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STOKES SYSTEM WITH VARIABLE EXPONENTS OF NONLINEARITY

Some nonlinear Stokes system is considered. The initial-boundary value problem for the system is investigated and the existence and uniqueness of the weak solution for the problem is proved.

Key words and phrases: evolution Stokes system, initial-boundary value problem, weak solution.

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Introduction

Let $n \in \mathbb{N}$ and $T > 0$ be fixed numbers, $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary $\partial\Omega$, $Q_{0,T} := \Omega \times (0,T)$, $\Sigma_{0,T} := \partial\Omega \times (0,T)$, $\Omega_\tau := \{(x,t) \mid x \in \Omega, t = \tau\}$, $\tau \in [0,T]$. We seek a weak solution $\{u, \pi\}$ of the problem

$$u_t - \sum_{i,j=1}^{n} \Big(A_{ij}(x,t) u_{x_i} \Big)_{x_j} + G(x,t)|u|^{q(x,t)-2}u + \nabla\pi(x,t) = F(x,t), \qquad (x,t) \in Q_{0,T}, \quad (1)$$

$$\operatorname{div} u = 0, \qquad (x,t) \in Q_{0,T}, \quad (2)$$

$$\int\limits_{\Omega} \pi(x,t)\, dx = 0, \quad t \in (0,T), \quad (3)$$

$$u|_{\Sigma_{0,T}} = 0, \quad (4)$$

$$u|_{t=0} = u_0(x), \quad x \in \Omega. \quad (5)$$

Here $u = (u_1, \ldots, u_n) : Q_{0,T} \to \mathbb{R}^n$ is the velocity field, $|u| = (|u_1|^2 + \ldots + |u_n|^2)^{1/2}$, $\operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \ldots + \frac{\partial u_n}{\partial x_n}$, $\pi : Q_{0,T} \to \mathbb{R}$ is the pressure, $\nabla\pi = (\frac{\partial \pi}{\partial x_1}, \ldots, \frac{\partial \pi}{\partial x_n})$, and $q(x,t)$ is the variable exponent of the nonlinearity of system (1).

The linearized version of the Navier-Stokes system is called the Stokes system. It is well known that these equations describe the time evolution of the solutions to the mathematical

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models of the viscous incompressible fluids. For more details about the physical meaning of the Navier-Stokes and Stokes systems see [1], [2], etc. The initial-boundary value problem for the Stokes system are considered in [3], [4], [5], [6], [7], [8], [9], [10] (see also the references given there).

We perturb the classical Stokes equations by the monotonous nonlinear term with the exponent of the nonlinearity $q = q(x,t)$. This exponent is Lipschitz continuous function only with respect to the time variable t. We seek a weak solution to the initial-boundary value problem (1)-(5). As we know this problem is not studied yet. The paper is organized as follows. In Section 1, we formulate the considered problem and main results. The auxiliary statements are given in Section 2. Finally, in Section 3 we prove the main results.

1 STATEMENT OF PROBLEM AND FORMULATION OF MAIN RESULTS

Let $||\cdot||_B \equiv ||\cdot;B||$ be a norm of some Banach space B, B^* be a dual space, $\langle\cdot,\cdot\rangle_B$ be a scalar product between B^* and B, $B^n := B\times\ldots\times B$ be n-th Cartesian product of the B, $||z;B^n|| := ||z_1||_B+\ldots+||z_n||_B$ for $z=(z_1,\ldots,z_n)\in B^n$, $(\cdot,\cdot)_H$ be a scalar product in some Hilbert space H, $|\cdot|_H := \sqrt{(\cdot,\cdot)_H}$.

Suppose that $N\in\mathbb{N}$, $\mathcal{O}$ is a measurable set in $\mathbb{R}^N$ (for example, $\mathcal{O}=\Omega$ or $\mathcal{O}=Q_{0,T}$), $\mathcal{B}_+(\mathcal{O}) := \{\mathfrak{q}\in L^\infty(\mathcal{O}) \mid \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y)>0\}$. For every $\mathfrak{q}\in\mathcal{B}_+(\mathcal{O})$, by definition, put

$$\mathfrak{q}_0 := \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y),\quad \mathfrak{q}^0 := \operatorname*{ess\,sup}_{y\in\mathcal{O}} \mathfrak{q}(y),\qquad \mathfrak{q}'(y) := \frac{\mathfrak{q}(y)}{\mathfrak{q}(y)-1}\ \text{ for a.e. } y\in\mathcal{O}\quad ,$$

$$\rho_{\mathfrak{q}}(v;\mathcal{O}) := \int\limits_{\mathcal{O}} |v(y)|^{\mathfrak{q}(y)}\,dy,\quad v:\mathcal{O}\to\mathbb{R}.$$

Assume that $\mathfrak{q}\in\mathcal{B}_+(\mathcal{O})$ and $\mathfrak{q}_0>1$. The set $L^{\mathfrak{q}(y)}(\mathcal{O}) := \{v:\mathcal{O}\to\mathbb{R} \mid \rho_{\mathfrak{q}}(v;\mathcal{O})<+\infty\}$ with the Luxemburg norm $||v;L^{\mathfrak{q}(y)}(\mathcal{O})|| := \inf\{\lambda>0 \mid \rho_{\mathfrak{q}}(v/\lambda;\mathcal{O})\le 1\}$ is called a generalized Lebesgue space. It is well known that $L^{\mathfrak{q}(y)}(\mathcal{O})$ is the Banach space which is reflexive and separable.

Let $\Lambda_t(Q_{0,T})$ be a set of the functions $q: Q_{0,T}\to\mathbb{R}$ for which there exists an extension outside $Q_{0,T}$ (we denote it q again) such that the following conditions are satisfied:
(i) $q\in C(\mathbb{R}_t;L^\infty(\mathbb{R}^n_x))\cap\mathcal{B}_+(\mathbb{R}^{n+1}_{x,t})$; (ii) $q_0>1$; (iii) there exists a constant $L>0$ such that

$$|q(x,t)-q(x,s)|\le L|t-s|,\quad x\in\mathbb{R}^n,\quad t,s\in\mathbb{R}.$$

For the sake of convenience we shall write $u(t)$ instead of $u(\cdot,t)$ and $L^p(0,T)$ instead of $L^p((0,T))$ etc. Let us consider the set of the solenoidal functions (functions for which the incompressibility constraint $\operatorname{div} u=0$ holds) $C_{\rm div} := \{u\in[D(\Omega)]^n \mid \operatorname{div} u=0\}$. Here $u=(u_1,\ldots,u_n)$ and $\operatorname{div} u := \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\ldots+\frac{\partial u_n}{\partial x_n}$. Let $r\in[1,+\infty)$, $s\in\mathbb{N}$,

$$X_r \ \text{ is a closure of } \ C_{\rm div} \ \text{ in } \ [L^r(\Omega)]^n,\qquad H := X_2, \tag{6}$$

$$Z_s \ \text{ is a closure of } \ C_{\rm div} \ \text{ in } \ [H^s_0(\Omega)]^n. \tag{7}$$

