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CENTER CONDITIONS FOR A CUBIC DIFFERENTIAL SYSTEM WITH AN INVARIANT CONIC

We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one irreducible invariant conic. The presence of a center at $O(0,0)$ is proved by constructing integrating factors.

Key words and phrases: cubic differential system, the problem of the center, invariant conic, integrating factor.

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Introduction

We consider the cubic system of differential equations

$$\dot{x} = y + p_2(x,y) + p_3(x,y) \equiv P(x,y), \quad \dot{y} = -x + q_2(x,y) + q_3(x,y) \equiv Q(x,y), \qquad (1)$$

where $p_j(x,y)$ and $q_j(x,y)$ are homogeneous polynomials of degree j, $j \in \{2,3\}$ and $P(x,y)$, $Q(x,y) \in \mathbb{R}[x,y]$ are coprime polynomials. The origin $O(0,0)$ is a singular point for (1) with purely imaginary eigenvalues ($\lambda_{1,2} = \pm i$), i.e. a focus or a center. The goal of this work is to find verifiable conditions under which $O(0,0)$ is a center.

The problem of distinguishing between a center and a focus (the problem of the center) is open for general cubic differential systems (1). It is completely solved for: quadratic systems $\dot{x} = y + p_2(x,y),\ \dot{y} = -x + q_2(x,y)$; cubic symmetric systems $\dot{x} = y + p_3(x,y),\ \dot{y} = -x + q_3(x,y)$; the Kukles system $\dot{x} = y,\ \dot{y} = -x + q_2(x,y) + q_3(x,y)$ and some families of polynomial systems of higher degree.

The problem of the center was solved in some particular cases of system (1) and for some families of systems (1) with invariant algebraic curves (see, for example, [2], [3], [4], [5], [10], [11], [12], [14], [15], [16], [17]).

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1 INVARIANT ALGEBRAIC CURVES

We study the problem of the center for a cubic differential system (1) assuming that the system has a real irreducible invariant algebraic curve.

Definition 1. *An algebraic invariant curve of (1) is the solution set in $\mathbb{C}^2$ of an equation $\Phi(x,y)=0$, where Φ is a polynomial in x,y with complex coefficients such that*

$$\frac{d\Phi}{dt}=\frac{\partial\Phi}{\partial x}P+\frac{\partial\Phi}{\partial y}Q=\Phi(x,y)K(x,y) \tag{2}$$

for some polynomial in x,y, $K=K(x,y)$ with complex coefficients, called the cofactor of the invariant algebraic curve $\Phi(x,y)=0$.

According to [4] an invariant straight line of system (1) can have the form

$$C+Ax+By=0,\quad A,B,C\in\mathbb{C},\ (A,B)\neq 0$$

and every irreducible invariant conic has the form

$$1+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2=0 \tag{3}$$

with $(a_{20},a_{11},a_{02})\neq 0,\ a_{20},a_{11},a_{02},a_{10},a_{01}\in\mathbb{C}$.

It is known that a singular point $O(0,0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral [13]

$$x^2+y^2+\sum_{k=3}^{\infty}F_k(x,y)=C$$

in the neighborhood of $O(0,0)$ or an analytic integrating factor of the form [1]

$$\mu(x,y)=1+\sum_{k=1}^{\infty}\mu_k(x,y), \tag{4}$$

where F_k and μ_k are homogeneous polynomials of degree k.

In [4] the problem of the center was solved for cubic differential systems (1) with: four invariant straight lines; three invariant straight lines; two invariant straight lines and one real irreducible invariant conic. In [10], [14] the problem of the center was solved for a cubic differential system (1) with two parallel invariant straight lines that can be reduced to a Liénard type system and in [7] it was solved for a cubic system (1) with two invariant straight lines and one irreducible invariant cubic of the form

$$x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3=0. \tag{5}$$

The center conditions were obtained for a cubic system (1) with: two distinct invariant straight lines in [6]; one invariant straight line and one invariant cubic of the form (5) in [9]; one irreducible invariant cubic of the form (5) in [8]. The presence of a center in these papers was proved by using the method of Darboux integrability and the rational reversibility.

