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ELEMENTARY REMARKS TO THE RELATIVE GROWTH OF SERIES BY THE SYSTEM OF MITTAG-LEFFLER FUNCTIONS

For a regularly converging in $\mathbb{C}$ series $F_\varrho(z)=\sum\limits_{n=1}^{\infty}a_nE_\varrho(\lambda_nz)$, where $0<\varrho<+\infty$ and $E_\varrho(z)=\sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k/\varrho)}$ is the Mittag-Leffler function, it is investigated the asymptotic behavior of the function $E_\varrho^{-1}(M_{F_\varrho}(r))$, where $M_f(r)=\max\{|f(z)|:|z|=r\}$. For example, it is proved that if $\varlimsup\limits_{n\to\infty}\frac{\ln\ln n}{\ln\lambda_n}\le\varrho$ and $a_n\ge0$ for all $n\ge1$, then $\varlimsup\limits_{r\to+\infty}\frac{\ln E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln r}=\frac{1}{1-\overline{\gamma}\varrho}$, where $\overline{\gamma}=\varlimsup\limits_{n\to\infty}\frac{\ln\lambda_n}{\ln\ln(1/a_n)}$.

A similar result is obtained for the Laplace-Stiltjes type integral $I_\varrho(r)=\int\limits_0^{\infty}a(x)E_\varrho(rx)dF(x)$.

Key words and phrases: relative growth, entire function, Mittag-Leffler function, regularly converging series.

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Introduction

Let $f(z)=\sum\limits_{k=0}^{\infty}f_kz^k$, $g(z)=\sum\limits_{k=0}^{\infty}g_kz^k$ be entire functions, $M_f(r)=\max\{|f(z)|:|z|=r\}$ and $\Phi_f(r)=\ln M_f(r)$. The study of growth of the function $\Phi_f^{-1}(\ln M_g(r))$ in terms of the exponential type has begun in [1, 2] and was continued in [3]. As a result, it is proved that if $|f_{k-1}/f_k|\nearrow+\infty$ as $k\to\infty$, then

$$\varlimsup_{r\to+\infty}\frac{\Phi_f^{-1}(\ln M_g(r))}{r}=\varlimsup_{k\to\infty}\left(\frac{|g_k|}{|f_k|}\right)^{1/k}.$$

We remark that $\Phi_f^{-1}(x)=M_f^{-1}(e^x)$ and thus $\Phi_f^{-1}(\ln M_g(r))=M_f^{-1}(M_g(r))$. The order

$$\varrho[g]_f=\varlimsup_{r\to+\infty}\frac{\ln M_f^{-1}(M_g(r))}{\ln r}$$

and the lower order

$$\lambda[g]_f=\varliminf_{r\to+\infty}\frac{\ln M_f^{-1}(M_g(r))}{\ln r}$$

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of the function f with respect to the function g are used in [4]. Research of relative growth of entire functions was continued by many mathematicians (see bibliography in [5]).

Let (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series $F(z)=\sum\limits_{n=1}^{\infty}a_nf(\lambda_nz)$ by the system $\big(f(\lambda_nz)\big)$ regularly convergent in $\mathbb{C}$, i. e.

$$\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty$$

for all $r\in[0,+\infty)$. Many authors have studied the representation of analytic functions by series by the system $\big(f(\lambda_nz)\big)$ and the growth of such functions. We will specify here only the monographs of A. F. Leont'ev [6] and B. V. Vinnitskyi [3]. Recently M. M. Sheremeta [7, 8] studied the growth of the function F with respect to the function f. In particular, for the series by the system of Mittag-Leffler functions he proved the following statement.

