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ON THE DECOMPOSITION PROBLEM FOR FUNCTIONS OF SMALL EXPONENTIAL TYPE

The technique of decomposition for functions into the sum or product of two functions is often used to facilitate the study of properties of functions. Some decomposition problems in the weighted Hardy space, Paley-Wiener space, and Bergman space are well known. Usually, in these spaces, functions are represented as the sum of two functions, each of them is "big" only in the first or only in the second quarter. The problem of decomposition of functions has practical applications, particularly in information theory. In these applications, it is often necessary to find those solutions of the decomposition problem whose growth on the negative real semi-axis is "small". In this article we consider the decomposition problem for an entire function of any small exponential type in $\{z : \operatorname{Re} z < 0\}$. We obtain conditions for the existence of solutions of the above problem.

Key words and phrases: Hardy space, Paley-Wiener space, decomposition.

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Introduction

Decomposition for spaces of analytic functions into the sum or product of two spaces with prescribed properties are interesting and practically important in the theory of analytic function spaces. The results of such investigation can be found in the papers of D. Dryanov [9], B. Mourrain [19], C. Carlota, A. Cornelas [2], E. Milicka [18], J. Zhao, M. Kostic, W-S. Du [25] and many others.

Let the Paley - Wiener space W^p_σ, $\sigma > 0$, be the space of entire functions f of exponential type $\le \sigma$ belonging to $L^p(\mathbb{R})$. Space W^p_σ also can be defined [15] as the space of entire functions satisfying the condition

$$\sup_{\varphi\in(0;2\pi)} \left\{ \int\limits_0^{+\infty} |(fre^{i\varphi})|^p e^{-p\sigma r|\sin\varphi|} dr \right\}^{1/p} < +\infty. \tag{1}$$

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Theorem 1. [20] *The space W^2_σ coincides with the space of functions represented as*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\sigma}^{\sigma} \varphi(it) e^{itz} dt, \varphi \in L^2(-i\sigma; i\sigma). \quad (2)$$

The above theorem is one of the fundamental result in the theory of the Paley-Wiener spaces and has a number of generalizations and analogs. The Paley-Wiener theorem relates to the pointwise growth rate of analytic functions to the support of the Fourier transforms of their boundary values [13]. K. Flornes [12] considers the sampling and interpolation in the Paley-Wiener spaces, in [14] there where given the proofs of a Paley - Wiener type theorem and the inversion formula for the Jacobi transform. C. Eoff [10] investigates the discrete nature of the Paley - Wiener space. S. Favorov considers a local version of Wiener's theorem on absolutely convergent Fourier series and found the application of this theorem in the theory of quasicrystals [11]. The following theorem is an analog of the Paley-Wiener theorem for the case $p = 1$, which was proved by G. Ber.

Theorem 2. [1] *The space W^1_σ coincides with the space of function f represented by* (2), *where*

$$\varphi(t) = \sum_{k=-\infty}^{+\infty} c_k e^{-\frac{ik\pi t}{\sigma}}, \quad (c_k) \in l^1 \quad (3)$$

and

$$\sum_{m=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} (-1)^{k+m} c_{k+m} \frac{k}{1+k^2} \right| < +\infty.$$

Ber's theorem was reformulated in a useful for applications form [4].

Theorem 3. *A function f belongs to W^1_σ, $\sigma > 0$, if and only if f represented by*

$$f(z) = \frac{\sigma}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{\sin \sigma z}{\sigma z - \pi k}, \quad (4)$$

where $(c_k) \in l^2$ and

$$\sum_{k=-\infty}^{+\infty} \left| f\left(\frac{\pi k}{\sigma}(1-\delta) \right) \right| < +\infty,$$

for $\delta \in (0;1)$.

We use the basic theorems of Paley - Wiener theory for the research of the decomposition problem in the space of analytic function. The investigation of this type was described in the papers of R. S. Yulmukhametov [22], [23], Yu. I. Lyubarskii [16], I. E. Chyzhykov [3], Z. Wen [24].

