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HADAMARD COMPOSITION OF SERIES IN SYSTEMS OF FUNCTIONS

For regularly converging in $\mathbb{C}$ series $A_j(z) = \sum\limits_{n=1}^{\infty} a_{n,j} f(\lambda_n z)$, $1 \le j \le p$, where f is an entire transcendental function, the asymptotic behavior of a Hadamard composition $A(z) = (A_1 * ... * A_p)_m(z) = \sum\limits_{n=1}^{\infty} \left(\sum\limits_{k_1+\dots+k_p=m} c_{k_1\dots k_p} a_{n,1}^{k_1} \cdot \ldots \cdot a_{n,p}^{k_p} \right) f(\lambda_n z)$ of genus m is investigated. The function A_1 is called dominant, if $|c_{m0\dots0}||a_{n,1}|^m \ne 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for $2 \le j \le p$. The generalized order of a function A_j is called the quantity $\varrho_{\alpha,\beta}[A_j] = \varlimsup\limits_{r\to+\infty} \dfrac{\alpha(\ln \mathfrak{M}(r, A_j))}{\beta(\ln r)}$, where $\mathfrak{M}(r, A_j) = \sum\limits_{n=1}^{\infty} |a_{n,j}| M_f(r\lambda_n)$, $M_f(r) = \max\{|f(z)| : |z| = r\}$ and the functions α and β are positive, continuous and increasing to $+\infty$.

Under certain conditions on α, β, $M_f(r)$ and (λ_n), it is proved that if among the functions A_j there exists a dominant one, then $\varrho_{\alpha,\beta}[A] = \max\{\varrho_{\alpha,\beta}[A_j] : 1 \le j \le p\}$. In terms of generalized orders, a connection is established between the growth of the maximal terms of power expansions of the functions $(A_1^{(k)} * ... * A_p^{(k)})_m$ and $((A_1 * ... * A_p)_m)^{(k)}$. Unresolved problems are formulated .

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Introduction

Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1}$$

be an entire transcendental function, $M_f(r) = \max\{|f(z)| : |z| = r\}$ and (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

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in the system $f(\lambda_n z)$ regularly converges in $\mathbb{C}$, i. e. for all $r\in[0,+\infty)$

$$\mathfrak{M}(r,A):=\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty. \tag{3}$$

Many authors have studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We will specify here only on the monographs of A.F. Leont'ev [5] and B.V. Vynnytskyi [12], where references to other works can be find.

Since series (2) regularly converges in $\mathbb{C}$, the function A is entire. Generalized orders are used to study its growth. For this purpose, as in [7] by L we denote a class of continuous non-negative on $(-\infty,\,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,\,+\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si}\subset L^0$. For $\alpha\in L$ and $\beta\in L$ quantity $\varrho_{\alpha,\beta}[f]=\varlimsup\limits_{r\to+\infty}\dfrac{\alpha(\ln M_f(r))}{\beta(\ln r)}$ is called [7] generalized (α,β)-order of the entire function f.

In the papers [11, 10] the relationship between the growth of functions $M_f(r)$, $\mathfrak{M}(r,A)$ and $M_f^{-1}(\mathfrak{M}(r,A))$ was studied. When studying the logarithmic convexity of the function $\ln M_f(r)$ was used, from which it follows that

$$\Gamma_f(r):=\frac{d\ln M_f(r)}{d\ln r}\nearrow+\infty,\quad r\to+\infty,$$

(in points where the derivative does not exist, under $\dfrac{d\ln M_f(r)}{d\ln r}$ we mean right-hand derivative). For example, in [10] the following theorem was proved.

Theorem. *Let $\alpha\in Lsi$, $\beta\in L^0$, $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,\,+\infty)$ and*

$$\varrho_{\alpha,\beta}[A]:=\varrho_{\alpha,\beta}[\mathfrak{M}]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln\mathfrak{M}(r,A))}{\beta(\ln r)}.$$

If $a_n\ge 0$ for all $n\ge 1$, series (2) regularly converges in $\mathbb{C}$, $\ln n=O(\Gamma_f(\lambda_n))$ and $\ln\lambda_n= =o\left(\beta^{-1}(c\alpha\left(\dfrac{1}{\ln\lambda_n}\ln\dfrac{1}{a_n}\right)\right)$ as $n\to\infty$ each $c\in(0,\,+\infty)$ then $\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[f]$.

Let $G_j(z)=\sum\limits_{n=0}^{\infty}g_{n,j}z^n$, $1\le j\le p$, be an entire transcendental functions. As in [6], we say that the function $G(z)=\sum\limits_{n=0}^{\infty}g_nz^n$ is similar to the Hadamard composition of the functions g_j if $g_n=w(g_{n,1},...,g_{n,p})$ for all n, where $w:\mathbb{C}^p\to\mathbb{C}$ is a continuous function. Clearly, if $p=2$ and $w(g_{n,1},\,g_{n,2})=g_{n,1}g_{n,2}$ then $g=(g_1*g_2)$ is [4] the Hadamard composition (product) of the functions g_1 and g_2. Properties of this composition obtained by J.Hadamard find the applications [3, 1] in the theory of the analytic continuation of the functions represented by power series.

The article [9] considers the case when w is a homogeneous polynomial. Recall that a polynomial is named homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial $P(x_1,...,x_p)$ is homogeneous of the degree m if and only if $P(tx_1,...,tx_p) = t^m P(x_1,...,x_p)$ for all t from the field above that a polynomial is defined. Function (1) is called a Hadamard composition of genus $m \geq 1$ of functions (2) if $a_n = P(a_{n,1},...,a_{n,p})$, where

$$P(x_1,...,x_p) = \sum_{k_1+\cdots+k_p=m} c_{k_1...k_p} x_1^{k_1} \cdot ... \cdot x_p^{k_p}, \quad k_j \in \mathbb{Z}_+, \tag{4}$$

is a homogeneous polynomial of degree $m \geq 1$ with constant coefficients $c_{k_1...k_p}$. We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus $m = 2$. Hadamard composition of genus $m \geq 1$ of functions f_j is denoted by $(G_1 * ... * G_p)_m$, i. e. $(G_1 * ... * G_p)_m(z) = \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \left(\sum_{k_1+\cdots+k_p=m} c_{k_1...k_p} g_{n,1}^{k_1} \cdot ... \cdot g_{n,p}^{k_p} \right) z^n$.