Proposition 1. *Let*

$$E_\varrho(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k/\varrho)},\quad 0<\varrho<+\infty,$$

be the Mittag-Leffler function, and the series

$$F_\varrho(z)=\sum_{n=1}^{\infty}a_nE_\varrho(\lambda_nz),\quad 0<\varrho<+\infty, \tag{1}$$

regularly convergent in $\mathbb{C}$. *Suppose that* $\ln n=O(\lambda_n^\varrho)$ *as* $n\to\infty$ *and* $a_n\ge 0$ *for all* $n\ge 1$. *If* $\varrho>1/2$ *and* $\ln n=o(\ln\ln(1/a_n))$ *as* $n\to\infty$, *then*

$$\lim_{r\to+\infty}\frac{\ln\, E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln\, r}=1. \tag{2}$$

In [8] it was conjectured that this statement is also true in the case when $0<\varrho\le 1/2$.

Considering the general case $0<\varrho<+\infty$ by a slightly different method here we obtain results that generalize and supplement Proposition 1.

1 Main results

We need some results from the theory of entire Dirichlet series. Suppose that (μ_n) is an increasing to $+\infty$ sequence of non-negative numbers and

$$D(s)=\sum_{n=1}^{\infty}a_n\exp(s\mu_n),\quad s=\sigma+it, \tag{3}$$

is an entire Dirichlet series. For $\sigma<+\infty$ we put $M_D(\sigma)=\sup\{|D(\sigma+it)|:\, t\in\mathbb{R}\}$. The quantities

$$P_R[D]:=\overline{\lim_{\sigma\to+\infty}}\frac{\ln\,\ln\, M_D(\sigma)}{\ln\,\sigma},\quad p_R[D]:=\underline{\lim_{\sigma\to+\infty}}\frac{\ln\,\ln\, M_D(\sigma)}{\ln\,\sigma}$$

are called the logarithmic R-order and logarithmic lower R-order, respectively. We remark that $P_R[D] \geq p_R[D] \geq 1$. If $P_R[D] = p > 1$, then the quantities

$$T_R[D] := \varlimsup_{\sigma\to+\infty} \frac{\ln M_D(\sigma)}{\sigma^p}, \quad t_R[D] := \varliminf_{\sigma\to+\infty} \frac{\ln M_D(\sigma)}{\sigma^p}$$

are called the logarithmic R-type and logarithmic lower R-type, respectively. Also, we put

$$K_R[D] := \varlimsup_{n\to\infty} \frac{\ln \mu_n}{\ln\left(\dfrac{1}{\mu_n}\ln\dfrac{1}{|a_n|}\right)}, \quad k_R[D] := \varliminf_{n\to\infty} \frac{\ln \mu_n}{\ln\left(\dfrac{1}{\mu_n}\ln\dfrac{1}{|a_n|}\right)}$$

and

$$Q_R[D] := \varlimsup_{n\to\infty} \mu_n^p \left(\ln\frac{1}{|a_n|}\right)^{1-p}, \quad q_R[D] := \varliminf_{n\to\infty} \mu_n^p \left(\ln\frac{1}{|a_n|}\right)^{1-p}.$$

In [9] the following lemmas are proved.

Lemma 1. *If* $\varlimsup\limits_{n\to\infty} \dfrac{\ln\ln n}{\ln \mu_n} \leq 1$, *then* $P_R[D] = K_R[D] + 1$. *If, moreover,*

$$\frac{\ln|a_n| - \ln|a_{n+1}|}{\mu_{n+1} - \mu_n} \nearrow +\infty$$

and $\ln \mu_{n+1} \sim \ln \mu_n$ *as* $n \to \infty$, *then* $p_R[D] = k_R[D] + 1$.

Lemma 2. *If* $\ln n = o\left(\mu_n^{p/(p-1)}\right)$ *as* $n\to\infty$, *then* $T_R[D] = (p-1)^{p-1}p^{-p}Q_R[D]$. *If, moreover,*

$$\frac{\ln|a_n| - \ln|a_{n+1}|}{\mu_{n+1} - \mu_n} \nearrow +\infty$$

and $\mu_{n+1} \sim \mu_n$ *as* $n\to\infty$, *then* $t_R[D] = (p-1)^{p-1}p^{-p}q_R[D]$.

Using Lemma 1, we prove the following theorem.