The purpose of this work is to obtain the center conditions for a cubic differential system (1) with an irreducible invariant conic of the form (3) by constructing integrating factors.

2 Cubic systems with one invariant conic

Let us consider the cubic system (1) in the form

$$\begin{aligned}\dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y),\\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y),\end{aligned} \tag{6}$$

where $P(x,y)$ and $Q(x,y)$ are coprime polynomials in $\mathbb{R}[x,y]$. The origin $O(0,0)$ is a singular point which is a center or a focus (a fine focus) for (6).

Assume that the cubic system (6) has a real invariant conic curve of the form (3). By rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$), we can make the curve to be

$$\Phi \equiv 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 = 0, \tag{7}$$

where $(a_{20}, a_{02}) \neq 0,\ (a_{20}, a_{10}) \neq 0,\ (a_{02}, a_{01}) \neq 0,\ a_{20}, a_{02}, a_{10}, a_{01} \in \mathbb{R}$.

In this Section we determinate the condition under which the cubic system (6) has an irreducible invariant conic of the form (7).

Theorem 1. *The cubic differential system* (6) *has an invariant conic of the form* (7) *if and only if one of the following six sets of conditions holds*

(c_1) $k = (aa_{10} + a_{01}a_{10} - ga_{01})/2$, $l = a_{01}a_{10} + ba_{01} - fa_{10}$, $p = (-l)/2$, $n = (a_{01}a_{10}^2 - 2fa_{01}^2 + ba_{01}a_{10} + 2dfa_{01} - 4ma_{01} + fa_{10}^2 - 2cfa_{10} + 4fm)/(2a_{01})$, $q = (2ga_{01}^2 - 2aa_{01}a_{10} - a_{01}^2a_{10} - 2ca_{01}^2 - da_{01}a_{10} + 2cda_{01} - a_{10}^3 + 3ca_{10}^2 - 2c^2a_{10} - 4ma_{10} + 4cm)/(2a_{01})$, $r = 0$, $s = (aa_{10}^2 + (d-2a)a_{01}^2 + 2ada_{01} - 2aca_{10} + 4am - a_{01}^3 + (g-c)a_{01}a_{10} + 2ma_{01})/(2a_{01})$;

(c_2) $k = aa_{10} + a_{01}a_{10} - ga_{01}$, $l = (a_{01}a_{10} + ba_{01} - fa_{10})/2$, $q = (-k)/2$, $m = (aa_{01}a_{10} + a_{01}^2a_{10} + ga_{01}^2 - 2dga_{01} - 2ga_{10}^2 + 2cga_{10} - 4na_{10} + 4gn)/(2a_{10})$, $p = (3da_{01}^2 - a_{01}^3 - (a_{10} + 2b + c)a_{01}a_{10} - 2d^2a_{01} - 4na_{01} - 2da_{10}^2 + 2fa_{10}^2 + 2cda_{10} + 4dn)/(2a_{10})$, $s = 0$, $r = (ba_{01}^2 + (f-d)a_{01}a_{10} - 2bda_{01} - a_{10}^3 + (c-2b)a_{10}^2 + 2bca_{10} + 2na_{10} + 4bn)/(2a_{10})$;

(c_3) $k = (-ga_{01})/2$, $l = (ba_{01})/2$, $p = [(c-b)a_{01}]/2$, $n = [3a_{01}^3 + (4a - 2d - 4f)a_{01}^2 + 4(df - af - 2m)a_{01} + 8fm]/(4a_{01})$, $q = [(g-2c)a_{01}^2 + 2(cd - ac)a_{01} + 4cm]/(2a_{01})$, $r = [a_{01}(2f - a_{01})]/4$, $s = [-a_{01}^3 + 2(ad - a^2 + m)a_{01} + 4am + (d-3a)a_{01}^2]/(2a_{01})$;