Take a function $q \in \Lambda_t(Q_{0,T})$ and denote

$$V^t := Z_1 \cap [L^{q(x,t)}(\Omega)]^n \quad \text{for every} \quad t \in [0,T], \tag{8}$$

$$U(Q_{0,T}) := L^2(0,T;Z_1) \cap [L^{q(x,t)}(Q_{0,T})]^n, \tag{9}$$

$$\mathcal{D}_{\mathrm{div}} := \{u \in [D(Q_{0,T})]^n \quad | \quad \operatorname{div} u = 0\}. \tag{10}$$

Since Z_1 and $[L^{q(x,t)}(\Omega)]^n$ are continuously embedded in the locally convex space $[L^1(\Omega)]^n$ (see [11, c. 17]), from Remark 5.12 [11, c. 22], we get that, V^t is Banach space with standard norm for the intersection of the spaces. Easy to show, that V^t is reflexive and separable. We will make similar consideration for the space $U(Q_{0,T})$. We also consider the space

$$W(Q_{0,T}) := \{u \in U(Q_{0,T}) \quad | \quad u_t \in [U(Q_{0,T})]^*\}$$

with the norm $||u;W(Q_{0,T})|| := ||u;U(Q_{0,T})|| + ||u_t;[U(Q_{0,T})]^*||$. The notation u_t stands for the distributional time derivative which is defined by the rule

$$\langle u_t, \varphi\rangle_{\mathcal{D}_{\mathrm{div}}} := -\int\limits_{Q_{0,T}} u(x,t)\varphi_t(x,t)\,dxdt \quad \text{for} \quad \varphi \in \mathcal{D}_{\mathrm{div}}. \tag{11}$$

Assume that the following conditions are fulfilled.

(A): A_{ij} are n-order square matrix with the elements from $L^\infty(Q_{0,T})$; $A_{ij} = A_{ji}$ $(i,j=\overline{1,n})$; for a.e. $(x,t) \in Q_{0,T}$ and for every $\xi^1,\dots,\xi^n \in \mathbb{R}^n$, we get

$$a_0\sum_{i=1}^n |\xi^i|^2 \le \sum_{i,j=1}^n \Big(A_{ij}(x,t)\xi^i,\xi^j\Big)_{\mathbb{R}^n} \le a^0\sum_{i=1}^n |\xi^i|^2 \quad (0 < a_0 \le a^0 < +\infty);$$

(G): G is n-order square matrix, $G = diag(g_1,\dots,g_n)$, $g_l \in L^\infty(Q_{0,T})$, and $0 < g_0 \le g_l(x,t) \le g^0 < +\infty$ for a.e. $(x,t) \in Q_{0,T}$, where $l = \overline{1,n}$;

(F): $F \in L^2(0,T;H)$;

(U): $u_0 \in H$.

We define the operators $A(t) : V^t \to [V^t]^*$, $\mathcal{A} : U(Q_{0,T}) \to [U(Q_{0,T})]^*$ by the rules

$$\langle A(t)z,w\rangle_{V^t} := \int\limits_\Omega \Big[\sum_{i,j=1}^n \Big(A_{ij}(x,t)z_{x_i}(x), w_{x_j}(x)\Big)_{\mathbb{R}^n} +$$

$$+\Big(G(x,t)|z(x)|^{q(x,t)-2}z(x), w(x)\Big)_{\mathbb{R}^n}\Big]\,dx, \quad z,w \in V^t, \quad t \in (0,T), \tag{12}$$

$$\langle \mathcal{A}u,v\rangle_{U(Q_{0,T})} := \int\limits_0^T \langle A(t)u(t),v(t)\rangle_{V^t}\,dt, \quad u,v \in U(Q_{0,T}), \tag{13}$$

Suppose that

$$h := \min\Big\{2, \frac{q^0}{q^0-1}\Big\}. \tag{14}$$

Let $(\cdot,\cdot)_{\mathbb{R}^n}$ be a scalar product in the space $\mathbb{R}^n$,

$$(u,v)_\Omega := \int\limits_\Omega (u(x),v(x))_{\mathbb{R}^n}\,dx, \quad u=(u_1,\dots,u_n), v=(v_1,\dots,v_n) : \Omega \to \mathbb{R}^n. \tag{15}$$

Definition 1. *The pair of the functions $\{u,\pi\}$ is called a weak solution of problem (1)-(5), if $u \in W(Q_{0,T}) \cap L^\infty(0,T;H)$, $\pi \in L^h(Q_{0,T})$, u satisfies (5) in H,*

$$\langle u_t + \mathcal{A}u, z\rangle_{U(Q_{0,T})} = \int_0^T (F(t), z(t))_\Omega \, dt \tag{16}$$

holds for $z \in U(Q_{0,T})$, π satisfies (1) in $\mathcal{D}^*_{div}$, and π satisfies (3) in $D^*(0,T)$.

Theorem 1 (existence)**.** *Let $q \in \Lambda_t(Q_{0,T})$, conditions **(A)-(U)** hold. Then problem (1)-(5) has a weak solution $\{u,\pi\}$. Moreover, $u \in C([0,T];H)$.*

Theorem 2 (uniqueness)**.** *Let $q \in \Lambda_t(Q_{0,T})$, conditions **(A)-(G)** hold. Then, problem (1)-(5) can't have more the one weak solution.*

2 Auxiliary statements

For the Banach spaces X and Y the notation $X \circlearrowleft Y$ means the continuous embedding; the notation $X \,\bar{\circlearrowleft}\, Y$ means the continuous and densely embedding; the notation $X \overset{K}{\subset} Y$ means the compact embedding.