Here we study the properties of Hadamard compositions of genus $m \geq 1$ for entire functions represented by series in a system of functions.

1 Definition and convergence

Function (2) is called a Hadamard composition of genus m of the functions

$$A_j(z) = \sum_{n=1}^{\infty} a_{n,j} f(\lambda_n z), \quad 1 \leq j \leq p, \tag{5}$$

if $a_n = P(a_{n,1},...,a_{n,p})$, where P is defined by (4). Then as above

$$A(z) = (A_1 * ... * A_p)_m(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) = \sum_{n=1}^{\infty} \left(\sum_{k_1+\cdots+k_p=m} c_{k_1...k_p} a_{n,1}^{k_1} \cdot ... \cdot a_{n,p}^{k_p} \right) f(\lambda_n z). \tag{6}$$

At first we prove the following lemma.

Lemma 1. *Let* $\ln n = o(\Gamma_f(\lambda_n))$ *as* $n \to \infty$. *Then series* (2) *regularly converges in* $\mathbb{C}$ *if and only if*

$$\varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) = +\infty. \tag{7}$$

Proof. If series (2) regularly converges in $\mathbb{C}$ then $|a_n| M_f(r\lambda_n) \to 0$ as $n \to \infty$, i. e. $|a_n| M_f(r\lambda_n) \leq 1$ for all $r \in [0, +\infty)$ and $n \geq n_0(r)$, whence $\frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) \geq r$ for all $n \geq n_0(r)$. In view of the arbitrariness of r we get (7).

On the other hand, if $r \in [1, +\infty)$ is an arbitrary number and (7) holds then for every $K > r$ and all $n \geq n_0 = n_0(K)$ we have $\frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) \geq K$, i. e. $|a_n| M_f(K\lambda_n) \leq 1$.

Therefore,

$$\sum_{n=n_0}^{\infty}|a_n|M_f(r\lambda_n)=\sum_{n=n_0}^{\infty}|a_n|M_f(K\lambda_n)\frac{M_f(r\lambda_n)}{M_f(K\lambda_n)}\le\sum_{n=n_0}^{\infty}\frac{M_f(r\lambda_n)}{M_f(K\lambda_n)}$$
$$=\sum_{n=n_0}^{\infty}\exp\left\{-\int\limits_{r\lambda_n}^{K\lambda_n}\frac{d\ln\, M_f(t)}{d\ln\, t}d\ln\, t\right\}=\sum_{n=n_0}^{\infty}\exp\left\{-\int\limits_{r\lambda_n}^{K\lambda_n}\Gamma_f(t)d\ln\, t\right\}$$
$$\le\sum_{n=n_0}^{\infty}\exp\{-\Gamma_f(r\lambda_n)\ln(K/r)\}\le\sum_{n=n_0}^{\infty}\exp\{-\Gamma_f(\lambda_n)\ln(K/r)\}<+\infty,$$

because $\ln\, n=o(\Gamma_f(\lambda_n))$ as $n\to\infty$. Lemma 1 is proved. □

We say that the function A_1 is dominant, if $|c_{m0\dots0}||a_{n,1}|^m\ne0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\le j\le p$. Using Lemma 1 we prove the following theorem.

Theorem 1. *Let* $\ln\, n=o(\Gamma_f(\lambda_n))$ *as* $n\to\infty$. *If all series* (5) *regularly converge in* $\mathbb{C}$ *then their Hadamard composition of genus* $m\ge1$ *regularly converges in* $\mathbb{C}$.

If the function A_1 *is dominant and Hadamard composition of genus* $m=1$ *of the functions* A_j *regularly converges in* $\mathbb{C}$ *then each series* (5) *regularly converges in* $\mathbb{C}$.

Proof. At first we remark that the condition $\dfrac{d\ln\, M_f(r)}{d\ln\, r}\nearrow+\infty$ as $r\to+\infty$ implies the condition $\dfrac{d\ln\, M_f^{-1}(x)}{d\ln\, x}\searrow+\infty$ as $x\to+\infty$, i. e. the function M_f^{-1} is slowly increasing.

Since series (2) is Hadamard composition of genus $m\ge1$ of regularly convergent in $\mathbb{C}$ series (5) then

$$|a_n|\le\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}||a_{n,1}|^{k_1}\cdot\ldots\cdot|a_{n,p}|^{k_p}\tag{8}$$

and by Lemma 1 $|a_{n,j}|\le1/M_f(K\lambda_n)$ for every $K>0$, all $1\le j\le p$ and all $n\ge n_0=n_0(K)$. Therefore,

$$|a_n|\le\sum_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}\left(\frac{1}{M_f(K\lambda_n)}\right)^{k_1}\cdot\ldots\cdot\left(\frac{1}{M_f(K\lambda_n)}\right)^{k_p}=C\left(\frac{1}{M_f(K\lambda_n)}\right)^m,$$

where $C=\sum\limits_{k_1+\dots+k_p=m}|c_{k_1\dots k_p}|$, and thus,

$$\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)\ge\frac{1}{\lambda_n}M_f^{-1}\left(\frac{M_f^m(K\lambda_n)}{C}\right)=\frac{1+o(1)}{\lambda_n}M_f^{-1}(M_f^m(K\lambda_n))$$
$$\ge\frac{1+o(1)}{\lambda_n}M_f^{-1}(M_f(K\lambda_n))=(1+o(1)K,\quad n\to\infty,$$

whence in view of the arbitrariness of K (7) follows, i. e. series (2) regularly converges in $\mathbb{C}$