(c_4) $b = l = 0$, $a = [a_{01}(2m - a_{01}^2 + da_{01} - 2a_{20})]/(4a_{02})$, $q = (ga_{01}a_{02} + 2pa_{20})/(2a_{02})$, $c = (4ga_{02}^2 - ga_{01}^2a_{02} + 2pa_{01}a_{20})/(4a_{02}a_{20})$, $f = (a_{01}^3a_{02} - da_{01}^2a_{02} - 4a_{01}a_{02}^2 + 2a_{01}a_{02}a_{20} + 2ra_{01}a_{20} + 4da_{02}^2)/(4a_{02}a_{20})$, $k = (-ga_{01})/2$, $n = (2a_{02}^2 - a_{01}^2a_{02} + da_{01}a_{02} - 2a_{02}a_{20} + 2ra_{20})/(2a_{02})$, $s = [a_{20}(2m - a_{01}^2 + da_{01} + 2a_{02} - 2a_{20})]/(2a_{02})$;

(c_5) $k = (aa_{10} + a_{01}a_{10} - ga_{01})/2$, $l = (a_{01}a_{10} + ba_{01} - fa_{10})/2$, $m = (2c\gamma a_{10}^2 - a_{10}^3(a_{01}^2 + \gamma) + 2\gamma a_{10}(a_{01}^2 + aa_{01} - da_{01} + \gamma) - 2g\gamma^2)/(4\gamma a_{10})$, $n = (2a_{01}^2a_{10}^2 - \gamma a_{01}^2 - 2fa_{01}a_{10}^2 + 2d\gamma a_{01} + 2\gamma a_{10}^2 + 2b\gamma a_{10} - 2c\gamma a_{10} - \gamma^2)/(4\gamma)$, $p = [a_{10}^2(f - 2a_{01}) + a_{10}a_{01}(c-b) + \gamma(a_{01} - d)]/(2a_{10})$, $q = (a_{10}^3a_{01} - ca_{10}^2a_{01} + \gamma a_{10}(d - a - 2a_{01}) + g\gamma a_{01})/(2\gamma)$, $r = (2fa_{01}a_{10} - a_{01}^2a_{10} - \gamma a_{10} - 2b\gamma)/(4a_{10})$, $s = [a_{10}(2g\gamma - 2aa_{01}a_{10} - a_{01}^2a_{10} - \gamma a_{10})]/(4\gamma)$, $\gamma = a_{01}^2 - 4a_{02}$;