2.1 Projection operator

Suppose that H and Z_s are determined from (6) and (7) respectively, where $s \in \mathbb{N}$. From [12, Ch. 1, §6.1], we obtain the embeddings

$$Z_s \,\bar{\circlearrowleft}\, Z_1 \,\bar{\circlearrowleft}\, H \cong H^* \,\bar{\circlearrowleft}\, Z_1^* \,\bar{\circlearrowleft}\, Z_s^*.$$

Moreover, $Z_s \subset [H_0^s(\Omega)]^n$ and $Z_s \overset{K}{\subset} H$. Let w^μ, $\mu \in \mathbb{N}$, be eigenfunctions (associated to the eigenvalues $\lambda_\mu > 0$) of the spectral problem

$$\int_\Omega \sum_{|\alpha|=s} (D^\alpha w^\mu, D^\alpha v)_{\mathbb{R}^n} \, dx = \lambda_\mu \int_\Omega (w^\mu, v)_{\mathbb{R}^n} \, dx \qquad \forall\, v \in Z_s. \tag{17}$$

For the sake of convenience we have assumed that $\{w^\mu\}_{\mu\in\mathbb{N}}$ is an orthonormal set in H.

Proposition 1. *(see [12, Ch. 1, §6.3]). If $s \in \mathbb{N}$ and $s \geq \frac{n}{2}$, then the set $\{w^\mu\}_{\mu\in\mathbb{N}}$ of all eigenfunctions of problem (17) is a basis for the space Z_s.*

Let $m \in \mathbb{N}$ be a fixed number, and $\mathfrak{M}$ be a set of all linear combinations of the elements from $\{w^1,\dots,w^m\}$. Define an unique orthogonal projection $P_m : H \to \mathfrak{M}$ by the rule (see [13, p. 527])

$$P_m h := \sum_{j=1}^m (h, w^j)_H \, w^j, \quad h \in H. \tag{18}$$

Since $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{V} \equiv Z_s$, $s \in \mathbb{N}$, then let us define an operator $\widehat{P}_m : \mathcal{V} \to \mathcal{V}$ by the rule

$$\widehat{P}_m v := P_m v \quad \text{for every} \quad v \in \mathcal{V}. \tag{19}$$

For a conjugate operator $\widehat{P}_m^* : \mathcal{V}^* \to \mathcal{V}^*$ we have $\widehat{P}_m^*(\mathcal{V}^*) \subset \mathcal{V}$ (see [14, p. 865]).

Proposition 2. *(see Lemma 3.9 [14, p. 865-866]). If $\psi_1^m,\dots,\psi_m^m \in \mathbb{R}$, $F \in \mathcal{V}^*$. and $z^m := \sum\limits_{s=1}^{m} \psi_s^m w^s \in \mathcal{V}$ satisfies*

$$\begin{cases} \langle z^m, w^1\rangle_{\mathcal{V}} = \langle F, w^1\rangle_{\mathcal{V}}, \\ \quad\vdots \\ \langle z^m, w^m\rangle_{\mathcal{V}} = \langle F, w^m\rangle_{\mathcal{V}}, \end{cases}$$

then the following equality is satisfied $z^m = \widehat{P}_m^* F$ in $\mathcal{V}^*$.

Proposition 3. *(see Lemma 1 [10, p. 111]). Suppose that P_m and $\widehat{P}_m$ are determined from (18) and (19) respectively, where $\mathcal{V} = Z_s$, $s \in \mathbb{N}$, and $\{w^\mu\}_{\mu\in\mathbb{N}}$ is an orthonormal basis for the space H that consists of all eigenfunctions of problem (17). Then, for every $w \in L^r(0,T;Z_s^*)$ and $r > 1$, we have the inequality*

$$||\widehat{P}_m^* w; L^r(0,T;Z_s^*)|| \le ||w; L^r(0,T;Z_s^*)||. \tag{20}$$

2.2 Cauchy's problem for system of ordinary differential equations

Take $\ell \in \mathbb{N}$ and $Q = (0,T)\times\mathbb{R}^\ell$. In this section we seek a weak solution $\varphi : [0,T] \to \mathbb{R}^\ell$ of the problem

$$\varphi'(t) + L(t,\varphi(t)) = M(t), \quad t \in [0,T], \qquad \varphi(0) = \varphi^0, \tag{21}$$

where $M : [0,T] \to \mathbb{R}^\ell$ and $L : Q \to \mathbb{R}^\ell$ are some functions (for the sake of convenience we have assumed that $L(t,0) = 0$ for every $t \in [0,T]$), and $\varphi^0 = (\varphi_1^0,\dots,\varphi_\ell^0) \in \mathbb{R}^\ell$.

Definition 2. *We shall say that a function $L : Q \to \mathbb{R}^\ell$ satisfies the Carathéodory condition if for every $\xi \in \mathbb{R}^\ell$ the function $(0,T) \ni t \mapsto L(t,\xi) \in \mathbb{R}^\ell$ is measurable and if for a.e. $t \in (0,T)$ the function $\mathbb{R}^\ell \ni \xi \mapsto L(t,\xi) \in \mathbb{R}^\ell$ is continuous.*

Definition 3. *We shall say that a function $L : Q \to \mathbb{R}^\ell$ satisfies the L^p-Carathéodory condition if L satisfies the Carathéodory condition and for every $R > 0$ there exists a function $h_R \in L^p(0,T)$ such that*

$$|L(t,\xi)| \le h_R(t)$$

for a.e. $t \in (0,T)$ and for every $\xi \in \overline{D_R} := \{y \in \mathbb{R}^\ell \mid \ |y| \le R\}$.

Lemma 1. *Suppose that $q \in \Lambda_t(Q_{0,T})$, condition **(G)** holds, $m \in \mathbb{N}$, $\xi = (\xi_1,\dots,\xi_m) \in \mathbb{R}^m$, $w^1,\dots,w^m \in L^{q^0}(\Omega)$, $w(x,\xi) = \sum\limits_{l=1}^{m} \xi_l w^l(x)$, and $z \in [L^{q^0}(\Omega)]^n$. Then the function*

$$I(t,\xi) := \int\limits_\Omega \big(G(x,t)|w(x,\xi)|^{q(x,t)-2} w(x,\xi), z(x)\big)_{\mathbb{R}^n}\, dx, \quad t \in (0,T), \quad \xi \in \mathbb{R}^m,$$

satisfies the L^∞-Carathéodory condition.

Proof. We use the methods of Lemma 3.25 [14, p. 874]).