Theorem 1. *Let* $0 < \varrho < +\infty$, $\varlimsup\limits_{n\to\infty} \dfrac{\ln\ln n}{\ln \lambda_n} \leq \varrho$, $a_n \geq 0$ *for all* $n \geq 1$ *and series* (1) *regularly convergent in* $\mathbb{C}$.

If

$$\overline{\gamma} := \varlimsup_{n\to\infty} \frac{\ln \lambda_n}{\ln\ln(1/a_n)} < \frac{1}{\varrho}, \tag{4}$$

then

$$\varlimsup_{r\to+\infty} \frac{\ln E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln r} = \frac{1}{1-\overline{\gamma}\varrho}. \tag{5}$$

If $\dfrac{\ln a_n - \ln a_{n+1}}{\lambda_{n+1}^\varrho - \lambda_n^\varrho} \nearrow +\infty$, $\ln \lambda_{n+1} \sim \ln \lambda_n$ *as* $n\to\infty$ *and*

$$\underline{\gamma} := \varliminf_{n\to\infty} \frac{\ln \lambda_n}{\ln\ln(1/a_n)} < \frac{1}{\varrho}, \tag{6}$$

then

$$\varliminf_{r\to+\infty} \frac{\ln E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln r} = \frac{1}{1-\underline{\gamma}\varrho}. \tag{7}$$

Proof. It is well known [10, p. 115] that

$$M_{F_\varrho}(r)=E_\varrho(r)=(1+o(1))\varrho e^{r^\varrho},\quad r\to+\infty. \tag{8}$$

Therefore, $E_\varrho^{-1}(x)=(1+o(1))\ln^{1/\varrho}x$ as $x\to+\infty$ and

$$E_\varrho^{-1}(M_{F_\varrho}(r))=(1+o(1))\ln^{1/\varrho}\left(\sum_{n=1}^{\infty}a_ne^{r^\varrho}e^{\lambda_n^\varrho}\right)=(1+o(1))\ln^{1/\varrho}D(\sigma),\quad r\to+\infty, \tag{9}$$

where $\sigma=r^\varrho$ and $\mu_n=\lambda_n^\varrho$. Hence

$$\frac{\ln E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln r}=\frac{(1+o(1))}{\varrho}\frac{\ln\ln D(\sigma)}{(\ln\sigma)/\varrho}=(1+o(1))\frac{\ln\ln D(\sigma)}{\ln\sigma},\quad \sigma\to+\infty \tag{10}$$

and thus by Lemma 1 in view of (4)

$$\varlimsup_{r\to+\infty}\frac{\ln E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln r}=P_R[D]=K_R[D]+1=\varlimsup_{n\to\infty}\frac{\ln\mu_n}{\ln\left(\dfrac{1}{\mu_n}\ln\dfrac{1}{a_n}\right)}+1=$$

$$=\varlimsup_{n\to\infty}\frac{\ln\ln(1/a_n)}{\ln\ln(1/a_n)-\ln\mu_n}=\frac{1}{1-\varlimsup\limits_{n\to\infty}\dfrac{\ln\mu_n}{\ln\ln(1/a_n)}}=\frac{1}{1-\varlimsup\limits_{n\to\infty}\dfrac{\varrho\ln\lambda_n}{\ln\ln(1/a_n)}}=\frac{1}{1-\varrho\overline{\gamma}},$$

i. e. (5) holds. The first part of Theorem 1 is proved.

Now we remark that the conditions $\dfrac{\ln a_n-\ln a_{n+1}}{\lambda_{n+1}^\varrho-\lambda_n^\varrho}\nearrow+\infty$ and $\ln\lambda_{n+1}\sim\ln\lambda_n$ as $n\to\infty$ imply the conditions $\dfrac{\ln a_n-\ln a_{n+1}}{\mu_{n+1}-\mu_n}\nearrow+\infty$ and $\ln\mu_{n+1}\sim\ln\mu_n$ as $n\to\infty$. Therefore, by Lemma 1 from (10) in view of (6) as above we obtain