Now we suppose the function A_1 is dominant and $m\ge1$. We put

$$\Sigma_n'=\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}(a_{n,1})^{k_1}\cdot\ldots\cdot(a_{n,p})^{k_p}$$
$$=\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}(a_{n,1})^{k_1}\cdot\ldots\cdot(a_{n,p})^{k_p}-c_{m0\dots0}(a_{n,1})^m.$$

Since for each monomial of the polynomial Σ'_n the sum of the exponents is equal to m, we have

$$\frac{|a_{n,1}|^{k_1}\cdot\ldots\cdot|a_{n,p}|^{k_p}}{|a_{n,1}|^m}=\frac{|a_{n,2}|^{k_2}\cdot\ldots\cdot|a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}}\to 0,\quad n\to\infty$$

and, thus $\Sigma'_n = o(|a_{n,1}|^m)$ as $n\to\infty$. Therefore,

$$|a_n|\geq|c_{m0\ldots0}||a_{n,1}|^m-|\Sigma'_n|=|c_{m0\ldots0}||a_{n,1}|^m-o(|a^m_{n,1|})\geq\frac{|c_{m0\ldots0}|}{2}|a_{n,1}|^m,\quad n\geq n^*_0$$

i. e. $|a_{n,1}|\leq c|a_n|$ for $n\geq n^*_0$ provided $m=1$, where $c=\text{const}>0$. Thus, we have

$$\begin{aligned}\varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_{n,j}|}\right)&\geq\varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_{n,1}|}\right)\geq\varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{c|a_n|}\right)\\&=\varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)=+\infty,\end{aligned}$$

i. e. all series (6) regularly converge in $\mathbb{C}$. Theorem 1 is proved. □

2 Maximal term and central index

Let series (2) regularly converges in $\mathbb{C}$, $\mu(r,A)=\max\{|a_n|M_f(r\lambda_n):n\geq1\}$ be the maximal term and $\nu(r,A)=\max\{n\geq1:|a_n|M_f(r\lambda_n)=\mu(r,A)\}$ be the central index of series (3). The following lemma is true.

Lemma 2. *The functions* $\ln\,\mu(r,A)$, $\lambda_{\nu(r,A)}$ *and* $\nu(r,A)$ *are non-decreasing and*

$$\ln\,\mu(r,A)-\ln\,\mu(r_0,A)=\int\limits_{r_0}^{r}\frac{\Gamma_f(t\lambda_{\nu(t,A)})}{t}dt,\quad 0\leq r_0\leq r<+\infty. \tag{9}$$

Proof. For $h>0$ we have

$$\begin{aligned}\mu(r+h,A)&=|a_{\nu(r+h,A)}|M_f((r+h)\lambda_{\nu(r+h,A)})\\&=|a_{\nu(r+h,A)}|M_f(r\lambda_{\nu(r+h,A)})\frac{M_f((r+h)\lambda_{\nu(r+h,A)})}{M_f(r\lambda_{\nu(r+h,A)})}\\&\leq\mu(r,A)\exp\{\ln\,M_f((r+h)\lambda_{\nu(r+h,A)})-\ln\,M_f(r\lambda_{\nu(r+h,A)})\}\\&=\mu(r,A)\exp\left\{\int\limits_{r\lambda_{\nu(r+h,A)}}^{(r+h)\lambda_{\nu(r+h,A)}}\Gamma_f(t)d\ln\,t\right\}\\&\leq\mu(r,A)\exp\left\{\Gamma_f((r+h)\lambda_{\nu(r+h,A)})\ln\,(1+h/r)\right\},\end{aligned}$$

i. e.

$$\ln\,\mu(r+h,A)-\ln\,\mu(r,A)\leq\Gamma_f((r+h)\lambda_{\nu(r+,A)})\ln\,(1+h/r). \tag{10}$$

Similarly,

$$\mu(r,A)=|a_{\nu(r,A)}|M_f((r+h)\lambda_{\nu(r,A)})\frac{M_f(r\lambda_{\nu(r,A)})}{M_f((r+h)\lambda_{\nu(r,A)})}$$
$$\le\mu(r+h,A)\exp\left\{-\int\limits_{r\lambda_{\nu(r,A)}}^{(r+h)\lambda_{\nu(r,A)}}\Gamma_f(t)d\ln t\right\}$$
$$\le\mu(r+h,A)\exp\left\{-\Gamma_f(r\lambda_{\nu(r,A)})\ln(1+h/r)\right\},$$

i. e.

$$\ln\mu(r+h,A)-\ln\mu(r,A)\ge\Gamma_f(r\lambda_{\nu(r,A)})\ln(1+h/r). \tag{11}$$

From (10) and (11) we obtain

$$\Gamma_f(r\lambda_{\nu(r,A)})\frac{\ln(1+h/r)}{h}\le\frac{\ln\mu(r+h,A)-\ln\mu(r,A)}{h}\le\Gamma_f((r+h)\lambda_{\nu(r+h,A)})\frac{\ln(1+h/r)}{h}.$$

Hence it follows that the functions $\ln\mu(r,A)$, $\lambda_{\nu(r,A)}$ and $\nu(r,A)$ are non-decreasing. Our reasoning is also correct if $h<0$. Therefore, if (r_1,r_2) is an interval of constancy of the function $\nu(r,A)$ then at $h\to 0$ we obtain

$$\frac{d\ln\mu(r,A)}{dr}=\frac{\Gamma_f(r\lambda_{\nu(r,A)})}{r},\quad r\in(r_1,r_2).$$

Since the function $\Gamma_f(r\lambda_{\nu(r,A)})$ has a finite number of discontinuities on each finite interval, we obtain the equality (9). □

3 GROWTH OF $\mathfrak{M}(r,A)$, $\mu(r,A)$ AND $\lambda_{\nu(r,A)}$

At first we prove the following theorem.