Step 1. Since

$$|w|^{q(x,t)-1}\cdot|z|\le C_1\Big(|z|^{q^0}+|w|^{\frac{q(x,t)-1}{q^0-1}q^0}\Big)\le C_2\Big(|z|^{q^0}+|w|^{q^0}+1\Big),\tag{22}$$

the Fubini Theorem [15, p. 91] yields that $I(\cdot,\xi)\in L^1(0,T)$. Then $[0,T]\ni t\mapsto I(t,\xi)\in\mathbb{R}$ is the measurable function.

Step 2. Let us prove that the function $\mathbb{R}\ni\xi_1\mapsto I(t,\xi_1,\dots,\xi_m)\in\mathbb{R}$ is continuous at the point $\xi_1^0\in\mathbb{R}$. Take $\xi=(\xi_1,\xi_2,\dots,\xi_m)$, $\xi^0=(\xi_1^0,\xi_2,\dots,\xi_m)$, where $|\xi-\xi^0|\le 1$.

By Theorem 2.1 [16, p. 2], we get

$$\big||\eta_1|^{q(x,t)-2}\eta_1-|\eta_2|^{q(x,t)-2}\eta_2\big|\le C_3(|\eta_1|+|\eta_2|)^{q(x,t)-1-\beta(x,t)}|\eta_1-\eta_2|^{\beta(x,t)},$$

where $0<\beta(x,t)\le\min\{1,q(x,t)-1\}$, $\eta_1,\eta_2\in\mathbb{R}$, $C_3>0$ is independent of η_1,η_2,x,t. Hence,

$$\begin{aligned}|I(t,\xi)-I(t,\xi^0)|&=\Big|\int\limits_\Omega\Big(G\Big(|w(x,\xi)|^{q(x,t)-2}w(x,\xi)-|w(x,\xi^0)|^{q(x,t)-2}w(x,\xi^0)\Big),z\Big)_{\mathbb{R}^n}\,dx\Big|\le\\ &\le C_4\int\limits_\Omega\big(|w(x,\xi)|+|w(x,\xi^0)|\big)^{q(x,t)-1-\beta(x,t)}|w(x,\xi)-w(x,\xi^0)|^{\beta(x,t)}|z|\,dx=\\ &=C_4(I_1(t)+I_2(t)),\end{aligned}\tag{23}$$

where $I_1(t):=\int\limits_{\Omega_1(t)}h(x,t,\xi,\xi^0)\,dx$, $I_2(t)=\int\limits_{\Omega_2(t)}h(x,t,\xi,\xi^0)\,dx$,
$\Omega_1(t)=\{x\in\Omega\mid q(x,t)\le 2\}$, $\Omega_2(t)=\{x\in\Omega\mid\ q(x,t)>2\}$, and

$$h(x,t,\xi,\xi^0)=\big(|w(x,\xi)|+|w(x,\xi^0)|\big)^{q(x,t)-1-\beta(x,t)}|w(x,\xi)-w(x,\xi^0)|^{\beta(x,t)}|z(x)|,\quad x\in\Omega.$$

Taking $\beta(x,t)=q(x,t)-1$, $x\in\Omega_1(t)$, gives (see also (22))

$$\begin{aligned}I_1(t)&=\int\limits_{\Omega_1(t)}|w(x,\xi)-w(x,\xi^0)|^{q(x,t)-1}|z(x)|\,dx=\int\limits_{\Omega_1(t)}|\xi_1-\xi_1^0|^{q(x,t)-1}|w^1(x)|^{q(x,t)-1}|z(x)|\,dx\le\\ &\le|\xi_1-\xi_1^0|^{q_0-1}\int\limits_{\Omega_1(t)}|w^1(x)|^{q(x,t)-1}|z(x)|\,dx=C_5|\xi_1-\xi_1^0|^{q_0-1}\underset{\xi_1\to\xi_1^0}{\longrightarrow}0.\end{aligned}$$

Taking $\beta(x,t)=1$, $x\in\Omega_2$, gives

$$\begin{aligned}I_2(t)&=\int\limits_{\Omega_2(t)}\big(|w(x,\xi)|+|w(x,\xi^0)|\big)^{q(x,t)-2}|w(x,\xi)-w(x,\xi^0)|\cdot|z(x)|\,dx=\\ &=|\xi_1-\xi_1^0|\int\limits_{\Omega_2(t)}\big(|w(x,\xi)|+|w(x,\xi^0)|\big)^{q(x,t)-2}|w^1(x)|\cdot|z(x)|\,dx\le C_6(\xi_1^0)|\xi_1-\xi_1^0|\underset{\xi_1\to\xi_1^0}{\longrightarrow}0.\end{aligned}$$

Therefore, by (23), we obtain that $|I(t,\xi)-I(t,\xi^0)|\underset{\xi_1\to\xi_1^0}{\longrightarrow}0$. Continuing in the same way, we see that I is continuous with respect to $\xi_2,\dots,\xi_m$.

Step 3. Taking into account the results of Step 1 and Step 2, we obtain that the function I satisfies the Carathéodory condition. Since $g\in L^\infty(Q_{0,T})$, the L^∞-Carathéodory condition holds. □

Proposition 4. *(the Carathéodory-LaSalle theorem, see Theorem 3.24 [14, p. 872]). Suppose that $p \geq 2$, function $L : Q \to \mathbb{R}^\ell$ satisfies L^p-Carathéodory condition, $M \in L^p(0,T;\mathbb{R}^\ell)$, and $\varphi^0 \in \mathbb{R}^\ell$. If there exist nonnegative functions $\alpha, \beta \in L^1(0,T)$ such that for every $\xi \in \mathbb{R}^\ell$ and for a.e. $t \in [0,T]$ the inequality*

$$(L(t,\xi),\xi)_{\mathbb{R}^\ell} \geq -\alpha(t)|\xi|^2 - \beta(t) \tag{24}$$

holds, then problem (21) has a global weak solution $\varphi \in W^{1,p}(0,T;\mathbb{R}^\ell)$.

2.3 Additional statements

Let $\mathbb{Z}_{\geq -1} := \{s \in \mathbb{Z} \mid s \geq -1\}$. The following Propositions are needed for the sequel.