(c_6) $a = [a_{10}^3a_{01}a_{02}(a_{20}-a_{02}) + a_{10}^2a_{01}a_{02}(ga_{02}-ca_{20}) + a_{10}a_{01}^2a_{20}(a_{01}a_{20}-a_{01}a_{02}+da_{02}-fa_{20}) + ba_{01}^3a_{20}^2]/(a_{10}^3a_{02}^2)$, $k = [a_{10}^2a_{20}a_{02}a_{01}(a_{10}-c) + a_{10}a_{01}^2a_{20}(a_{01}a_{20}-a_{01}a_{02}+da_{02}-fa_{20}) + ba_{01}^3a_{20}^2]/(2a_{10}^2a_{02}^2)$, $m = [2ca_{10}^4a_{02} - 2a_{10}^5a_{02} + a_{10}^3(2a_{01}^2a_{02} - a_{01}^2a_{20} - 2da_{01}a_{02} + fa_{01}a_{20} - 4a_{02}^2 + 6a_{02}a_{20}) + a_{10}^2(4ga_{02}^2 - ba_{01}^2a_{20} - 4ca_{02}a_{20}) + 4a_{10}a_{01}a_{20}(a_{01}a_{20} - a_{01}a_{02} + da_{02} - fa_{20}) + 4ba_{01}^2a_{20}^2]/(2a_{10}^3a_{02})$, $n = [a_{10}^3a_{02} - ca_{10}^2a_{02} + a_{10}(2a_{02}^2 - a_{01}^2a_{02} - a_{01}^2a_{20} + da_{01}a_{02} + fa_{01}a_{20}) + ba_{20}(4a_{02}-a_{01}^2)]/(2a_{10}a_{02})$, $p = [a_{10}^3a_{02}(2f-3a_{01}) + a_{10}^2a_{01}a_{02}(c-2b) + a_{10}(a_{01}^3a_{02} - a_{01}^3a_{20} - da_{01}^2a_{02} + fa_{01}^2a_{20} - 4a_{01}a_{02}^2 + 4a_{01}a_{02}a_{20} + 4da_{02}^2 - 4fa_{02}a_{20}) + ba_{01}a_{20}(4a_{02}-a_{01}^2)]/(2a_{10}^2a_{02})$, $q = [a_{10}^3a_{02}a_{20}(f-3a_{01}) + a_{10}^2a_{01}a_{02}a_{20}(2c-b) + 2a_{10}a_{20}(a_{01}^3a_{02} - a_{01}^3a_{20} - da_{01}^2a_{02} + fa_{01}^2a_{20} - 2a_{01}a_{02}^2 + 2a_{01}a_{02}a_{20} + 2da_{02}^2 - 2fa_{02}a_{20}) + 2ba_{01}a_{20}^2(2a_{02}-a_{01}^2)]/(2a_{10}^2a_{02}^2)$, $r = [a_{10}(2a_{02}-a_{01}^2+fa_{01}) + b(4a_{02}-a_{01}^2)]/(2a_{10})$, $l = (a_{01}a_{10} + ba_{01} - fa_{10})/2$, $s = [a_{10}^4a_{02}a_{20}(c-a_{10}) + a_{10}^3a_{20}(a_{01}^2a_{02} - a_{01}^2a_{20} - da_{01}a_{02} + fa_{01}a_{20} - 2a_{02}^2 + 4a_{02}a_{20}) + a_{10}^2a_{20}(4ga_{02}^2 - ba_{01}^2a_{20} - 4ca_{02}a_{20}) + 4a_{10}a_{01}a_{20}^2(a_{01}a_{20} - a_{01}a_{02} + da_{02} - fa_{20}) + 4ba_{01}^2a_{20}^3]/(2a_{10}^3a_{02}^2)$.

Proof. By Definition 1, the curve (7) is an invariant conic for system (6) if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$ such that

$$P(x,y)\frac{\partial\Phi}{\partial x} + Q(x,y)\frac{\partial\Phi}{\partial y} \equiv \Phi(x,y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y). \tag{8}$$

Identifying the coefficients of the monomials x^iy^j in (8), we find that $c_{01} = a_{10}$, $c_{10} = -a_{01}$, $c_{20} = (a+a_{01})a_{10} - ga_{01}$, $c_{11} = a_{01}^2 - da_{01} - 2a_{02} - a_{10}^2 + ca_{10} + 2a_{20}$, $c_{02} = (f-a_{01})a_{10} - ba_{01}$ and $a_{20}, a_{02}, a_{10}, a_{01}$ satisfy the following system of equations

$$\begin{aligned}
&U_{40} \equiv a_{20}(2k - aa_{10} - a_{01}a_{10} + ga_{01}) = 0,\\
&U_{31} \equiv 2a_{20}^2 + a_{20}(a_{01}^2 - da_{01} - 2a_{02} - a_{10}^2 + ca_{10} - 2m) + 2sa_{02} = 0,\\
&U_{22} \equiv a_{20}(a_{01}a_{10} + ba_{01} - fa_{10} + 2p) - a_{02}(aa_{10} + a_{01}a_{10} - ga_{01} + 2q) = 0,\\
&U_{13} \equiv 2a_{20}(r - a_{02}) + a_{02}(2a_{02} - a_{01}^2 + da_{01} + a_{10}^2 - ca_{10} - 2n) = 0,\\
&U_{04} \equiv a_{02}(a_{01}a_{10} + ba_{01} - fa_{10} - 2l) = 0,\\
&U_{30} \equiv a_{20}(2a + a_{01}) - a_{10}^2(a + a_{01}) + a_{10}(ga_{01} + k) - sa_{01} = 0,\\
&U_{21} \equiv 2a_{02}(a_{10} - g) + a_{20}(2c - 3a_{10}) + a_{01}^2(g - 2a_{10}) +\\
&\qquad + a_{01}(da_{10} - aa_{10} - q) + a_{10}(a_{10}^2 - ca_{10} + m) = 0,\\
&U_{12} \equiv a_{02}(3a_{01} - 2d) + 2a_{20}(f - a_{01}) - a_{01}^3 + da_{01}^2 +\\
&\qquad + a_{01}(2a_{10}^2 + ba_{10} - ca_{10} - n) + a_{10}(p - fa_{10}) = 0,\\
&U_{03} \equiv a_{01}^2(a_{10} + b) - a_{01}(fa_{10} + l) - a_{02}(a_{10} + 2b) + ra_{10} = 0.
\end{aligned} \tag{9}$$

