез називають глобальним. Синтезуюче позиційне керування задовольняє наперед заданим обмеженням на абсолютну величину. Результати роботи може бути застосовано для дослідження систем більш високої розмірності. Ефективність запропонованого підходу проілюстровано з використанням модельного прикладу.