Proposition 5. *(the generalized De Rham theorem, see Theorem 4.1 [17], Remark 4.3 [17], and Lemma 2 [18]). Suppose that Ω be an open bounded connected and Lipschitz subset of $\mathbb{R}^n$, $T > 0$, $s_1, s_2 \in \mathbb{Z}_{\geq -1}$, $h_1, h_2 \in [1,\infty]$, and $\mathcal{F} \in W^{s_1,h_1}(0,T;[W^{s_2,h_2}(\Omega)]^n)$. Then, if*

$$\langle \mathcal{F}(\cdot), v\rangle_{[D(\Omega)]^n} = 0 \quad in \quad D^*(0,T) \tag{25}$$

for all $v \in C_{\mathrm{div}}$, then there exists an unique

$$\pi \in W^{s_1,h_1}(0,T;W^{s_2+1,h_2}(\Omega)) \tag{26}$$

such that

$$\nabla\pi = \mathcal{F} \quad in \quad [D^*(Q_{0,T})]^n, \tag{27}$$

$$\int\limits_\Omega \pi(\cdot)\, dx = 0 \quad in \quad D^*(0,T). \tag{28}$$

Moreover, there exists a positive number C_7 (independent of $\mathcal{F}, \pi$) such that

$$||\pi; W^{s_1,h_1}(0,T;W^{s_2+1,h_2}(\Omega))|| \leq C_7||\mathcal{F}; W^{s_1,h_1}(0,T;[W^{s_2,h_2}(\Omega)]^n)||. \tag{29}$$

Proposition 6. *(the Aubin theorem, see [19] and [20, p. 393]). If $s, h \in (1,\infty)$ are fixed numbers, $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are the Banach spaces, and $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, then*

$$\{u \in L^s(0,T;\mathcal{W}) \mid u_t \in L^h(0,T;\mathcal{B})\} \overset{K}{\subset} [L^s(0,T;\mathcal{L}) \cap C([0,T];\mathcal{B})].$$

Proposition 7. *(Lemma 1.18 [11, p. 39]). If $u^m \underset{m\to\infty}{\longrightarrow} u$ in $L^p(Q_{0,T})$ $(1 \leq p \leq \infty)$, then there exists a subsequence (we call it $\{u^m\}_{m\in\mathbb{N}}$ again) such that $u^m \underset{m\to\infty}{\longrightarrow} u$ a.e. in $Q_{0,T}$.*

Proposition 8. *(Theorem 1 [21, p. 108]). If $q \in \Lambda_t\,(Q_{0,T})$, then for every $u \in W(Q_{0,T})$ we have that $u \in C([0,T];H)$ and the following formula of integration by parts is true*

$$\langle u_t, \chi_{t_1,t_2}u\rangle_{U(Q_{0,T})} = \frac{1}{2}\int\limits_\Omega |u(x,t_2)|^2\, dx - \frac{1}{2}\int\limits_\Omega |u(x,t_1)|^2\, dx, \quad 0 \leq t_1 < t_2 \leq T, \tag{30}$$

where

$$\chi_{t_1,t_2}(t) = \begin{cases} 1, & t \in [t_1,t_2]; \\ 0, & t \notin [t_1,t_2]. \end{cases} \tag{31}$$

It‘s clear that if $u=(u_1,\dots,u_n)\in[L^2(\mathcal{O})]^n$, where $\mathcal{O}=\Omega$ or $\mathcal{O}=Q_{0,T}$, then

$$|| \, |u|; L^2(\mathcal{O})||^2 = \int\limits_{\mathcal{O}} |u|^2 \, dy = \sum_{l=1}^{n} ||u_l; L^2(\mathcal{O})||^2 \le n||u; [L^2(\mathcal{O})]^n||^2,$$

and so $|| \, |u|; L^2(\mathcal{O})|| \le \sqrt{n}||u; [L^2(\mathcal{O})]^n||$.

Lemma 2. *Let conditions* **(A)**-**(G)** *hold,* $\{w^j\}_{j\in\mathbb{N}} \subset V^t$, $m\in\mathbb{N}$, $L=(L_1,L_2,\dots,L_m)$,

$$L_\mu(t,\xi) = \langle A(t)z^m, w^\mu\rangle_{V^t}, \quad \mu=\overline{1,m}, \quad t\in(0,T), \quad \xi\in\mathbb{R}^m,$$

and $z^m(x) = \sum\limits_{\mu=1}^{m} \xi_\mu w^\mu(x)$ *for* $x\in\Omega$. *Then*

$$\Big(L(t,\xi),\xi\Big)_{\mathbb{R}^m} \ge \int\limits_{\Omega} \Big[a_0 \sum_{i=1}^{n} |z^m_{x_i}|^2 + g_0|z^m|^{q(x,t)}\Big] \, dx, \quad t\in(0,T), \quad \xi\in\mathbb{R}^m. \tag{32}$$

Proof. It‘s clear that

$$(L(t,\xi),\xi)_{\mathbb{R}^m} = \langle A(t)z^m, z^m\rangle_{V^t}. \tag{33}$$

Using condition **(A)**, we get the following estimate:

$$\sum_{i,j=1}^{n} \Big(A_{ij}u^m_{x_i}, u^m_{x_j}\Big)_{\mathbb{R}^n} \ge a_0 \sum_{i=1}^{n} |u^m_{x_i}|^2. \tag{34}$$

It follows from condition **(G)** that

$$\Big(G|u^m|^{q(x,t)-2}u^m, u^m\Big)_{\mathbb{R}^n} = \sum_{l=1}^{n} g_l(x,t)|u^m|^{q(x,t)-2}|u^m_l|^2 \ge$$

$$\ge g_0 \sum_{l=1}^{n} |u^m|^{q(x,t)-2}|u^m_l|^2 = g_0|u^m|^{q(x,t)}. \tag{35}$$

If we use conditions **(A)** and **(G)**, then we get

$$\langle A(t)z^m, z^m\rangle_{V^t} = \int\limits_{\Omega} \Big[\sum_{i,j=1}^{n} \Big(A_{ij}(x,t)z^m_{x_i}(x), z^m_{x_j}(x)\Big)_{\mathbb{R}^n} +$$

$$+\Big(G(x,t)|z^m(x)|^{q(x,t)-2}z^m(x), z^m(x)\Big)_{\mathbb{R}^n}\Big] \, dx \ge \int\limits_{\Omega} \Big[a_0 \sum_{i=1}^{n} |z^m_{x_i}|^2 + g_0|z^m|^{q(x)}\Big] \, dx. \tag{36}$$