Let $a_{02} = 0$. Then $a_{20} \neq 0$ and the equations $U_{ij} = 0$, $i+j = 4$ of (9) yield $k = (aa_{10} + a_{01}a_{10} - ga_{01})/2$, $p = (fa_{10} - a_{01}a_{10} - ba_{01})/2$, $r = 0$, $a_{20} = (a_{10}^2 - a_{01}^2 + da_{01} - ca_{10} + 2m)/2$. In this case, we express s, q, n and l from the equations $U_{ij} = 0$, $i+j = 3$ of (9) and obtain the set of conditions (c_1) for the existence of the invariant conic

$$2 + 2a_{10}x + 2a_{01}y + (a_{10}^2 - a_{01}^2 + da_{01} - ca_{10} + 2m)x^2 = 0.$$

Assume that $a_{20} = 0$ and let $a_{02} \neq 0$. Then the equations $U_{ij} = 0$, $i+j = 4$ of (9) yield $l = (a_{01}a_{10} + ba_{01} - fa_{10})/2$, $q = (ga_{01} - aa_{10} - a_{01}a_{10})/2$, $s = 0$, $a_{02} = (a_{01}^2 - da_{01} - a_{10}^2 + ca_{10} + 2n)/2$.

We express $r, p, m,\ k$ from the equations $U_{ij}=0,\ i+j=3$ of (9) and obtain the set of conditions (c_2) for the existence of the invariant conic

$$2+2a_{10}x+2a_{01}y+(a_{01}^2-a_{10}^2-da_{01}+ca_{10}+2n)y^2=0.$$

Let $a_{20}a_{02}\neq 0$. In this case, from the equations $U_{ij}=0,\ i+j=4$ of (9) we find l,k,s,q,n. Suppose that $a_{10}=0$. Then $U_{03}\equiv b(a_{01}^2-4a_{02})=0$.

If $a_{02}=a_{01}^2/4$, then $r=[a_{01}(2f-a_{01})]/4$, $p=[(c-b)a_{01}]/2$, $a_{20}=[2m+(d-a-a_{01})a_{01}]/2$. We get the set of conditions (c_3) for the existence of the invariant conic

$$4+4a_{01}y+2[2m+(d-a-a_{01})a_{01}]x^2+a_{01}^2y^2=0.$$

If $b=0$ and $a_{02}\neq a_{01}^2/4$, then we express a, c and f from the equations $U_{ij}=0,\ i+j=3$ of (9). We have the set of conditions (c_4) for the existence of the invariant conic

$$1+a_{01}y+a_{20}x^2+a_{02}y^2=0.$$

Let $a_{20}a_{02}a_{10}\neq 0$. In this case, from the equations $U_{ij}=0,\ i+j=3$ of (9) we find $r,p,\ m$ and $U_{30}\equiv e_1e_2=0$, where $e_1=(a_{01}^2-4a_{02})a_{20}+a_{02}a_{10}^2$, $e_2=a_{10}^3a_{02}(aa_{02}+a_{01}a_{02}-a_{01}a_{20})+a_{10}^2a_{01}a_{02}(ca_{20}-ga_{02})+a_{10}a_{01}^2a_{20}(a_{01}a_{02}-a_{01}a_{20}-da_{02}+fa_{20})-ba_{01}^3a_{20}^2$.