We consider the decomposition of functions as a problem of the some function representation into the sum of two functions each of them characterized by the module which is "big" only in the upper or lower half-planes. B. V. Vynnytskyi and his collaborators investigated this problem for Paley - Wiener space W^1_σ [21], [5].

Let $E^p[\mathbb{C}(\alpha;\beta)], 0<\beta-\alpha<2\pi, 1\le p<+\infty$, be the space of analytical functions f in $\mathbb{C}(\alpha;\beta)=\{z:\alpha<\arg z<\beta\}$ such that

$$\sup_{\alpha<\varphi<\beta}\left\{\int_0^{+\infty}|f(re^{i\varphi})|dr\right\}<+\infty.$$

Also everywhere on $\partial\mathbb{C}(\alpha;\beta)$ function $f\in E^p[\mathbb{C}(\alpha;\beta)]$ [3] has angular boundary values and $f\in L^p[\mathbb{C}(\alpha;\beta)]$.

T. I. Hishchak investigated for $f\in W^1_\sigma$ the decomposition problem

$$f=\chi-\mu, \tag{5}$$

where χ, μ are entire functions with an exponential growth on the left half-plane and $\chi\in E^1[\mathbb{C}(0;\frac{\pi}{2})]$, $\mu\in E^1[\mathbb{C}(\frac{-\pi}{2};0)]$. She obtained the following useful result [7].

Theorem 4. *Let $f\in W^1_\sigma$. The functions $\chi(z)=\chi_1(z)+i\chi_2(-iz)$ and $\mu=\chi-f$, where*

$$\chi_1(z)=\frac{1}{\sqrt{2\pi}}\int_0^\sigma\varphi(t)e^{itz}dt,\quad \chi_2(z)=-\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^0\varphi(t)e^{itz}dt.$$

is a solution of the decomposition problem (5) *if and only if both of the following conditions are fulfilled*

$$\sum_{m=1}^{+\infty}\left|\sum_{k=-\infty}^{+\infty}c_k\frac{k}{(m-\frac{i}{2}-k)(m-\frac{i}{2}-ik)}\right|<+\infty, \tag{6}$$

$$\sum_{m=1}^{+\infty}\left|\sum_{k=-\infty}^{+\infty}c_k\frac{k}{(m+\frac{i}{2}+ik)(m+\frac{i}{2}-k)}\right|<+\infty. \tag{7}$$

For the case $p=2$ there exists the elementary solution of the decomposition problem (5) based on the Paley - Wiener theorem:

$$\chi(z)=\frac{1}{\sqrt{2\pi}}\int_0^\sigma\varphi(it)e^{itz}dt,\quad \mu(z)=-\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^0\varphi(it)e^{itz}dt.$$

The case $p=1$ is much more complicated and interesting for research and application. This problem is generated by studies of completeness of system of functions [21] and considered in [5]. The above decomposition problem is interesting in the signal processing theory [8].

We say that an entire function is an entire function of exponential type α in the half-plane $\mathbb{C}_-=\{z:\operatorname{Re}z<0\}$ if

$$(\forall\delta>0)(\exists A>0)(\forall z\in\mathbb{C}_-):|f(z)|\le Ae^{(\alpha+\delta)|z|} \tag{8}$$

and inequality (8) is false if replace the number α by a smaller one.

1 The main results

The aim of our research is to find the solutions of the decomposition problem for functions with small exponential type in the lower half-plane. Function χ in the representation of Hishchak T. I. is the entire function of exponential type σ in half-plane $\{z : \operatorname{Re} z > 0\}$.

We offer a solving the above decomposition problem (5) for an entire function of any small exponential type in $\mathbb{C}_-$. We propose a new representation of the function. This representation different from the one received by Hishchak. Let $f = \hat{\chi} + \hat{\mu}$, where

$$\hat{\chi}(z) = \chi_1(z) + i\hat{\chi}_2(-iz) \tag{9}$$

and

$$\chi_1(z) = \frac{1}{\sqrt{2\pi}} \int_0^\sigma \varphi(t)e^{itz}, \qquad \hat{\chi}_2(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^0 \varphi(t)e^{itz}dt.$$