Thus, (33)-(36) imply that (32) holds. $\square$

3 Proofs of main results

Proof of Theorem 1. The solution will be constructed via Faedo-Galerkin's method. Let $r_- = \min\{2, q_0\}$, $r_+ = \max\{2, q^0\}$, $r'_- = \frac{r_-}{r_- - 1}$, $r'_+ = \frac{r_+}{r_+ - 1}$,

$$V_+ := Z_1 \cap [L^{q^0}(\Omega)]^n, \quad V_- := Z_1 \cap [L^{q_0}(\Omega)]^n \tag{37}$$

(see notation (7)). Note that

$$C_{\mathrm{div}} \mathrel{\bar{\circlearrowleft}} V_+ \mathrel{\bar{\circlearrowleft}} V^t \mathrel{\bar{\circlearrowleft}} V_- \mathrel{\bar{\circlearrowleft}} H \mathrel{\bar{\circlearrowleft}} V_-^* \mathrel{\bar{\circlearrowleft}} [V^t]^* \mathrel{\bar{\circlearrowleft}} V_+^*, \quad t \in [0, T], \tag{38}$$

$$L^{r_+}(0, T; V_+) \mathrel{\bar{\circlearrowleft}} U(Q_{0,T}) \mathrel{\bar{\circlearrowleft}} L^{r_-}(0, T; V_-) \mathrel{\bar{\circlearrowleft}} L^1(0, T; V_+^*), \tag{39}$$

$$L^{r'_-}(0, T; V_-^*) \mathrel{\bar{\circlearrowleft}} [U(Q_{0,T})]^* \mathrel{\bar{\circlearrowleft}} L^{r'_+}(0, T; V_+^*) \mathrel{\bar{\circlearrowleft}} L^1(0, T; V_+^*). \tag{40}$$

Thus, the elements from $U(Q_{0,T})$ and $[U(Q_{0,T})]^*$ are distributions on $(0, T)$ with value in V_+^*. Then, similarly to Proposition 2.6.2 [22, p. 58], for $u \in W(Q_{0,T})$ we have that u_t (see (11)) is the distributional derivative in sense of the set of functions on $(0, T)$ with value in $V_+^* + [L^1(\Omega)]^n$. Let

$$s \in \mathbb{N}, \quad s \geq \max\Big\{2, \ \frac{n}{2}, \ n\Big(\frac{1}{2} - \frac{1}{q^0}\Big)\Big\}. \tag{41}$$

Note that (41) implies that $Z_s \mathrel{\bar{\circlearrowleft}} V_+ \mathrel{\bar{\circlearrowleft}} V^t$.

<u>Step 1</u> (construction of approximation). Let $\{w^\mu\}_{\mu\in\mathbb{N}}$ is taken from Proposition 1, $s \in \mathbb{N}$ satisfies (41). By definition, put

$$u^m(x, t) := \sum_{\mu=1}^{m} \varphi_\mu^m(t) w^\mu(x), \quad (x, t) \in Q_{0,T}, \quad m \in \mathbb{N},$$

where the unknown function $\varphi := (\varphi_1^m, \dots, \varphi_m^m)$ satisfies (see notation (12) and (15))

$$(u_t^m(t), w^\mu)_\Omega + \langle A(t) u^m(t), w^\mu \rangle_{V^t} = (F(t), w^\mu)_\Omega, \quad t \in (0, T), \quad \mu = \overline{1, m}, \tag{42}$$

$$\varphi_1^m(0) = \alpha_1^m, \quad \dots, \quad \varphi_m^m(0) = \alpha_m^m. \tag{43}$$

Here the numbers $\alpha_1^m, \dots, \alpha_m^m \in \mathbb{R}$ we choose such that $u_0^m \underset{m\to\infty}{\longrightarrow} u_0$ strongly in H, where $u_0^m(x) := \sum_{j=1}^m \alpha_j^m w^j(x), \quad x \in \Omega$. It's clear that (43) implies that

$$u^m(0) = u_0^m. \tag{44}$$

Let us show that the mentioned function φ exists. Let L be a vector-valued function from Lemma 2. Then Cauchy problem (42)-(43) takes form (21) if

$$M(t) = ((F(t), w^1)_\Omega, \dots, (F(t), w^m)_\Omega), \quad t \in (0, T).$$

It follows from condition **(F)** that $M \in L^2(0, T; \mathbb{R}^m)$. Conditions **(A)**-**(G)** and Lemma 1 yield that the function L satisfies the L^∞-Carathéodory condition. Using estimate (32), conditions $a_0 > 0$ and $g_0 > 0$, we receive $(L(t, \varphi^m), \varphi^m)_{\mathbb{R}^m} \geq 0$. Then estimate (24) with

$\alpha(t) \equiv 0$ and $\beta(t) \equiv 0$ holds, and from the Carathéodory-LaSalle theorem (see Proposition 4) we have the existence of the solution

$$\varphi \in W^{1,2}(0,T;\mathbb{R}^m) \tag{45}$$

of problem (21) and so problem (42)-(43).

Step 2 (first estimates). Multipling the μ-th equation of (42) by $\varphi_\mu^m(t)$ and summing over $\mu = \overline{1,m}$, we get

$$\sum_{\mu=1}^{m}\Big(u_t^m(t), w^\mu \varphi_\mu^m(t)\Big)_\Omega + \Big(L(t,\varphi^m(t)), \varphi^m(t)\Big)_{\mathbb{R}^m} = \sum_{\mu=1}^{m}\Big(F(t), w^\mu \varphi_\mu^m(t)\Big)_\Omega, \quad t \in (0,T).$$

After integrating for $t \in (0,\tau) \subset (0,T)$ and some transformation, we receive

$$\int\limits_{Q_{0,\tau}} (u_t^m, u^m)_{\mathbb{R}^n}\, dxdt + \int\limits_0^\tau \Big(L(t,\varphi^m), \varphi^m\Big)_{\mathbb{R}^m}\, dt = \int\limits_{Q_{0,\tau}} (F, u^m)_{\mathbb{R}^n}\, dxdt, \quad \tau \in (0,T]. \tag{46}$$

Using (44), (45), we obtain

$$\int\limits_{Q_{0,\tau}} (u_t^m, u^m)_{\mathbb{R}^n}\, dxdt = \int\limits_{Q_{0,\tau}} \frac{1}{2}\frac{\partial}{\partial t}(|u^m|^2)\, dxdt = \frac{1}{2}\int\limits_{\Omega_\tau} |u^m|^2\, dx - \frac{1}{2}\int\limits_\Omega |u_0^m|^2\, dx. \tag{47}$$

