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ON SOLUTIONS OF THE NONHOMOGENEOUS CAUCHY PROBLEM FOR PARABOLIC TYPE DIFFERENTIAL EQUATIONS IN A BANACH SPACE

For a differential equation of the form $u'(t) + Au(t) = f(t), t \in (0, \infty)$, where A is the infinitesimal generator of a bounded analytic C_0-semigroup of linear operators in a Banach space $\mathfrak{B}$, $f(t)$ is a $\mathfrak{B}$-valued polynomial, the behavior in the preassigned points of solutions of the Cauchy problem $u(0) = u_0 \in \mathfrak{B}$ depending on $f(t)$ is investigated.

Key words and phrases: Banach space, C_0-semigroup of linear operators, abstract parabolic equation, nonhomogeneous Cauchy problem, bounded and bounded holomorphic semigroups..

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The paper is dedicated to memory of S.D. Ivasishen, my scientific and spiritual Teacher

Introduction

The study of linear differential equations whose coefficients are unbounded operators in a Banach or Hilbert space is expedient not only because they include a number of partial differential equations but also because it offers the possibility to look at ordinary as well as partial differential operators from a single point of view. The origin of the theory of such equations dates from the work of Hille (1948) [1], in which the first existence theorems were obtained for the Cauchy problem for an equation $u' = Au$ with unbounded operator A in a Banach space. They were formulated in terms of semigroups of operators. Appreciating their role in mathematics, E. Hille had written: "I hail a semigroup when I see one and I seem to see them everywhere". During the last 50 years, the theory of operator differential equations, boundary value problems for them and semigroups related to them was enriched with significant results. It became a field of independent interest, attracting the attention of many mathematicians.

We consider the Cauchy problem for a nonhomogeneous equation of the form

$$u'(t) + Au(t) = f(t), \quad t \in [0, \infty),$$

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where A is the generator of a bounded holomorphic semigroup of linear operators in a Banach space $\mathfrak{B}$, and $f(t)$ is a strongly differentiable $\mathfrak{B}$-valued function. The purpose of the present paper is to investigate a behavior in the preassigned points of solutions of the Cauchy problem $u(0) = u_0 \in \mathfrak{B}$ depending on $f(t)$.

1. Preliminaries

Let $\mathfrak{B}$ be a Banach space over the field $\mathbb{C}$ of complex numbers with norm $\|\cdot\|$. Recall that a one-parameter family $\{U(t)\}_{t\geq 0}$ of bounded linear operators on $\mathfrak{B}$ forms a C_0-semigroup in $\mathfrak{B}$ if:

1) $U(0) = I \quad (I$ is the identity operator in $\mathfrak{B})$;
2) $\forall t, s > 0 : U(t+s) = U(t)U(s)$;
3) $\forall x \in \mathfrak{B} : \lim\limits_{t\to 0} \|U(t)x - x\| = 0$.

(As for the theory of C_0-semigroups see, for example, [2], [3], [4] and [5], [6], [7]).

The linear operator A defined as

$$Ax = \lim_{t\to 0} \frac{1}{t}(U(t)x - x),\ \mathcal{D}(A) = \left\{ x \in \mathfrak{B} : \lim_{t\to 0} \frac{1}{t}(U(t)x - x) \text{ exists } \right\},$$

($\mathcal{D}(\cdot)$ denotes the domain of an operator) is called the generating operator or, simply, the generator of $\{U(t)\}_{t\geq 0}$. This operator is closed, $\mathcal{D}(A)$ is dense in $\mathfrak{B}$ and $U(t)$-invariant, that is, $\forall x \in \mathcal{D}(A) : U(t)x \in \mathcal{D}(A)\ (t \geq 0)$ and $AU(t)x = U(t)Ax$. Moreover,

$$\frac{d}{dt}U(t)x = AU(t)x,\ x \in \mathcal{D}(A).$$

A C_0-semigroup $\{U(t)\}_{t\geq 0}$ in $\mathfrak{B}$ is called (strongly) differentiable if for any $x \in \mathfrak{B}$, the $\mathfrak{B}$-valued function $U(t)x$ is strongly differentiable on $(0, \infty)$. As is known (see [3]), for such a semigroup

$$\forall x \in \mathfrak{B}, \forall t > 0 : U(t)x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n),$$

the vector-valued function $U(t)x$ is infinitely differentiable on $(0, \infty)$, and

$$\forall x \in \mathfrak{B}, \forall t > 0, \forall n \in \mathbb{N} : \frac{d^n U(t)x}{dt^n} = A^n U(t)x.$$

Let now $\theta \in \left(0, \frac{\pi}{2}\right]$. A C_0-semigroup $\{U(t)\}_{t\geq 0}$ in $\mathfrak{B}$ is called holomorphic with angle θ (or, simply,holomorphic) if the operator-valued function $U(\cdot)$ is defined in the sector $S_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}$ and:

1) $\forall z_1, z_2 \in S_\theta : U(z_1 + z_2) = U(z_1)U(z_2)$;
2) $\forall x \in \mathfrak{B} : U(z)x$ is holomorphic in S_θ;
3) $\forall x \in \mathfrak{B} : \|U(z)x - x\| \to 0$ as $z \to 0$ in any closed subsector of S_θ.