Obviously, $\varepsilon > 0$ is the exponential type of $\hat{\chi}_2$ in the $\mathbb{C}_-$ and ε can be an arbitrary small. Then function $\hat{\mu}$ is defined by formula

$$\hat{\mu} = f - \hat{\chi} =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^0 \varphi(t)e^{itz}dt + i\frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^0 \varphi(t)e^{tz}dt.$$

Theorem 5. *Let $f \in W_\sigma^1$. Functions $\hat{\chi}, \hat{\mu}$ are the solution of the decomposition problem* (5) *if and only if conditions* (6) *and* (7) *are fulfilled.*

Proof. First let us prove that the function $\hat{\chi}(x - i\frac{\pi}{2\sigma})$ belongs to $L^1(0; +\infty)$ if and only if χ belongs to $L^1(0; +\infty)$ and defined in (6). The series in (3) converges uniformly on every interval of the positive real half-axis by the Weierstrass M-test. So, the series can be integrated term by term

$$\chi_1(z) = \frac{1}{\sqrt{2\pi}} \int_0^\sigma \sum_{k=-\infty}^{+\infty} c_k e^{\frac{-ik\pi t}{\sigma}} e^{itz}dt = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{e^{i\sigma(z-\frac{k\pi}{\sigma})} - 1}{i(z - \frac{k\pi}{\sigma})}$$

and

$$\hat{\chi}_2(-iz) = -\frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^0 \sum_{k=-\infty}^{+\infty} c_k e^{\frac{-ik\pi t}{\sigma}} e^{tz}dt = -\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{1 - e^{-\varepsilon(z-\frac{k\pi}{\sigma}i)}}{z - \frac{k\pi}{\sigma}i}.$$

Therefore

$$\hat{\chi}(z) = \chi_1(z) + i\hat{\chi}_2(-iz) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \left(\frac{e^{i\sigma(z-\frac{k\pi}{\sigma})} - 1}{i(z - \frac{k\pi}{\sigma})} - i\frac{1 - e^{-\varepsilon(z-\frac{k\pi}{\sigma}i)}}{z - \frac{k\pi}{\sigma}i} \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \left(\frac{(-1)^k e^{i\sigma z}}{i(z - \frac{k\pi}{\sigma})} + \frac{(-1)^k i e^{-\varepsilon z}}{z - \frac{k\pi}{\sigma}i} - \frac{k\pi(i-1)}{i\sigma(z - \frac{k\pi}{\sigma})(z - \frac{k\pi}{\sigma}i)} \right).$$

For $z = x - i\frac{\pi}{2\sigma}$ we have

$$\hat{\chi}\left(x - i\frac{\pi}{2\sigma}\right) = -\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k e^{i\sigma(x-i\frac{\pi}{2\sigma})}}{x - i\frac{\pi}{2\sigma} - \frac{k\pi}{\sigma}} +$$

$$+\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k i e^{-\varepsilon(x-i\frac{\pi}{2\sigma})}}{x - i\frac{\pi}{2\sigma} - \frac{k\pi}{\sigma}i} +$$

$$+\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} \frac{k\pi(i-1)}{\sigma(x - i\frac{\pi}{2\sigma} - \frac{k\pi}{\sigma})(x - i\frac{\pi}{2\sigma} - \frac{k\pi}{\sigma}i)} =$$

$$= -\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k e^{i\sigma x + \frac{\pi}{2}}}{x - \frac{\pi}{\sigma}(\frac{i}{2} + k)} + \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k i e^{-\varepsilon x + i\frac{\pi}{2}}}{x - i\frac{\pi}{\sigma}(\frac{1}{2} + k)} +$$

$$+\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} \frac{k\pi(i+1)}{\sigma(x - \frac{\pi}{\sigma}(\frac{i}{2} + k))(x - i\frac{\pi}{\sigma}(\frac{1}{2} + k))}.$$

The above addends denote by $S_1(x)$, $S_2(x)$, $S_3(x)$ so

$$\hat{\chi}\left(x - i\frac{\pi}{2\sigma}\right) = -S_1(x) + S_2(x, \varepsilon) + S_3(x). \tag{10}$$