Clearly,

$$|(F,u^m)_{\mathbb{R}^n}| \le |F|\cdot|u^m| \le \frac{|F|^2}{2} + \frac{|u^m|^2}{2}. \tag{48}$$

Using (32), (47)-(48), from equality (46), we obtain the following estimate

$$\frac{1}{2}\int\limits_\Omega |u^m(x,\tau)|^2\, dx + \int\limits_{Q_{0,\tau}} \Big[a_0 \sum_{i=1}^n |u_{x_i}^m|^2 + g_0 |u^m|^{q(x,t)}\Big]\, dxdt \le$$
$$\le \frac{1}{2}\int\limits_\Omega |u_0^m|^2\, dx + \frac{1}{2}\int\limits_{Q_{0,\tau}} |F|^2\, dxdt + \frac{1}{2}\int\limits_{Q_{0,\tau}} |u^m|^2\, dxdt. \tag{49}$$

Take $y(t) := \int_\Omega |u^m(x,t)|^2\, dx, \quad t \in [0,T]$. Then, from (49), we get an estimate

$$\frac{1}{2}\, y(\tau) \le C_8 + \frac{1}{2}\int\limits_0^\tau y(t)\, dt, \quad \tau \in [0,T].$$

Therefore, the Gronwall lemma implies that $y(\tau) \le C_9$, and so

$$\int\limits_\Omega |u^m(x,\tau)|^2\, dx \le C_9, \quad \tau \in (0,T]. \tag{50}$$

It follows from (49) and (50) that

$$\int\limits_{Q_{0,\tau}} \Big[\sum_{i=1}^n |u_{x_i}^m|^2 + |u|^2 + |u|^{q(x,t)}\Big]\, dxdt \le C_{10}, \quad \tau \in (0,T], \tag{51}$$

This estimate yields that

$$\int\limits_{Q_{0,\tau}} \Big|G|u^m|^{q(x,t)-2}u^m\Big|^{q'(x,t)}\,dxdt \le C_{11}\int\limits_{Q_{0,\tau}} |u^m|^{q(x,t)}\,dxdt \le C_{12}. \tag{52}$$

From (50) and (51) it follows the estimates

$$||u^m; L^\infty(0,T,H)|| + ||u^m; U(Q_{0,T})|| \le C_{13}, \tag{53}$$

Here the constants $C_8, \dots, C_{13}$ are independent of m.

By (52)-(53) we have existence of the subsequence $\{u^{m_k}\}_{k\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_k} \underset{k\to\infty}{\longrightarrow} u \quad * - \text{weakly in} \quad L^\infty(0,T;H) \quad \text{and weakly in} \quad U(Q_{0,T}),$$

$$G|u^m|^{q(x,t)-2}u^m \underset{k\to\infty}{\longrightarrow} \chi_1 \quad \text{weakly in} \quad [L^{q'(x,t)}(Q_{0,T})]^n. \tag{54}$$

Step 3 (additional estimates). From (13) and (51) it follows an inequality

$$\langle \mathcal{A}u^m, v\rangle_{U(Q_{0,T})} = \int\limits_{Q_{0,T}} \Big[\sum_{i,j=1}^n (A_{ij}u^m_{x_i}, v_{x_j})_{\mathbb{R}^n} + (G|u^m|^{q(x,t)-2}u^m, v)_{\mathbb{R}^n}\Big]\,dxdt \le$$

$$\le C_{14}\Big(||u^m; L^2(0,T;Z_1)|| \cdot ||v; L^2(0,T)Z_1)|| +$$

$$+ ||G|u^m|^{q(x,t)-2}u^m; [L^{q'(x,t)}(Q_{0,T})]^n|| \cdot ||v; [L^{q(x,t)}(Q_{0,T})]^n||\Big) \le C_{15}||v; U(Q_{0,T})||$$

and so

$$||\mathcal{A}u^m; [U(Q_{0,T})]^*|| \le C_{16}. \tag{55}$$

Since s satisfies (41), from (39) and the construction of the space $U(Q_{0,T})$, we obtain

$$U(Q_{0,T}) \circlearrowleft L^2(0,T;H) \circlearrowleft [U(Q_{0,T})]^*, \tag{56}$$

$$L^{r_+}(0,T;Z_s) \circlearrowleft L^{r_+}(0,T;V_+) \circlearrowleft U(Q_{0,T}).$$

Therefore,

$$[U(Q_{0,T})]^* \circlearrowleft L^{r'_+}(0,T;Z_s^*). \tag{57}$$

Using (39) and (53), we obtain

$$||u^m; L^{r_-}(0,T;V_-)|| \le C_{17}||u; U(Q_{0,T})|| \le C_{18}. \tag{58}$$

Using Proposition 2 and notation (12)-(13) and (18)-(19), in same way as in [12, Ch. 1, §5.3], we rewrite (42) as

$$u^m_t = \widehat{P}^*_m(F - \mathcal{A}u^m).$$

Thus, from estimate (20), embeddings (57) and (56), and estimate (55), we get

$$||u^m_t; L^{r'_+}(0,T;Z_s^*)|| = ||\widehat{P}^*_m(F - \mathcal{A}u^m); L^{r'_+}(0,T;Z_s^*)|| \le$$

$$\leq ||F - \mathcal{A}u^m; L^{r'_+}(0,T;Z_s^*)|| \leq C_{19}||F - \mathcal{A}u^m; [U(Q_{0,T})]^*|| \leq$$

$$\leq C_{20}\Big(||F; L^2(0,T;H)|| + ||\mathcal{A}u^m; [U(Q_{0,T})]^*||\Big) \leq C_{21}. \tag{59}$$

Here the constant $C_{16}, \ldots, C_{21} > 0$ are independent of m.

Since $V_- \overset{K}{\subset} H \circlearrowleft Z_s^*$, from (58), (59), the Aubin theorem (see Proposition 6), and Proposition 7, we obtain

$$u^{m_k} \underset{k\to\infty}{\longrightarrow} u \quad \text{in} \quad L^{r_-}(0,T;H) \cap C([0,T];Z_s^*) \quad \text{and a.e. in} \quad Q_{0,T}.$$

Therefore, (5) holds and $\chi_1 = G|u|^{q(x,t)-2}u$ (see (54)).