$$\varliminf_{r\to+\infty}\frac{\ln E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln r}=p_R[D]=k_R[D]+1=\frac{1}{1-\varliminf\limits_{n\to\infty}\dfrac{\varrho\ln\lambda_n}{\ln\ln(1/a_n)}}=\frac{1}{1-\varrho\underline{\gamma}},$$

i. e. (7) holds. The proof of Theorem 1 is complete. □

Remark 1. *Since* $P_R[D]\ge p_R[D]\ge1$ *and the condition* $\ln\lambda_n=o(\ln\ln(1/a_n))$ *as* $n\to\infty$ *implies* $\overline{\gamma}=0$, *from Theorem 1 it follows that if* $\varlimsup\limits_{n\to\infty}\dfrac{\ln\ln n}{\ln\lambda_n}\le\varrho$, $a_n\ge0$ *for all* $n\ge1$ *and* $\ln\lambda_n=o(\ln\ln(1/a_n))$ *as* $n\to\infty$, *then* (2) *holds.*

The growth of the function $E_\varrho^{-1}(M_{F_\varrho}(r))$ should be compared with the growth of the function r^p, where $p=P_R[D]$. Using Lemma 2, we prove the following theorem.

Theorem 2. *Let* $0<\varrho<+\infty$, $P_R[D]=p>1$, $A_p:=(p-1)^{p-1}p^{-p}$, $\ln n=o(\lambda_n^{\varrho p/(p-1)})$ *as* $n\to\infty$, $a_n\ge0$ *for all* $n\ge1$ *and series* (1) *regularly convergent in* $\mathbb{C}$. *Then*

$$\varlimsup_{r\to+\infty}\frac{E_\varrho^{-1}(M_{F_\varrho}(r))}{r^p}=A_p^{1/\varrho}\varlimsup_{n\to\infty}\lambda_n^p\left(\ln\frac{1}{a_n}\right)^{(1-p)/\varrho}. \tag{11}$$

If, moreover, $\dfrac{\ln a_n - \ln a_{n+1}}{\lambda_{n+1}^{\varrho} - \lambda_n^{\varrho}} \nearrow +\infty$ *and* $\lambda_{n+1} \sim \lambda_n$ *as* $n \to \infty$, *then*

$$\varliminf_{r\to+\infty} \frac{E_{\varrho}^{-1}(M_{F_\varrho}(r))}{r^p} = A_p^{1/\varrho} \varliminf_{n\to\infty} \lambda_n^p \left(\ln \frac{1}{a_n}\right)^{(1-p)/\varrho}. \tag{12}$$

Proof. In view of (9) we have

$$\frac{E_{\varrho}^{-1}(M_{F_\varrho}(r))}{r^p} = (1+o(1))\frac{\ln^{1/\varrho} D(\sigma)}{\sigma^{p/\varrho}} = (1+o(1))\left(\frac{\ln D(\sigma)}{\sigma^p}\right)^{1/\varrho}, \quad \sigma = r^{\varrho} \to +\infty.$$

Since $\mu_n = \lambda_n^{\varrho}$, hence by Lemma 2 we get

$$\varlimsup_{r\to+\infty} \frac{E_{\varrho}^{-1}(M_{F_\varrho}(r))}{r^p} = T_R[D]^{1/\varrho} = \left(A_p \varlimsup_{n\to\infty} \lambda_n^{\varrho p} \left(\ln \frac{1}{|a_n|}\right)^{1-p}\right)^{1/\varrho},$$

i. e. (11) holds. The proof of (12) is similar. □

2 Relative growth of Laplace-Stieltjes type integrals

Let V be the class of the function F which are nonnegative, nondecreasing, unbounded and continuous on the right on $[0,+\infty)$. Suppose that $0<\varrho<+\infty$ and positive on $[0,+\infty)$ function a is such that the Laplace-Stieltjes type integral

$$I_{\varrho}(r) = \int_0^{\infty} a(x)E_{\varrho}(rx)dF(x) \tag{13}$$

exists for every $r \in [0,+\infty)$. As in [11, p. 21] we say that the function a has the regular variation regard to F if there exists $\xi \geq 0$, $\eta \geq 0$ and $h>0$ such that $\int_{x-\xi}^{x+\eta} a(t)dF(t) \geq ha(x)$ for all $x \geq \xi$.