T. I. Hishchak [7] proves that

$$\int_{\frac{\pi}{2}}^{+\infty} |S_1(x)| dx < +\infty.$$

Noting that

$$\sum_{k=-\infty}^{+\infty} \frac{|c_k|}{|x - i\frac{\pi}{\sigma}(\frac{1}{2} + k)|} = \sum_{k=-\infty}^{+\infty} \frac{|c_k|}{\sqrt{x^2 + (\frac{\pi}{2\sigma} + \frac{\pi k}{\sigma})^2}} \le \frac{1}{x}\sum_{k=-\infty}^{+\infty} |c_k|.$$

Then consider the integral over $x \in [\frac{\pi}{2}; +\infty)$ for the function $S_2(x, \varepsilon)$

$$\int_{\frac{\pi}{2}}^{+\infty} |S_2(x, \varepsilon)| dx = \frac{1}{\sqrt{2\pi}}\int_{\frac{\pi}{2}}^{+\infty} \left|\sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k i e^{-\varepsilon x + \frac{\pi i}{2}}}{x - i\frac{\pi}{\sigma}(\frac{1}{2} + k)}\right| dx =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{\pi}{2}}^{+\infty} \sum_{k=-\infty}^{+\infty} \frac{|c_k| e^{\varepsilon x}}{|x - i\frac{\pi}{\sigma}(\frac{1}{2} + k)|} dx \le \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty} |c_k| \int_{\frac{\pi}{2}}^{+\infty} \frac{e^{-\varepsilon x}}{x} dx < +\infty.$$

In [7] proved that $\chi(x - i\frac{\pi}{2\sigma}) \in L^1(\frac{\pi}{\sigma}; +\infty)$ if and only if the following condition holds

$$\int_{\frac{\pi}{\sigma}}^{+\infty} \left|\sum_{k=-\infty}^{+\infty} c_k \frac{k}{(x - \frac{\pi}{\sigma}(\frac{i}{2} + k))(x - i\frac{\pi}{\sigma}(\frac{1}{2} + k))}\right| < +\infty.$$

This inequality implies

$$\int_{\frac{\pi}{2}}^{+\infty} |S_3(x)|dx < +\infty.$$

Then we proved that $\hat{\chi}(x - i\frac{\pi}{2\sigma}) \in L^1(0; +\infty)$.

Now suppose that if $\hat{\chi}(x - i\frac{\pi}{2\sigma}) \in L^1(0; +\infty)$ then $f \in W^1_\sigma$, $\hat{\chi} \in L^1(\frac{\pi}{\sigma}; +\infty)$. Using (10), we conclude

$$\chi\left(x - \frac{i\pi}{2\sigma}\right) = \hat{\chi}\left(x - i\frac{\pi}{2\sigma}\right) + S_2(x) - S_2(x, \varepsilon).$$

In [7], T. Hishchak proved that $S_1 \in L^1(\frac{\pi}{\sigma}; +\infty)$; in proof of the sufficiency we have received that $S_2 \in L^1(\frac{\pi}{\sigma}; +\infty)$ without using (6). Hence $S_3 \in L^1(\frac{\pi}{\sigma}; +\infty)$.

Finally, we use a theorem of the Phraugmen - Lindelof type [6], [17] to our prove and show that $\hat{\chi}(x-i\frac{\pi}{\sigma}) \in \mathbb{C}(0; \frac{\pi}{2})$. So, $\hat{\chi}(x-\frac{i\pi}{2\sigma}) \in L^1(\partial\mathbb{C}(0; \frac{\pi}{2}))$ and we obtaine $\hat{\chi}(x-\frac{i\pi}{2\sigma}) \in L^1(\frac{\pi}{\sigma}; +\infty)$.