Step 4 (passing to the limit). Take $\psi \in C^1([0,T])$ such that $\psi(T) = 0$. When we multiply equality (42) by $\psi(t)$, integrate for $t \in (0,T)$, and the first term integrate by parts. We obtain the following

$$\int\limits_{Q_{0,T}} \Big[-\Big(u^m, w^\mu\Big)_{\mathbb{R}^n} \psi_t + \sum_{i,j=1}^{n} \Big(A_{ij} u^m_{x_i}, w^\mu_{x_j}\Big)_{\mathbb{R}^n} \psi + \Big(G|u^m|^{q(x,t)-2}u^m, w^\mu\Big)_{\mathbb{R}^n} \psi\Big] dxdt +$$

$$= \int\limits_{\Omega} \Big(u_0^m, w^\mu\Big)_{\mathbb{R}^n} \psi(0)\, dx + \int\limits_{Q_{0,T}} \Big(F, w^\mu\Big)_{\mathbb{R}^n} \psi\, dxdt.$$

Taking $m = m_k$ and letting $k \to \infty$, thanks to arbitrariness of ψ, we get

$$\langle \mathcal{F}, z\rangle_{U(Q_{0,T})} = 0 \qquad \forall\, z \in U(Q_{0,T}), \tag{60}$$

where $\mathcal{F} := F - u_t - \mathcal{A}u$. Hence, $u_t \in [U(Q_{0,T})]^*$, (16) holds and $u \in C([0,T];H)$. Taking $z(x,t) = w(x)\varphi(t)$, $x \in \Omega$, $t \in (0,T)$, from (60), we obtain

$$\int\limits_0^T \langle \mathcal{F}(t), w\rangle_{[D(\Omega)]^n}\, \varphi(t)\, dt = 0, \qquad w \in C_{\mathrm{div}}, \quad \varphi \in D(0,T),$$

and so (25) holds. Clearly, from (40) we get

$$\mathcal{F} \in L^2(0,T;[H^{-1}(\Omega)]^n) + [L^{\frac{q^0}{q^0-1}}(Q_{0,T})]^n \subset W^{0,h}(0,T;[W^{-1,h}(\Omega)]^n),$$

where h is taken from (14). Then, the generalized De Rham theorem (see Proposition 5) yields that there exists $\pi \in W^{0,h}(0,T;W^{0,h}(\Omega)) = L^h(Q_{0,T})$ such that (27)-(28) hold. Thus, π satisfies (1) in $[D^*(Q_{0,T})]^n$ and (3) in $D^*(0,T)$. Theorem 1 is proved. $\Box$

Proof of Theorem 2. Let $\{u_1, \pi_1\}$ and $\{u_2, \pi_2\}$ be weak solutions of problem (1)-(5). Set $u := u_1 - u_2$. Take (16) for u_1:

$$\langle u_{1\,t} + \mathcal{A}u_1, z\rangle_{U(Q_{0,T})} = \int\limits_0^T (F(t), z(t))_\Omega\, dt. \tag{61}$$

Take (16) for u_2:

$$\langle u_{2\,t} + \mathcal{A}u_2, z\rangle_{U(Q_{0,T})} = \int\limits_0^T (F(t), z(t))_\Omega \, dt. \tag{62}$$

Subtracting (62) from (61), setting $z = \chi_{0,\tau}u$ (see notation (31)), $\tau \in (0,T]$, we obtain

$$\langle u_t, \chi_{0,\tau}u\rangle_{U(Q_{0,T})} + \int\limits_0^\tau \langle A(t)u_1(t) - A(t)u_2(t), u_1(t) - u_2(t)\rangle_{V^t} \, dt = \int\limits_0^\tau (F(t), u(t))_\Omega \, dt, \ \tau \in (0,T].$$

Using (30) and simple transformations, in the same way as (49), from this equality, we get

$$\frac{1}{2}\int\limits_{\Omega_\tau} |u|^2 \, dx + \int\limits_{Q_{0,\tau}} \Big[a_0 \sum_{i=1}^n |u_{x_i}|^2 + (G|u_1|^{q(x,t)-2}u_1 - G|u_2|^{q(x,t)-2}u_2, u_1 - u_2)_{\mathbb{R}^n}\Big] \, dxdt \leq$$

$$\leq C_{22} \int\limits_{Q_{0,\tau}} |u|^2 \, dxdt, \quad \tau \in (0,T]. \tag{63}$$

Let $y(\tau) := \int_{\Omega_\tau} |u|^2 \, dx$, $\tau \in (0,T]$. Then, from (63) it follows that $\frac{1}{2}y(\tau) \leq C_{22}\int_0^\tau y(t) \, dt$, $\tau \in (0,T]$. Using the Gronwall lemma, we see that $y(\tau) \leq 0$ for $\tau \in [0,T]$, and so $u_1 = u_2$.

Since π_1 and π_2 satisfy (1) in $\mathcal{D}_{\mathrm{div}}$, we obtain

$$(u_1 - u_2)_t + \mathcal{A}u_1 - \mathcal{A}u_2 + \nabla(\pi_1 - \pi_2) = 0.$$

Then the equality $u_1 = u_2$ yields that $\nabla(\pi_1 - \pi_2) = 0$. Therefore, for $t \in (0,T)$ we have that $\pi_1(t) - \pi_2(t) = C(t)$. It follows from condition (3) with π_1 and π_2 that $C(t) = 0$. Thus, $\pi_1 = \pi_2$ and Theorem 2 is proved. $\Box$

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Хома М.В., Бугрій О.М. *Системи Стокса зі змінними показниками нелінійності* // Буковинський матем. журнал — 2022. — Т.10, №2. — С. 28–42.

У статті розглянуто мішану задачу для нелінійної системи рівнянь гідродинаміки, яку прийнято називати системою Стокса. Ми збурюємо класичні рівняння Стокса монотонним нелінійним доданком зі змінним показником нелінійності – функцією $q = q(x,t)$. Цей показник нелінійності q залежить від просторової та часової змінної і, зокрема, задовольняє умову Ліпшиця за змінною t. У роботі досліджуємо існування та єдиність узагальненого розв'язку розглядуваної задачі. Доведення теореми існування розв'язку ґрунтується

на методі Фаедо-Гальоркіна. При побудові гальоркінських наближень використано теорему Каратеодорі-Ла Салля про глобальну розв'язність задачі Коші для системи звичайних диференціальних рівнянь. Побудувавши гальоркінські наближення для нашої системи, доводимо їх обмеженість у відповідних функційних просторах функцій зі змінним показником сумовності. Затим показуємо збіжність наближень до узагальненого розв'язку задачі. Теорему єдиності розв'язку мішаної задачі доводимо методом від супротивного.