As in the proof of Theorem 1 we have

$$E_{\varrho}^{-1}(I_{\varrho}(r)) = (1+o(1))\ln^{1/\varrho}\left(\int_0^{\infty} a(x)e^{r^{\varrho}x^{\varrho}}dF(x)\right) =$$

$$= (1+o(1))\ln^{1/\varrho} I(\sigma), \quad I(\sigma) = \int_0^{\infty} a_1(x)e^{x\sigma}dF_1(x), \tag{14}$$

where $a_1(x) = a(x^{1/\varrho})$, $F_1(x) = F(x^{1/\varrho})$ and $\sigma = r^{\varrho} \to +\infty$.

For the integral $I(\sigma)$ in [11, p. 73] and [12] the following analogs of Lemmas 1 and 2 are proved.

Lemma 3. *If* $F_1 \in V$, $\varlimsup\limits_{x\to+\infty} \dfrac{\ln\ln F_1(x)}{\ln x} \leq 1$ *and a function* a_1 *has the regular variation regard to* F_1, *then*

$$P_R[I] := \varlimsup_{\sigma\to+\infty} \frac{\ln\ln I(\sigma)}{\ln \sigma} = \varlimsup_{x\to+\infty} \frac{\ln x}{\ln\left(\dfrac{1}{x}\ln\dfrac{1}{a_1(x)}\right)} + 1. \tag{15}$$

If, moreover, the function $v(x)=-(\ln a_1(x))'$ is continuous and increasing on $[x_0,+\infty)$, then

$$p_R[I]:=\varliminf_{\sigma\to+\infty}\frac{\ln\ln I(\sigma)}{\ln\sigma}=\varliminf_{x\to+\infty}\frac{\ln x}{\ln\left(\frac{1}{x}\ln\frac{1}{a_1(x)}\right)}+1.$$

Lemma 4. *If $F_1\in V$, $p>1$, $\ln F_1(x)=o(x^{p/(p-1)})$ as $x\to+\infty$ and a_1 has regular variation regard to F_1, then*

$$T_R[I]:=\varlimsup_{\sigma\to+\infty}\frac{\ln I(\sigma)}{\sigma^p}=(p-1)^{p-1}p^{-p}\varlimsup_{x\to+\infty}x^p\left(\ln\frac{1}{a_1(x)}\right)^{1-p}.$$

If, moreover, the function $v(x)=-(\ln a_1(x))'$ is continuous and increasing on $[x_0,+\infty)$, then

$$t_R[I]:=\varliminf_{\sigma\to+\infty}\frac{\ln I(\sigma)}{\sigma^p}=(p-1)^{p-1}p^{-p}\varliminf_{x\to+\infty}x^p\left(\ln\frac{1}{a_1(x)}\right)^{1-p}.$$

Using asymptotic equality (14) and Lemmas 3 and 4, we can prove the following two theorems.

Theorem 3. *If $F\in V$, $0<\varrho<+\infty$, $\varlimsup\limits_{x\to+\infty}\dfrac{\ln\ln F(x)}{\ln x}\le\varrho$ and a function a has the regular variation regard to F, then*

$$\varlimsup_{r\to+\infty}\frac{\ln E_\varrho^{-1}(I_\varrho(r))}{\ln r}=\frac{1}{1-\overline{\gamma}\varrho},\quad \overline{\gamma}:=\varlimsup_{x\to+\infty}\frac{\ln x}{\ln\ln(1/a(x))}.\tag{16}$$

If, moreover, the function $v^*(x)=-\dfrac{(\ln a(x))'}{x^{\varrho-1}}$ is continuous and increasing on $[x_0,+\infty)$, then

$$\varliminf_{r\to+\infty}\frac{\ln E_\varrho^{-1}(I_\varrho(r))}{\ln r}=\frac{1}{1-\underline{\gamma}\varrho},\quad \underline{\gamma}:=\varliminf_{x\to+\infty}\frac{\ln x}{\ln\ln(1/a(x))}.$$