Since $\hat{\chi}(x - i\frac{\pi}{\sigma}) \in L^1(\frac{\pi}{\sigma}; +\infty)$, $\hat{\chi}(iy + \frac{\pi}{2\sigma}) \in L^1(\frac{\pi}{\sigma}; +\infty)$ and $\hat{\chi}$ is an entire function by properties of entire functions it follows that $\hat{\chi}(z - \frac{i\pi}{2\sigma} - \frac{\pi}{2\sigma}) \in L^1(\partial\mathbb{C}(0; \frac{\pi}{2}))$. If $\chi_1 \in W^2_\sigma$ and $\hat{\chi}_2 \in W^2_\sigma$, then function $\hat{\chi}$ is an entire function of exponential type less then σ in the half-plane $\mathbb{C}_+$. Note that if $\alpha = 0$ and $\beta = \frac{\pi}{2}$, $\gamma = \frac{3}{2}$. Then for any $\tau > 0$ we obtain

$$\int_0^{+\infty} \left|\hat{\chi}\left(re^{i\varphi} - \frac{i\pi}{2\sigma} - \frac{\pi}{2\sigma}\right)\right| e^{-\tau r^{\frac{3}{2}}} dr =$$

$$= \int_0^{+\infty} \left|\hat{\chi}\left(re^{i\varphi} - \frac{i\pi}{2\sigma} - \frac{\pi}{2\sigma}\right)\right| e^{-2r\sigma} e^{-\tau r^{\frac{3}{2}} + 2r\sigma} dr \le$$

$$\le \left(\int_0^{+\infty} \left|\hat{\chi}\left(re^{i\varphi} - \frac{i\pi}{2\sigma} - \frac{\pi}{2\sigma}\right)\right|^2 e^{-4r\sigma} dr \int_0^{+\infty} e^{-\tau r^{\frac{3}{2}} + 2r\sigma} dr\right)^{\frac{1}{2}} \le$$

$$\le b_1 \int_0^{+\infty} e^{-\tau r^{\frac{3}{2}} + 2r\sigma} dr \le b < +\infty,$$

where b is independent of φ.

Therefore $\hat{\chi} \in E^1[\mathbb{C}(0; \frac{\pi}{2})]$. It remains to show that function μ in the decomposition $f = \hat{\chi} + \mu$ belongs to $E^1[\mathbb{C}(-\frac{\pi}{2}; 0)]$.

Indeed,

$$\mu(z) = f(z) - \hat{\chi}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{0} \varphi(t)e^{itz}dt + \frac{1}{\sqrt{2\pi}} \int_0^{\sigma} \varphi(t)e^{itz}dt - \frac{1}{\sqrt{2\pi}} \int_0^{\sigma} \varphi(t)e^{itz}dt +$$

$$+ i\frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{0} \varphi(t)e^{tz}dt = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{0} \varphi(t)e^{itz}dt + \frac{1}{\sqrt{2\pi}} \int_{-\varepsilon}^{0} \varphi(t)e^{tz}dt.$$

Then

$$\mu(z)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\sigma}^{0}\sum_{k=-\infty}^{+\infty}c_k e^{-\frac{ik\pi t}{\sigma}}e^{itz}dt+i\frac{1}{\sqrt{2\pi}}\int\limits_{-\varepsilon}^{0}\sum_{k=-\infty}^{+\infty}c_k e^{-\frac{ik\pi t}{\sigma}}e^{tz}dt=$$

$$=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}c_k\left(\frac{1-e^{ik\pi-i\sigma z}}{i(z-\frac{k\pi}{\sigma})}+i\frac{1-e^{\frac{i\varepsilon k\pi}{\sigma}-\varepsilon z}}{z-i\frac{k\pi}{\sigma}}\right)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}c_k\frac{(-1)^k ie^{-i\sigma z}}{z-\frac{k\pi}{\sigma}}-$$

$$-\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}\frac{(-1)^k ie^{-\sigma z}}{z-i\frac{k\pi}{\sigma}}-\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}c_k\frac{(1+i)\pi k}{\sigma(z-\frac{\pi k}{\sigma})(z-i\frac{k\pi}{\sigma})}.$$

Similarly as for function $\hat{\chi}$, we can prove that $\mu(z)\in E^1[\mathbb{C}(-\frac{\pi}{2};0)]$. □