Theorem 2. *If $\alpha(\ln x)\in L_{si}$, $\beta(\ln x)\in L_{si}$ and $\overline{\lim\limits_{n\to\infty}}\dfrac{\ln n}{\Gamma_f(\lambda_n)}<+\infty$ then*

$$\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[\mu]:=\overline{\lim_{r\to+\infty}}\frac{\alpha(\ln\mu(r,A))}{\beta(\ln r)}.$$

If $\alpha(e^x)\in L_{si}$, $\beta(\ln x)\in L_{si}$ and $\alpha(x)=o(\beta(x))$ as $x\to+\infty$ then

$$\varrho_{\alpha,\beta}[\mu]=\overline{\lim_{r\to+\infty}}\frac{\alpha(\Gamma_f(r\lambda_{\nu(r,A)}))}{\beta(\ln r)}.$$

Proof. From (3) it follows that for $q>1$ and $r\ge 1$

$$\mu(r,A)\le\mathfrak{M}(r,A)\le\sum_{n=1}^{\infty}|a_n|M_f(qr\lambda_n)\frac{M_f(r\lambda_n)}{M_f(qr\lambda_n)}\le\mu(qr,A)\sum_{n=1}^{\infty}\exp\left\{-\int\limits_{r\lambda_n}^{qr\lambda_n}\Gamma_f(t)d\ln t\right\}$$
$$\le\mu(qr,A)\sum_{n=1}^{\infty}\exp\{-\Gamma_f(r\lambda_n)\ln q\}\le\mu(qr,A)\sum_{n=1}^{\infty}\exp\{-\Gamma_f(\lambda_n)\ln q\}$$

Since, there exist $K>0$ such that $\ln n\le K\Gamma_f(\lambda_n)$ for all $n\ge n_0(K)$ for $q=e^{K+1}$ we have $\Gamma_f(\lambda_n)\ln q\ge\dfrac{K+1}{K}\ln n$ and, thus, $\sum\limits_{n=1}^{\infty}\exp\{-\Gamma_f(\lambda_n)\ln q\}\le C=\text{const}<+\infty$. Therefore, $\ln\,\mu(r,A)\le\ln\,\mathfrak{M}(r,A)\le\ln\,\mu(qr,A)+\ln\,C$ and in view of conditions $\alpha(\ln\,x)\in L_{si}$ and $\beta(\ln\,x)\in L_{si}$ we get the equality $\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[\mu]$. The first part of Theorem 2 is proved.

Equality (9) implies

$$\ln\,\mu(er,A)=\ln\,\mu(r)+\int\limits_{r}^{er}\frac{\Gamma_f(t\lambda_{\nu(t,A)})}{t}dt\ge\Gamma_f(r\lambda_{\nu(r,A)})$$

and

$$\ln\,\mu(r,A)=\ln\,\mu(1,A)+\int\limits_{1}^{r}\frac{\Gamma_f(t\lambda_{\nu(t,A)})}{t}dt\le\ln\,\mu(1,A)+\Gamma_f(r\lambda_{\nu(r,A)})\ln\,r.$$

Therefore,

$$\frac{\ln\,\mu(r,A)}{\ln\,r}+o(1)\le\Gamma_f(r\lambda_{\nu(r,A)})\le\ln\,\mu(er,A),\quad r\to+\infty. \tag{12}$$

Since $\beta(\ln\,x)\in L_{si}$, hence we obtain

$$\varlimsup_{r\to+\infty}\frac{\alpha(\Gamma_f(r\lambda_{\nu(r,A)}))}{\beta(\ln\,r)}\le\varlimsup_{r\to+\infty}\frac{\alpha(\ln\,\mu(er,A))}{\beta(\ln\,r)}=\varrho_{\alpha,\beta}[\mu].$$

On the other hand, in view of the conditions $\alpha(e^x)\in L_{si}$ and $\alpha(x)=o(\beta(x))$ as $x\to+\infty$ we have

$$\begin{aligned}(1+o(1))\alpha(\ln\,\mu(r,A))&\le\alpha(\Gamma_f(r\lambda_{\nu(r,A)})\ln\,r)=\alpha(\exp\{\ln\,\Gamma_f(r\lambda_{\nu(r,A)})+\ln\,\ln\,r\})\\&\le\alpha(\exp\{2\max\{\ln\,\Gamma_f(r\lambda_{\nu(r,A)}),\,\ln\,\ln\,r\}\})\\&=(1+o(1))\alpha(\exp\{\max\{\ln\,\Gamma_f(r\lambda_{\nu(r,A)}),\,\ln\,\ln\,r\}\})\\=(1+o(1))\max\{\alpha(\Gamma_f(r\lambda_{\nu(r,A)}),\alpha(\ln\,r)\}&\le(1+o(1))(\alpha(\Gamma_f(r\lambda_{\nu(r,A)})+\alpha(\ln\,r))\\&=(1+o(1))(\alpha(\Gamma_f(r\lambda_{\nu(r,A)})+o(\beta(\ln\,r))),\quad r\to+\infty,\end{aligned}$$

i. e. $\varrho_{\alpha,\beta}[\mu]\le\varlimsup\limits_{r\to+\infty}\dfrac{\alpha(\Gamma_f(r\lambda_{\nu(r,A)}))}{\beta(\ln\,r)}$. The proof of Theorem 2 is complete. □

Remark that if all $f_k\ge 0$ then $M_f(r)=f(r)$. In this case the following theorem is true.

Theorem 3. *Let $\alpha\in L_{si}$, $\beta(\ln\,x)\in L_{si}$ and for all $r\ge r_0$*

$$\ln\,m<h\le\frac{d\ln\,\ln\,f(r)}{d\ln\,r}\le H<+\infty. \tag{13}$$

If series (5) regularly converge in $\mathbb{C}$ and $A(z)=(A_1...*A_p)_m(z)$ then $\varrho_{\alpha,\beta}[A]\le$ $\le\max\{\varrho_{\alpha,\beta}[A_j]:\,1\le j\le p\}$.*

Moreover, if the function A_1 is dominant then $\varrho_{\alpha,\beta}[A_j]\le\varrho_{\alpha,\beta}[A_1]=\varrho_{\alpha,\beta}[A]$.