If $e_1=0$, then $a_{20}=(a_{02}a_{10}^2)/(4a_{02}-a_{01}^2)$. We obtain the set of conditions (c_5) for the existence of the invariant conic

$$(4a_{02}-a_{01}^2)(1+a_{10}x+a_{01}y+a_{02}y^2)+a_{02}a_{10}^2x^2=0.$$

If $e_2=0$ and $e_1\neq 0$, then express a from the equation $e_2=0$. In this case we get the set of conditions (c_6) for the existence of the invariant conic

$$1+a_{10}x+a_{01}y+a_{20}x^2+a_{02}y^2=0.$$

□

3 Cubic systems with an integrating factor

Let the cubic system (6) have an irreducible invariant conic, i.e. at least one set of the conditions from Theorem 1 holds. In this section we find the center conditions for cubic system (6) with one invariant conic by constructing integrating factors of the form

$$\mu=\frac{1}{\Phi^h}=\frac{1}{(1+a_{10}x+a_{01}y+a_{20}x^2+a_{02}y^2)^h}, \tag{10}$$

where h is a real parameter.

According to [4] the function (10) is an integrating factor for system (6) if and only if the following identity holds

$$P(x,y)\frac{\partial\mu}{\partial x}+Q(x,y)\frac{\partial\mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)=0. \tag{11}$$

The identity (11) will be be used in finding integrating factors for cubic differential system (6) with an invariant conic (7).

Theorem 2. *The cubic system (6) has an integrating factor of the form (10) if and only if one of the following sixteen sets of conditions holds*

(i) $g=[49k-\alpha(2\beta-7a)]/(7\beta)$, $k=(2\alpha\beta-7a\alpha+7g\beta)/49$, $l=-2p$, $n=[98(af\beta-bf\alpha-m\beta)+7(6f\alpha^2-b\alpha\beta-49fm-5f\beta^2)+2\alpha^2\beta]/(49\beta)$, $p=(7b\beta-7f\alpha-2\alpha\beta)/49$, $q=-4k$, $s=[49(2a^2\beta-2ab\alpha-a\beta^2+m\beta-7am)+7\alpha(6a\alpha+2b\beta-g\beta-\alpha\beta)+5\beta^3]/(49\beta)$, $196f(a\beta-b\alpha)+7(7m\beta-2b\alpha\beta-98fm+12f\alpha^2-10f\beta^2)+4\alpha^2\beta=0$, $r=0$, $\alpha=2b-c$, $\beta=2a-d$, $h=7/2$;

(ii) $a=(3\beta^3+\alpha^2\beta-250fa_{20}-25\beta a_{20}+10f\alpha^2)/(10\beta^2)$, $b=[\alpha(5f+2\beta)]/(5\beta)$, $d=2a-\beta$, $c=[\alpha(10f-\beta)]/(5\beta)$, $k=[(10f+\beta)(25\alpha a_{20}-\alpha^3)+\beta^3(10g+\alpha)]/(50\beta^2)$, $l=p=r=0$, $m=(-10fa_{20})/\beta$, $n=(15fa_{20})/\beta$, $q=-2k$, $s=(6250fa_{20}^2+625\beta a_{20}^2-500f\alpha^2a_{20}-50\alpha^2\beta a_{20}-25\beta^3a_{20}+10f\alpha^4-10g\alpha\beta^3+\alpha^4\beta-\alpha^2\beta^3)/(50\beta^3)$, $h=5/2$;