Now we propose a solution of the decomposition problem in the form

$$f=\breve{\chi}-\breve{\mu}, \tag{11}$$

where

$$\breve{\chi}(z)=\chi_1(z)+i\breve{\chi}_2(-iz)$$

and

$$\chi_1(z)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}c_k\frac{e^{i\sigma(z-\frac{k\pi}{\sigma})}-1}{i(z-\frac{k\pi}{\sigma})},\qquad \breve{\chi}_2(z)=-\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}c_k\frac{1-e^{-\varepsilon_k(z-\frac{k\pi}{\sigma}i)}}{z-\frac{k\pi}{\sigma}i}.$$

Here function $\chi_1(z)$ is the same as in (9) and function $\breve{\chi}_2(z)$ coincides with $\hat{\chi}_2(z)$ if $\varepsilon_k\equiv\varepsilon$ for every $k\in\mathbb{Z}$.

Theorem 6. *Let $f\in W^1_\sigma$. If for the consequence of positive values ε_k the inequality*

$$\sum_{k=-\infty}^{+\infty}|c_k|\int\limits_{\frac{\pi}{2}}^{+\infty}\frac{e^{-\varepsilon_k x}}{x}dx<+\infty$$

and conditions (6), (7) *hold, then functions $\breve{\chi},\breve{\mu}$ are the solution of the decomposition problem* (11).

Proof. Taking into account the proof of Theorem 5, we obtain

$$\breve{\chi}\left(x-i\frac{\pi}{2\sigma}\right)=-S_1(x)+S_2(x,\varepsilon_k)+S_3(x),$$

where $\varepsilon=(\varepsilon_k)_{k=-\infty}^{+\infty}$ and

$$S_2(x,\varepsilon_k)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{+\infty}c_k\frac{(-1)^k ie^{-\varepsilon_k x+i\frac{\pi}{2}}}{x-i\frac{\pi}{\sigma}(\frac{1}{2}+k)}.$$

In the proof of the previous theorem we showed that

$$\int_{\frac{\pi}{2}}^{+\infty} |S_1(x)|dx < +\infty, \qquad \int_{\frac{\pi}{2}}^{+\infty} |S_3(x)|dx < +\infty.$$

Therefore it remains to prove that

$$\int_{\frac{\pi}{2}}^{+\infty} |S_2(x,\varepsilon_k)|dx < +\infty.$$

Note that

$$\frac{1}{|x - i\frac{\pi}{\sigma}(\frac{1}{2}+k)|} = \frac{1}{\sqrt{x^2 + (\frac{\pi}{2\sigma} + \frac{\pi k}{\sigma})^2}} \leq \frac{1}{|x|}.$$

So,

$$\int_{\frac{\pi}{2}}^{+\infty} |S_2(x,\varepsilon_k)|dx = \frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{2}}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k i e^{-\varepsilon_k x + \frac{\pi i}{2}}}{x - i\frac{\pi}{\sigma}(\frac{1}{2}+k)} \right| dx \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{2}}^{+\infty} \sum_{k=-\infty}^{+\infty} \frac{|c_k| e^{\varepsilon_k x}}{|x - i\frac{\pi}{\sigma}(\frac{1}{2}+k)|} dx \leq \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} |c_k| \int_{\frac{\pi}{2}}^{+\infty} \frac{e^{-\varepsilon_k x}}{x} dx \leq M < +\infty,$$

where M independent of (ε_k). Therefore $\breve{\chi} \in E^1[\mathbb{C}(0; \frac{\pi}{2})]$. □

Example. Let

$$f(z) = \frac{1}{z^2}(\cos\sigma z - 1) \in W^1_\sigma.$$

Then we can represent the function by (4) at the point $z = \frac{\pi k}{\sigma}$, where

$$c_k = \frac{\sqrt{2\pi}}{\sigma}(-1)^k f\left(\frac{\pi k}{\sigma}\right).$$

Let

$$\varepsilon_k = \begin{cases} \frac{1}{\ln|k|}, & \text{if} \quad |k| > 1, \\ 1, & \text{if} \quad k \in \{-1; 0; 1\}. \end{cases}$$