Proof. From (13) it follows that for $\eta = \dfrac{\ln m}{H} + 1$ and for some $\xi \in (r/\eta, r)$

$$\ln\ln M_f(r) - \ln\ln M_f(r/\eta) = \ln\ln f(r) - \ln\ln f(r/\eta) = (\ln\ln f(\xi))'(1-1/\eta)r$$
$$\le (\ln\ln f(\xi))'(1-1/\eta)\eta\xi) = (\eta-1)\frac{d\ln\ln f(\xi)}{d\ln\xi} \le (\eta-1)H \le \ln m,$$

i. e. $M_f(r) \le M_f(r/\eta)^m$. Therefore, (8) implies

$$|a_n|M_f(r\lambda_n) \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|(|a_{n,1}|M_f(r\lambda_n/\eta))^{k_1}\cdot\ldots\cdot(|a_{n,p}|M_f(r\lambda_n/\eta))^{k_p},$$

i. e.

$$\mu(r,A) \le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|\mu(r/\eta,A_1)^{k_1}\cdot\ldots\cdot\mu(r/\eta,A_p)^{k_p}.$$

Since $\ln^+(c_1+\dots+c_n) \le \ln^+ c_1 + \dots + \ln^+ c_n + \ln n$, hence for all r enough large we get

$$\ln\mu(r,A) \le \sum_{k_1+\dots+k_p=m} \ln(|c_{k_1\dots k_p}|\mu(r/\eta,A_1)^{k_1}\cdot\ldots\cdot\mu(r/\eta,A_p)^{k_p}) + C_1$$
$$\le \sum_{k_1+\dots+k_p=m} (k_1\ln\mu(r/\eta,A_1)+\ldots+k_p\ln\mu(r/\eta,A_p)) + C_2,$$

where C_1 and C_2 are const > 0. But $\ln\mu(r,A_j) \le \alpha^{-1}(\varrho_{\alpha,\beta}[A_j]+\varepsilon)\beta(\ln r)) \le$ $\le \alpha^{-1}(\varrho+\varepsilon)\beta(\ln r))$ for all $\varepsilon > 0$ and all $r \ge r_0(\varepsilon))$, where $\varrho = \max\{\varrho_{\alpha,\beta}[A_j] : 1 \le j \le p\}$. Therefore, $\ln\mu(r,A) \le m\alpha^{-1}(\varrho+\varepsilon)\beta(\ln r-1)+C_2$, whence in view of the conditions $\alpha \in L_{si}$, $\beta(\ln x) \in L_{si}$ and of the arbitrariness of ε we obtain $\varrho_{\alpha,\beta}[A] \le \varrho$. The first part of Theorem 3 is proved.

If the function A_1 is dominant then, as above, we have $|a_n| \ge c|a_{n,1}|^m$ for $n \ge n_0^*$, where $c = \text{const} > 0$. From (13) it follows that for $\zeta = 1 - \dfrac{\ln m}{h}$ and for some $\xi \in (\zeta r, r)$, as above, we have

$$\ln\ln M_f(r) - \ln\ln M_f(\zeta r) = (\ln\ln f(\xi))'(1-\zeta)r \ge (1-\zeta)\frac{d\ln\ln f(\xi)}{d\ln\xi} \ge (1-\zeta)h \ge \ln m,$$

i. e. $M_f(r) \ge M_f(r\zeta)^m$. Therefore, $\mu(r,A) \ge c\mu(r\zeta,A)^m$, whence the inequality $\varrho_{\alpha,\beta}[A_j] \le$ $\le \varrho_{\alpha,\beta}[A_1] \le \varrho_{\alpha,\beta}[A]$ follows and $\varrho_{\alpha,\beta}[A] = \varrho_{\alpha,\beta}[A_1]$. The proof of Theorem 3 is complete. □

4 Growth of the derivative

Clearly, $A^{(k)}(z) = \sum\limits_{n=1}^{\infty} a_n^k\lambda_n f A^{(k)}(\lambda_n z)$ and $\mathfrak{M}(r,A^{(k)}) = \sum\limits_{n=1}^{\infty} |a_n|\lambda_n^k M_{f^{(k)}}(r\lambda_n)$. The following statement indicates that for each $k \ge 1$ the functions $A^{(k)}$ and A have the same growth.

Theorem 4. *If $\alpha \in L_{si}$, $\beta(\ln x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \to +\infty$ then $\varrho_{\alpha,\beta}[A^{(k)}] =$ $= \varrho_{\alpha,\beta}[A]$.*

Proof. It suffices to prove Theorem 3 for $k = 1$. From Cauchy formula $f'(z) = = \frac{1}{2\pi i}\int\limits_{|\tau-z|=\lambda_n}\frac{f(\tau)}{(\tau-z)^2}d\tau$ we obtain $M_{f'}(r)\le\frac{M_f(r+\lambda_n)}{\lambda_n}$ for all r and n. Therefore, choosing $r\lambda_n$ instead r we get

$$\mathfrak{M}(r,A')\le\sum_{n=1}^{\infty}|a_n|\lambda_n\frac{M_f(r\lambda_n+\lambda_n)}{\lambda_n}=\sum_{n=1}^{\infty}|a_n|M_f((r+1)\lambda_n)=\mathfrak{M}(r+1,A),$$

whence the inequality $\varrho_{\alpha,\beta}[A']\le\varrho_{\alpha,\beta}[A]$ follows.