If in addition the family $U(z)$ is bounded on every sector S_ψ with $\psi < \theta$, then $U(t)$ is called a bounded holomorphic semigroup with angle θ.

2. Main results

Consider now the nonhomogeneous Cauchy problem

$$u'(t) + Au(t) = f(t), \quad t \in (0, \infty), \tag{1}$$

$$u(0) = y_0, \quad y_0 \in \mathfrak{B}, \tag{2}$$

where A is the generator of a bounded holomorphic semigroup $\{U(t)\}_{t\geq 0}$ in $\mathfrak{B}$, and $f(t)$ is a strongly continuously differentiable on $[0, \infty)$ vector function with values in $\mathfrak{B}$. By a solution of problem (1),(2) we mean a continuously differentiable function $u(t) : [0, \infty) \mapsto \mathcal{D}(A)$ satisfying (1) and (2). As has been shown in [6], the general solution of this problem is represented in the form

$$u(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds. \tag{3}$$

We will be interested in a behavior in the given points of its solution depending on $f(t)$. In so doing, we will assume $0 \in \rho(A)$ ($\rho(\cdot)$ is the resolvent set of an operator). Then (see [5], [8])the semigroup $\{U(t)\}_{t\geq 0}$ is exponentially stable, that is,

$$\exists c > 0, \exists \omega > 0, \forall t \in [0, \infty) : \|U(t)\| \leq ce^{-\omega t} \tag{4}$$

(c and ω are constants).

Lemma 1. *For any $t \in (0, \infty)$, there exists the operator $(I - U(t))^{-1}$ (I is the identity operator in $\mathfrak{B}$), which is defined and bounded on the whole space $\mathfrak{B}$.*

Proof. Let $x \in \ker(I - U(t))$. Then, by virtue of the semigroup property 2), $U(nt)x = U^n(t)x = x(n \in \mathbb{N}_0) = \mathbb{N}\bigcup\{0\}$. It follows from (4) that $x = \lim_{n\to\infty} U(nt)x = 0$. So, $\ker(I - U(t)) = \{0\}$, that is, the operator $(I - U(t))^{-1}$ exists.

Show now that $\mathcal{R}(I - U(t)) = \mathfrak{B}$ ($\mathcal{R}(\cdot)$ is the range of an operator). It is not difficult to verify that

$$\forall y \in \mathfrak{B} : (U(nt) - I)y = (U(t) - I)\sum_{k=0}^{n-1} U(kt)y. \tag{5}$$

Moreover, the series $\sum\limits_{k=0}^{\infty} U(kt)y$ converges to some element $x \in \mathfrak{B}$, because

$$\left\| \sum_{k=n}^{\infty} U(kt)y \right\| \leq c\|y\| \sum_{k=n}^{\infty} e^{-\omega_k t} \to 0, \ n \to \infty.$$

The passage to the limit in (5) as $n \to \infty$ yields the equality $y = (I - U(t))x$, i.e. $y \in \mathcal{R}(I - U(t))$, as required. It remains to apply the closed graph theorem. □

Lemma 2. *Let in problem (1), (2) $f(t)$ be such that $\|f(t)\| \to 0$, $t \to \infty$. Then for a solution $u(t)$ of this problem, $\|u(t)\| \to 0$ as $t \to \infty$.*

Proof. According to what has been said above, $u(t)$ is represented in the form (3). It follows from (4) that the first summand in this representation tends to 0 as $t \to \infty$. Let us show that an analogous property holds for the second one, too. Indeed, choose in the equality

$$\int_0^t U(t-s)f(s)ds = \int_0^\tau U(t-s)f(s)ds + \int_\tau^t U(t-s)f(s)ds,$$

a sufficiently large τ such that $\|f(t)\| < \varepsilon$ as $t \geq \tau$ ($\varepsilon > 0$ is arbitrarily small). Then

$$\left\| \int_0^t U(t-s)f(s)ds \right\| \leq \max_{s\in[0,\tau]} \|f(s)\| e^{-\omega(t-\tau)} + \varepsilon \int_0^\infty \|U(s)\|ds.$$