The solution of decomposition problem (5) is given by functions

$$\breve{\chi}(z) = \frac{\sqrt{2}}{\pi} \left(\sum_{k=-\infty}^{+\infty} \frac{(-1)^k \tilde{c}_k e^{i\sigma z}}{i(z - \frac{k\pi}{\sigma})} + \left[\sum_{\substack{k=-\infty,\\ k\neq -1;0;1}}^{+\infty} \frac{\sigma^2 i((-1)^k - 1) e^{-\frac{1}{\ln k} z}}{k^2(z - i\frac{k\pi}{\sigma})} - \right.\right.$$

$$\left.\left. - \frac{\sigma i e^z}{z - \frac{k\pi}{\sigma} i} + \sum_{\substack{k=-1,\\ k\neq 0}}^{1} \frac{\sigma^2 i((-1)^k - 1)e^z}{k^2(z - i\frac{k\pi}{\sigma})} \right] + \sum_{k=-\infty}^{+\infty} \frac{k\pi(i-1)\tilde{c}_k}{i\sigma(z - \frac{k\pi}{\sigma})(z - i\frac{k\pi}{\sigma})},$$

where

$$\tilde{c}_k = \begin{cases} \frac{2\sigma^2}{\sqrt{\pi}k^2}(1-(-1)^k), & \text{if} \quad k \neq 0, \\ -\sigma(-1)^k, & \text{if} \quad k = 0, \end{cases}$$

and

$$\breve{\mu}(z) = \frac{\sqrt{2}}{\pi}\Bigg(\sum_{k=-\infty}^{+\infty} \frac{\tilde{c}_k i e^{-i\sigma z}}{(z-\frac{k\pi}{\sigma})} - \sum_{k=-\infty}^{+\infty} \frac{(-1)^k \tilde{c}_k e^{i\sigma z}}{i(z-\frac{k\pi}{\sigma})}$$

$$- \Bigg[\sum_{\substack{k=-\infty,\\ k\neq -1;0;1}}^{+\infty} \frac{\sigma^2 i((-1)^k-1)e^{-\frac{1}{\ln k}z}}{k^2(z-i\frac{k\pi}{\sigma})} - \frac{\sigma i e^z}{z-\frac{k\pi}{\sigma}i} +$$

$$- \sum_{\substack{k=-1,\\ k\neq 0}}^{1} \frac{\sigma^2 i((-1)^k-1)e^{z}}{k^2(z-i\frac{k\pi}{\sigma})}\Bigg] + \sum_{k=-\infty}^{+\infty} \frac{k\pi(i-1)\tilde{c}_k}{i\sigma(z-\frac{k\pi}{\sigma})(z-i\frac{k\pi}{\sigma})}\Bigg).$$

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Прийоми розщелення функцій на суму чи добуток двох функцій часто використовується для полегшення дослідження властивостей функцій. Відомі деякі розв'язки задач розщеплення в ваговому просторі Гарді та просторі Пелі - Вінера, де функція зображається у вигляді суми двох функцій, кожна з яких є "великою" лише у першій або лише у другій координатній чверті. Застосування результатів задач розщеплення, зокрема, в теорії інформації, може вимагати знаходження таких розв'язків проблеми розщеплення, зростання яких на від'ємній дійсній півосі було б "малим". У даній статті ми розглядаємо питання існування розв'язків проблеми розщеплення функцій, що належать простору Пелі-Вінера W^p_σ на суму або різницю двох цілих функцій одна з яких є функцією наперед визначеного довільно малого екпоненційного типу ε. Цілою функцією експоненційного типу $\alpha > 0$ в півплощині $\mathbb{C}_- = \{z : \operatorname{Re} z < 0\}$ називаємо цілу функцію для якої виконується

умова $(\forall\delta > 0)(\exists A > 0)(\forall z \in \mathbb{C}_-) : |f(z)| \leq Ae^{(\alpha+\delta)|z|}$ і дана умова не виконується якщо замінити число α на менше. Знайдені необхідні та достатні умови розв'язку проблеми розщеплення для цілих функцій довільно малого експоненційного типу в лівій півплощині.