On the other hand, $f(z)-f(0)=\int\limits_0^z f'(z)dz$, whence $M_f(r)\le|f(0)|+rM_{f'}(r)$ and, thus,

$$\mathfrak{M}(r,A)\le\sum_{n=1}^{\infty}|a_n|(|f(0)|+r\lambda_nM_{f'}(r\lambda_n))=(1+o(1))r\sum_{n=1}^{\infty}|a_n|\lambda_nM_{f'}(r\lambda_n)$$
$$=(1+o(1))r\mathfrak{M}(r,A'),\quad r\to+\infty,$$

whence, as above,

$$\alpha(\ln\mathfrak{M}(r,A))\le\alpha(\ln\mathfrak{M}(r,A))+\ln r+o(1))\le(1+o(1))\alpha(2\max\{\ln\mathfrak{M}(r,A),\ln r\})$$
$$=(1+o(1))\alpha(\max\{\ln\mathfrak{M}(r,A),\ln r\})\le(1+o(1))(\alpha(\mathfrak{M}(r,A)+\alpha(\ln r)),\quad r\to+\infty,$$

i. e. $\varrho_{\alpha,\beta}[A]\le\varrho_{\alpha,\beta}[A']$. Theorem 4 is proved. □

5 Growth of the ratio $\frac{\mu(r,(A_1^{(k)}*\ldots*A_p^{(k)})_m)}{\mu(r,((A_1*\ldots*A_p)_m)^{(k)})}$

Let $j\ge k$. Since

$$(A_1^{(j)}*\ldots*A_p^{(j)})_m(z)=\left(\sum_{k_1+\dots+k_p=m}c_{k_1\ldots k_p}a_{n,1}^{k_1}\cdot\ldots\cdot a_{n,p}^{k_p}\right)\lambda_n^{mj}f^{(j)}(\lambda_nz)$$

and

$$((A_1*\ldots*A_p)_m)^{(j)}(k)=\sum_{n=1}^{\infty}\left(\sum_{k_1+\dots+k_p=m}c_{k_1\ldots k_p}a_{n,1}^{k_1}\cdot\ldots\cdot a_{n,p}^{k_p}\right)\lambda_n^kf^{(k)}(\lambda_nz),$$

we have (with $a_n\sum\limits_{k_1+\dots+k_p=m}c_{k_1\ldots k_p}a_{n,1}^{k_1}\cdot\ldots\cdot a_{n,p}^{k_p}$)

$$\mu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)=|a_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}|\lambda^{mj}_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}M_{f^{(j)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})$$
$$=|a_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}|\lambda^{k}_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}M_{f^{(k)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})\times$$
$$\times\lambda^{mj-k}_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}\frac{M_{f^{(j)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})}{M_{f^{(k)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})}$$
$$\le\mu(r,((A_1*\ldots*A_p)_m)^{(k)})\lambda^{mj-k}_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}\frac{M_{f^{(j)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})}{M_{f^{(k)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})}.$$

Similarly,

$$\begin{aligned}\mu(r,((A_1*\ldots*A_p)_m)^{(k)}) =&|a_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})}|\lambda^k_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}M_{f^{(k)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})\\ =& |a_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})}|\lambda^{mj}_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}M_{f^{(j)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})\times\\ &\times\lambda^{k-mj}_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}\frac{M_{f^{(k)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})}{M_{f^{(j)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})}\\ \le& \mu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)\lambda^{k-mj}_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}\frac{M_{f^{(k)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})}{M_{f^{(j)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})}.\end{aligned}$$

Therefore, the following lemma is proved.

Lemma 3. *If functions* (5) *regularly converge in* $\mathbb{C}$ *and* $j\ge k$ *then*

$$\begin{aligned}\lambda^{mj-k}_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}\frac{M_{f^{(j)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})}{M_{f^{(k)}}(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)})})}&\le\frac{\mu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}{\mu(r,((A_1*\ldots*A_p)_m)^{(k)})}\\ &\le\lambda^{mj-k}_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)}\frac{M_{f^{(j)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})}{M_{f^{(k)}}(r\lambda_{\nu(r,(A_1^{(j)}*\ldots*A_p^{(j)})_m)})}.\end{aligned}\tag{14}$$

If we put

$$\mathfrak{G}(r,A,k):=\sqrt[(m-1)k]{\frac{r^{(m-1)k}\mu(r,(A_1^{(k)}*\ldots*A_p^{(k)})_m)}{\mu(r,((A_1*\ldots*A_p)_m)^{(k)})}}$$

then the following theorem is true.

Theorem 5. *Let* $\alpha(e^x)\in L_{si}$, $\beta(\ln x)\in L_{si}$, $\alpha(x)=o(\beta(x))$ *as* $x\to+\infty$ *and* $\overline{\lim\limits_{n\to\infty}}\dfrac{\ln n}{\Gamma_f(\lambda_n)}<+\infty$. *Suppose that all* $f_k\ge 0$ *and* $\ln m<h\le\dfrac{d\ln\ln f(r)}{d\ln r}\le H<+\infty$ *for all* $r\ge r_0$. *If series* (5) *regularly converge in* $\mathbb{C}$, $A(z)=(A_1*\ldots*A_p)_m(z)$ *and the function* A_1 *is dominant then*

$$\overline{\lim_{r\to+\infty}}\frac{\alpha(\Gamma_f(\mathfrak{G}(r,A,k)))}{\beta(\ln r)}=\varrho_{\alpha,\beta}[A].\tag{15}$$

Proof. If $j=k$ from (14) we get $r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}\le\mathfrak{G}(r,A,k)\le r\lambda_{\nu(r,(A_1^{(k)}*\ldots*A_p^{(k)})_m)}$, whence

$$\Gamma_f\left(r\lambda_{\nu(r,((A_1*\ldots*A_p)_m)^{(k)}))}\right)\le\Gamma_f(\mathfrak{G}(r,A,k))\le\Gamma_f\left(r\lambda_{\nu(r,(A_1^{(k)}*\ldots*A_p^{(k)})_m)}\right)$$

and, thus, by Theorem 2

$$\varrho_{\alpha,\beta}[\mu(\cdot,((A_1*\ldots*A_p)_m)^{(k)}))]\le\overline{\lim_{r\to+\infty}}\frac{\alpha(\Gamma_f(\mathfrak{G}(r,A,k)))}{\beta(\ln r)}\le\varrho_{\alpha,\beta}[\mu(\cdot,(A_1^{(k)}*\ldots*A_p^{(k)})_m)].\tag{16}$$