This inequality shows that the second summand in (3) is as small as desired. □

Lemma 3. *Suppose that in the problem* (1),(2),

$$f(t) = p_n(t) + g(t),$$

where

$$p_n(t) = \sum_{k=0}^n x_k t^k,\ x_k \in \mathfrak{B}, \quad \textit{and} \quad \|g(t)\| \to 0 \textit{ as } t \to \infty.$$

Then a solution $u(t)$ of this problem can be represented in the form

$$u(t) = q_n(t) + y(t),$$

where $\|y(t)\| \to 0$ $(t \to \infty)$, and

$$q_n(t) = \sum_{k=0}^n a_k t^k, \quad a_k = \sum_{i=0}^{n-k} (-1)^i \frac{(k+i)!}{k!} A^{-(i+1)} x_{k+i}.$$

Proof. Put

$$v(t) = U(t)y_0 + \int_0^t U(t-s)g(s)ds.$$

Then the representation (3) for $u(t)$ can be written as

$$u(t) = v(t) + \int_0^t U(t-s)p_n(t-s)ds = y(t) + \int_0^\infty U(s)p_n(t-s)ds,$$

where

$$y(t) = v(t) - \int_t^\infty U(s)p_n(t-s)ds = v(t) - \sum_{k=0}^n \int_t^\infty U(s)x_k(t-s)^k ds.$$

Since $v(t)$ is a solution of the problem (1),(2) with $f(t) = g(t)$, in view of Lemma2, we have $\|v(t)\| \to 0$ as $t \to \infty$. Integrating by parts k times the integral under the sign of $\sum$ in the expression for $y(t)$, we obtain

$$\int_t^\infty (t-s)^k U(s) x_k ds = (-1)^k k! U(t) A^{-(k+1)} x_k,$$

whence

$$y(t) = v(t) - U(t) \sum_{k=0}^n (-1)^k k! A^{-(k+1)} x_k.$$

For this reason $\|y(t)\| \to 0$ as $t \to \infty$. It remains to prove that

$$\int_0^\infty U(s) p_n(t-s) ds = q_n(t).$$

But

$$\int_0^\infty U(s) p_n(t-s) ds = \sum_{k=0}^n \int_0^\infty (t-s)^k U(s) x_k ds.$$

The integration by parts k times, taking into account the exponential stability of $\{U(t)\}_{t \geq 0}$, makes possible to conclude that

$$\int_0^\infty (t-s)^k U(s) x_k ds = \sum_{i=0}^k (-1)^i \frac{k!}{(k-i)!} t^{k-i} A^{-(i+1)} x_k.$$

It follows from this equality that $\int_0^\infty U(s) p_n(t-s) ds$ is a polynomial:

$$\int_0^\infty U(s) p_n(t-s) ds = \sum_{k=0}^n \sum_{j=0}^k (-1)^{k-j} \frac{k!}{j!} t^j A^{k-j+1} x_k$$

$$= \sum_{k=0}^n t^k \sum_{i=0}^{n-k} (-1)^i \frac{(k+i)!}{k!} A^{-(i+1)} x_{k+i} = \sum_{k=0}^n t^k a_k = q_n(t).$$

□

Corollary 1. *Suppose that in the problem* (1),(2), $f(t) = x_0 + t x_1$, $(x_0, x_1 \in \mathfrak{B})$. *Then its solution* $u(t)$ *can be represented in the form*

$$u(t) = U(t) y_0 + (I - U(t)) A^{-1} x_0 + (tA - I + U(t)) A^{-2} x_1. \tag{6}$$

The proof follows from Lemma3 if it is taken into account that in the case under consideration $g(t) \equiv 0$ and $n = 1$.

Theorem 1. *For arbitrary $t_1 > 0$, $y_0 \in \mathfrak{B}$, $y_i \in \mathcal{D}(A)(i = 1,2)$, there exists a unique function $f(t)$ of the form $f(t) = x_0 + tx_1$ such that the solution $u(t)$ of problem* (1),(2) *with this function on the right-hand side of* (1) *satisfies the conditions*

$$u(0) = y_0,\ u(t_1) = y_1,\ \lim_{t\to\infty} \frac{u(t)}{t} = y_2. \tag{7}$$

Proof. We shall seek vectors x_0 and x_1 for the solution $u(t)$ of problem (1),(2) with $f(t) = x_0 + tx_1$ to satisfy the relations (7). Because of (6),

$$\lim_{t\to\infty} \frac{u(t)}{t} = \lim_{t\to\infty} \frac{U(t)y_0}{t} + \lim_{t\to\infty} \frac{1}{t}(I - U(t))A^{-1}x_0 + \lim_{t\to\infty} \left(A - \frac{I - U(t)}{t}\right) A^{-2}x_1 = A^{-1}x_1.$$