Theorem 4. *If $F\in V$, $p>1$, $0<\varrho<+\infty$, $\ln F(x)=o(x^{\varrho p/(p-1)})$ as $x\to+\infty$ and a function a has the regular variation regard to F, then*

$$\varlimsup_{r\to+\infty}\frac{E_\varrho^{-1}(I_\varrho(r))}{r^p}=A_p^{1/\varrho}\varlimsup_{x\to+\infty}x^p\left(\ln\frac{1}{a(x)}\right)^{(1-p)/\varrho}.$$

If, moreover, the function $v^*(x)=-\dfrac{(\ln a(x))'}{x^{\varrho-1}}$ is continuous and increasing on $[x_0,+\infty)$, then

$$\varliminf_{r\to+\infty}\frac{E_\varrho^{-1}(I_\varrho(r))}{r^p}=A_p^{1/\varrho}\varliminf_{x\to+\infty}x^p\left(\ln\frac{1}{a(x)}\right)^{(1-p)/\varrho}.$$

By analogy, we will stop only on the proof of equality (16). It is easy to verify that the conditions of Theorem 4 imply the conditions of Lemma 3. Therefore, as in the proof of Theorem 1, in view of (14) and (15) we get

$$\varlimsup_{r\to+\infty}\frac{\ln E_\varrho^{-1}(I_\varrho(r))}{\ln r}=\varlimsup_{\sigma\to+\infty}\frac{\ln\ln I(\sigma)}{\ln\sigma}=$$

$$= \frac{1}{1-\varlimsup\limits_{x\to+\infty}\dfrac{\ln x}{\ln\ln\dfrac{1}{a_1(x)}}} = \frac{1}{1-\varlimsup\limits_{x\to+\infty}\dfrac{\ln x}{\ln\ln\dfrac{1}{a(x^{1/\varrho})}}} = \frac{1}{1-\varlimsup\limits_{x\to+\infty}\dfrac{\varrho\ln x}{\ln\ln\dfrac{1}{a(x)}}} = \frac{1}{1-\varrho\overline{\gamma}},$$

i. e. (16) holds.

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Мулява О. М. *Елементарні зауваження до відносного зростання рядів за системою функцій Міттаг-Леффлера* // Буковинський матем. журнал — 2022. — Т.10, №1. — С. 33–40.

Для регулярно збіжного в $\mathbb{C}$ ряду

$$F_\varrho(z)=\sum_{n=1}^{\infty}a_nE_\varrho(\lambda_nz),$$

де $0<\varrho<+\infty$ і $E_\varrho(z)=\sum\limits_{k=0}^{\infty}\dfrac{z^k}{\Gamma(1+k/\varrho)}$ – функція Міттаг-Леффлера, досліджено асимптотичне поводження функції $E_\varrho^{-1}(M_{F_\varrho}(r))$, де $M_f(r)=\max\{|f(z)|:\ |z|=r\}$. Доведено, наприклад, що якщо $a_n\ge 0$ для всіх $n\ge 1$ і

$$\varlimsup_{n\to\infty}\frac{\ln\ln n}{\ln\lambda_n}\le\varrho,$$

то

$$\varlimsup_{r\to+\infty}\frac{\ln\, E_\varrho^{-1}(M_{F_\varrho}(r))}{\ln\, r}=\frac{1}{1-\overline{\gamma}\varrho},$$

де

$$\overline{\gamma}=\varlimsup_{n\to\infty}\frac{\ln\,\lambda_n}{\ln\,\ln\,(1/a_n)}.$$

Подібний результат отримано для інтегралу типу Лапласа-Стілтьєса

$$I_\varrho(r)=\int\limits_0^\infty a(x)E_\varrho(rx)dF(x).$$

Ключові слова і фрази: відносне зростання, ціла функція, функція Міттаг-Лефлера, регулярно збіжний ряд.