By Theorems 2 and 4

$$\varrho_{\alpha,\beta}[\mu(\cdot,((A_1*\ldots*A_p)_m)^{(k)}))]=\varrho_{\alpha,\beta}[((A_1*\ldots*A_p)_m)^{(k)})]=\varrho_{\alpha,\beta}[(A_1*\ldots*A_p)_m]=\varrho_{\alpha,\beta}[A]$$

and by Theorems 2, 3 and 4

$$\begin{aligned}\varrho_{\alpha,\beta}[\mu(\cdot,(A_1^{(k)}*\ldots*A_p^{(k)})_m)]&=\varrho_{\alpha,\beta}[(A_1^{(k)}*\ldots*A_p^{(k)})_m]\le\max\{\varrho_{\alpha,\beta}[A_j^{(k)}]:\,1\le j\le p\}\\ &=\max\{\varrho_{\alpha,\beta}[A_j]:\,1\le j\le p\}=\varrho_{\alpha,\beta}[A_1]=\varrho_{\alpha,\beta}[A].\end{aligned}$$

Therefore, (16) implies (15). Theorem 5 is proved. □

As an example, consider series in the system of Mittag-Leffler functions. Let $0 < \varrho < +\infty$ and $E_\varrho(z) = \sum\limits_{k=0}^{\infty} \dfrac{z^k}{\Gamma(1+k/\varrho)}$ be the Mittag-Leffler function. Denote $A_\varrho(z) = \sum\limits_{n=1}^{\infty} a_n E_\varrho(\lambda_n z)$ and $A_{\varrho,j}(z) = \sum\limits_{n=1}^{\infty} a_{n,j} E_\varrho(\lambda_n z)$, $1 \le j \le p$. It is well known [2, p. 115] that $M_{E_\varrho}(r) = E_\varrho(r) = (1+o(1))\varrho e^{r^\varrho}$ as $r \to +\infty$ and this equality can be differentiated. Therefore, $\Gamma_{E_\varrho}(r) = (1+o(1))\varrho r^\varrho$ and $\dfrac{d \ln \ln E_\varrho(r)}{d \ln r} = (1+o(1))\varrho$ as $r \to +\infty$. Therefore, for $\alpha(x) = \ln^+ \ln x$ and $\beta(x) = x^+$ we obtain the following statement.

Corollary 1. *Let* $\ln n = O(\lambda_n^\varrho)$ *as* $n \to \infty$ *and the series* $\sum\limits_{n=1}^{\infty} a_{n,j} E_\varrho(\lambda_n z)$ *regularly converge in* $\mathbb{C}$. *If* $\ln m < \varrho$, $A_\varrho(z) = (A_{\varrho,1} * ... * A_{\varrho,p})_m(z)$ *and the function* $A_{\varrho,1}$ *is dominant then*

$$\varlimsup_{r\to+\infty} \frac{1}{\ln r} \ln \ln \frac{\mu(r,(A_{\varrho,1}^{(k)} * ... * A_{\varrho,p}^{(k)})_m)}{\mu(r,((A_{\varrho,1} * ... * A_{\varrho,p})_m)^{(k)})} = \varlimsup_{r\to+\infty} \frac{\ln \ln \ln \mathfrak{M}(r, A_\varrho)}{\ln r}$$

We remark that the condition $\alpha(e^x) \in L_{si}$ used in the proof of equality $\varrho_{\alpha,\beta}[\mu] = \varlimsup\limits_{r\to+\infty} \dfrac{\alpha(\Gamma_f(r\lambda_{\nu(r,A)})}{\beta(\ln r)}$. But in the case when $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ from (12) we get $\ln \ln \mu(r,A) \le \ln(\Gamma_f(r\lambda_{\nu(r,A)}) + o(1)) + \ln \ln r$ and, thus,

$$\varlimsup_{r\to+\infty} \frac{\ln \ln \mu(r,A)}{\ln r} = \varlimsup_{r\to+\infty} \frac{\ln \Gamma_f(r\lambda_{\nu(r,A)})}{\ln r}.$$

As a result, we arrive at the following statement.

Proposition 1. *Let* $\ln n = O(\lambda_n^\varrho)$ *as* $n \to \infty$ *and the series* $\sum\limits_{n=1}^{\infty} a_{n,j} E_\varrho(\lambda_n z)$ *regularly converge in* $\mathbb{C}$. *If* $\ln m < \varrho$, $A_\varrho(z) = (A_{\varrho,1} * ... * A_{\varrho,p})_m(z)$ *and the function* $A_{\varrho,1}$ *is dominant then*

$$\varrho \varlimsup_{r\to+\infty} \frac{\ln \mathfrak{G}(r, A_\varrho, k)}{\ln r} = \varlimsup_{r\to+\infty} \frac{\ln \ln \mathfrak{M}(r, A_\varrho)}{\ln r}.$$

Put $\sigma = r^\varrho$, $l_n = \lambda_n^\varrho$ and $D(\sigma) = \sum\limits_{n=1}^{\infty} |a_n| e^{l_n \sigma}$. Then $\mu(r, A_\varrho) = (1+o(1))\mu(\sigma, D)$ as $\sigma \to +\infty$, where $\mu(\sigma, D)$ is maximal term of entire Dirichlet series D. It is known [8, p. 26] that if $\ln n = O(l_n)$ as $n \to \infty$, $\alpha_1 \in L_{si}$, $\beta_1(\ln x) \in L_{si}$ and $\dfrac{d\beta_1^{-1}(c\alpha_1(x))}{d \ln x} = O(1)$ as $n \to \infty$ for each $c \in (0, +\infty)$ then

$$\varlimsup_{\sigma\to+\infty} \frac{\alpha_1(\ln D(\sigma))}{\beta_1(\sigma)} = \varlimsup_{\sigma\to+\infty} \frac{\alpha_1(\ln \mu(\sigma, D))}{\beta_1(\sigma)} = \varlimsup_{n\to\infty} \frac{\alpha_1(l_n)}{\beta_1\left(\dfrac{1}{l_n} \ln \dfrac{1}{|a_n|}\right)}.$$