Thus, $x_1 = Ay_2$. The representation (6) for $t = t_1$ implies also the equality

$$(I - U(t_1))A^{-1}x_0 = y_1 - U(t_1)y_0 - (t_1A - I + U(t_1))A^{-1}y_2.$$

By Lemma1, there exists a bounded operator $(I - U(t_1))^{-1}$. Therefore,

$$x_0 = (I - U(t_1))^{-1}(Ay_1 - AU(t_1)y_0 - (t_1A - I + U(t_1))y_2).$$

So, a function $f(t)$ of the desired form is found. Its uniqueness follows from the uniqueness of searching procedure for x_0 and x_1. □

Theorem 2. *Assume that $\mathfrak{B} = \mathfrak{H}$ is a Hilbert space with scalar product $(\cdot,\cdot)$, and let A be a positive definite selfadjoint operator in it (so, $(Ax, x) \ge \varepsilon(x, x)$ for an arbitrary $x \in \mathcal{D}(A)$ and some $\varepsilon > 0$). Then for any $t_2 > t_1 > 0, y_0 \in \mathfrak{H}$ and $y_1, y_2 \in \mathcal{D}(A)$, there exists a unique function $f(t)$ of the form $f(t) = x_0 + tx_1$ $(x_0, x_1 \in \mathfrak{H})$, such that the solution $u(t)$ of problem* (1),(2) *with this function satisfies the conditions*

$$u(0) = y_0,\ u(t_i) = y_i,\ i = 1, 2. \tag{8}$$

Proof. As in the previous theorem, we seek x_0 and x_1 so that for the solution $u(t)$ of problem (1),(2) with $f(t) = x_0 + tx_1$ (by virtue of Corollary1, it can be represented in the form (6)), to satisfy (8), i.e.

$$(I - U(t_i))A^{-1}x_0 + (t_iA - (I - U(t_i)))A^{-2}x_1 = y_i - U(t_i)y_0\ (i = 1, 2). \tag{9}$$

Applying to both sides of these equalities the operators $I - U(t_2)$ and $I - U(t_1)$ respectively and subtracting the second equality from the first one, we obtain

$$(t_1(I - U(t_2)) - t_2(I - U(t_1)))x_1 = (I - U(t_2))(Ay_1 - AU(t_1)y_0) - (I - U(t_1))(Ay_2 - AU(t_2)y_0). \tag{10}$$

Since

$$U(t) = \int_{\varepsilon}^{\infty} e^{-\lambda t} dE_\lambda$$

(E_λ is the resolution of identity of A), we have

$$t_1(I - U(t_2)) - t_2(I - U(t_1)) = \varphi(A) = \int_{\varepsilon}^{\infty} \varphi(\lambda) dE_\lambda,$$

where the function $\varphi(\lambda) = t_1(1 - e^{-\lambda t_2}) - t_2(1 - e^{-\lambda t_1})$ is such that

$$\varphi(0) = 0, \ \lim_{\lambda \to \infty} \varphi(\lambda) = t_1 - t_2 < 0, \ \varphi'(\lambda) = t_1 t_2 \left(e^{-\lambda t_2} - e^{-\lambda t_1}\right) < 0.$$

Then the function $\frac{1}{\varphi(\lambda)}$ is bounded on $[\varepsilon, \infty)$, and the operator $(\varphi(A))^{-1}$ (the function $(\varphi(\lambda))^{-1}$ of the operator A) is bounded on $\mathfrak{H}$. Applying it to both sides of (10), we arrive at the equality

$$x_1 = (\varphi(A))^{-1}((I - U(t_2))(Ay_1 - AU(t_1)y_0) - (I - U(t_1))(Ay_2 - AU(t_2)y_0)).$$

Taking into account that, by Lemma1, the operator $(I - U(t_1))^{-1}$ exists and it is defined on the whole $\mathfrak{H}$, we can find x_0 from (9). Namely,

$$x_0 = (I - U(t_1))^{-1}(Ay_1 - AU(t_1)y_0 - (t_1 A - (I - U(t_1)))A^{-1}x_1).$$

The uniqueness of a function $f(t)$ of the form mentioned above, which guarantees fulfilment of the conditions (8), follows from the construction itself of x_0 and x_1. □

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Для диференціального рівняння вигляду $u'(t)+Au(t) = f(t), t \in (0,\infty)$, де A – інфінітезімальний генератор обмеженої аналітичної C_0-півгрупи лінійних операторів у банаховому просторі $\mathfrak{B}$, $f(t)$ – $\mathfrak{B}$-значний поліном, досліджується поведінка у наперед заданих точках розв'язків задачі Коші в залежності від $f(t)$.