If we put $\alpha_1(x) = \alpha(x)$, $\beta_1(x) = \beta((\ln x)/\varrho)$ then here, under conditions $\alpha(e^x) \in L_{si}$, $\beta(\ln x) \in L_{si}$ and $\dfrac{d \exp\{\varrho\beta^{-1}(c\alpha_1(x)\}}{d \ln x} = O(1)$ as $n \to \infty$ for each $c \in (0, +\infty)$, we get

$$\varrho_{\alpha,\beta}[A_\varrho] = \varlimsup_{r\to+\infty} \frac{\ln \mathfrak{M}(r, A_\varrho)}{\beta(\ln r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\ln D(\sigma))}{\beta((\ln \sigma)\varrho)}$$

$$= \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n^\varrho)}{\beta\left(\dfrac{1}{\varrho} \ln \left(\dfrac{1}{\lambda_n^\varrho} \ln \dfrac{1}{|a_n|}\right)\right)} = \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\dfrac{1}{\varrho} \ln \left(\dfrac{1}{\lambda_n^\varrho} \ln \dfrac{1}{|a_n|}\right)\right)}, \qquad (17)$$

whence for $\alpha(x) = (\ln^+ \ln x)^q$, $0 < q < 1$, and $\beta(x) = x^+$ we obtain

$$\overline{\lim_{r\to+\infty}} \frac{\ln^q \ln \ln \mathfrak{M}(r, A_\varrho)}{\ln r} = \overline{\lim_{n\to\infty}} \frac{\varrho \ln^q \ln \lambda_n}{\ln\left(\dfrac{1}{\lambda_n^\varrho} \ln \dfrac{1}{|a_n|}\right)},$$

6 Discussion Open Problems

We were unable to solve the following actual problems.

1. Using inequalities (14) prove an analogue of Theorem 5 for $j > k$.

2. In the general case, find a formula for finding the generalized order $\varrho_{\alpha,\beta}[A] = \overline{\lim_{r\to+\infty}} \dfrac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(\ln r)}$ in terms of coefficients and prove an analogue of equality (17).

3. Is it possible to establish a connection between the growth of functions A_j and function A if there is no dominant function among the functions A_j?

References

[1] Bieberbach L. *Analytische Fortzetzung.* Springer-Verlag, Berlin, 1955. doi:10.1002/zamm 19550350918.

[2] Gol'dberg A.A., Ostrovskyi I.V. Distribution of values of meromorphic functions. Nauka, Moscow, 1976. (in Russian)

[3] Hadamard J. *La serie de Taylor et son prolongement analitique.* Scientia phys.- math. 1901, **12**, 43-62.

[4] Hadamard J. *Theoreme sur le series entieres.* Acta math. 1899, **22**, 55-63.

[5] Leont'ev A.F. Generalizations of exponential series. Nauka, Moscow, 1981. (in Russian)

[6] Mulyava O.M., Sheremeta M.M. *Compositions of Dirichlet series similar to the Hadamard compositions, and convergence classes.* Mat. Stud. 2019. **51** (1), 25-34. doi:10.15330/ms.51.1.25-34.

[7] Sheremeta M.N. *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion.* Izv. Vyssh. Uchebn. Zaved. Mat. 1967, no. 2, 100–108. (in Russian)

[8] Sheremeta M.M. Entire Dirichlet series. ICDO, Kyiv, 1993. (in Ukrainian).

[9] Sheremeta M.M. *Hadamard compositions of Gelfond-Leont'ev derivatives.* Axioms. 2022. **11**, 478. doi.org/10.3390/axioms1090478.

[10] Sheremeta M.M. *On the growth of series in systems of functions and Laplace-Stieltjes integrals.* Mat. Stud. 2021. **55**(2), 124-131.

[11] Sheremeta M.M. *Relative growth of series in system functions and Laplace-Stieltjes type integrals.* Axioms. 2021. - **10**, 43.

[12] Vynnytskyi B.V. Some approximation properties of generalized systems of exponentials. Drohobych, 1991. Dep. in UkrNIINTI 25.02.1991. (in Russian)

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Для регулярно збіжних в $\mathbb{C}$ рядів $A_j(z) = \sum\limits_{n=1}^{\infty} a_{n,j} f(\lambda_n z)$, $1 \le j \le p$, де f - ціла трансцендентна функція, досліджується асимптотичне поводження адамарової композиції $A(z) = (A_1 * ... * A_p)_m(z) = \sum\limits_{n=1}^{\infty} \left(\sum\limits_{k_1+\dots+k_p=m} c_{k_1\dots k_p} a_{n,1}^{k_1} \cdot ... \cdot a_{n,p}^{k_p} \right) f(\lambda_n z)$ роду m. Функція A_1 називається домінантною, якщо $|c_{m0\dots0}||a_{n,1}|^m \ne 0$ і $|a_{n,j}| = o(|a_{n,1}|)$ при $n \to \infty$ для $2 \le j \le p$. Узагальненим порядком функції A_j називається величина $\varrho_{\alpha,\beta}[A_j] = \varlimsup\limits_{r\to+\infty} \dfrac{\alpha(\ln \mathfrak{M}(r, A_j))}{\beta(\ln r)}$, де $\mathfrak{M}(r, A_j) = \sum\limits_{n=1}^{\infty} |a_{n,j}| M_f(r\lambda_n)$, $M_f(r) = \max\{|f(z)| : |z| = r\}$, а функції α і β є додатні, неперервні і зростаючі до $+\infty$.

За певних умов на α, β, $M_f(r)$ і (λ_n) доведено, що якщо серед функцій A_j існує домінанта, то $\varrho_{\alpha,\beta}[A] = \max\{\varrho_{\alpha,\beta}[A_j] : 1 \le j \le p\}$. У термінах узагальнених порядків встановлено зв'язок між ростом максимальних членів функцій $(A_1^{(k)} * ... * A_p^{(k)})_m$ і $((A_1 * ... * A_p)_m)^{(k)}$. Сформульовано нерозв'язані проблеми.