(iii) $b=[\alpha(fh+\beta)]/(h\beta)$, $c=[\alpha(2fh-h\beta+2\beta)]/(h\beta)$, $d=2a-\beta$, $g=[\alpha(3ah\beta^2-2fh\alpha^2+h\alpha^2\beta-h\beta^3-3\alpha^2\beta)]/(h\beta^3)$, $k=(gh\beta-ah\alpha+\alpha\beta)/(2h^2)$, $l=p=r=0$, $m=[2ah\beta^2+2a_{20}h^2\beta-2fh\alpha^2+\alpha^2\beta(h-3)+\beta^3(1-h)]/(2h^2\beta)$, $n=(-2a_{20}fh-2m\beta)/\beta$, $q=[(h\beta-2fh-3\beta)(4\alpha h^2a_{20}-3\alpha^3)+3\alpha\beta^2(h\beta-2ah-\beta)]/(2h^2\beta^2)$, $s=[\alpha^4(2fh-h\beta+3\beta)+\alpha^2\beta^2(h\beta-2ah-\beta)+2a_{20}h^2\beta^2(\beta-2ah)]/(2h^2\beta^3)$, $2a_{20}h^2(h\beta-2fh-3\beta)+\alpha^2(9\beta-2fh^2+6fh+h^2\beta-6h\beta)+\beta^2(2ah^2-6ah-h^2\beta+4h\beta-3\beta)=0$;

(iv) $f=[49l-\beta(2\alpha-7b)]/(2\alpha)$, $k=[2(2\alpha\beta-7a\alpha+7g\beta)]/49$, $l=(2\alpha\beta-7b\beta+7f\alpha)/49$, $q=(-k)/2$, $m=(2\alpha\beta^2-98ag\beta-7a\alpha\beta+98bg\alpha-343gn-35g\alpha^2+42g\beta^2-98n\alpha)/(49\alpha)$, $p=-4l$, $s=0$, $r=[14(3b\beta^2-7ab\beta+a\alpha\beta+7b^2\alpha)-343bn-49b\alpha^2-7f\alpha\beta+49n\alpha+5\alpha^3-7\alpha\beta^2]/(49\alpha)$, $196g(a\beta-b\alpha)+14(a\alpha\beta+49gn+5g\alpha^2-6g\beta^2)-49n\alpha-4\alpha\beta^2=0$, $\alpha=2b-c$, $\beta=2a-d$, $h=7/2$;

(v) $a=[\beta(5g+2\alpha)]/(5\alpha)$, $d=[\beta(10g-\alpha)]/(5\alpha)$, $k=q=s=0$, $l=(2\alpha\beta-5b\beta+5f\alpha)/25$, $m=(10bg\alpha^2-10g^2\beta^2-25gn\alpha-3g\alpha^3-g\alpha\beta^2-10n\alpha^2)/(5\alpha^2)$, $p=(100bg\alpha^2\beta+10b\alpha^3\beta-10f\alpha^4-100g^2\beta^3-250gn\alpha\beta-30g\alpha^3\beta-10g\alpha\beta^3-25n\alpha^2\beta-4\alpha^4\beta)/(25\alpha^3)$, $r=(50b^2\alpha^2-50bg\beta^2-125bn\alpha-25b\alpha^3-5f\alpha^2\beta+10g\alpha\beta^2+25n\alpha^2+3\alpha^4-\alpha^2\beta^2)/(25\alpha^2)$, $6\alpha^3g+5\alpha^2(n-4bg)+2\alpha g(25n+\beta^2)+20g^2\beta^2=0$, $\alpha=2b-c$, $h=5/2$;

(vi) $a=[\beta(gh+\alpha)]/(h\alpha)$, $d=[\beta(2gh-h\alpha+2\alpha)]/(h\alpha)$, $k=q=s=0$, $l=(\alpha\beta-bh\beta+fh\alpha)/(2h^2)$, $m=[\alpha^3g(1-h)+2\alpha^2h(bg-n)+\alpha g(h\beta^2-2h^2n-3\beta^2)-2g^2h\beta^2)/(\alpha^2h)$, $f=[\beta(3bh\alpha^2-2gh\beta^2-h\alpha^3+h\alpha\beta^2-3\alpha\beta^2)]/(h\alpha^3)$, $p=[\alpha^2\beta(2h^2-3h-1)+2\alpha^2h(2a-2ah-f)+4\alpha\beta bh(1-h)+8\alpha abh^2+\beta^3(3h-2h^2-1)+2\beta^2ah(4h-3)+4\beta h^2(hn-2a^2-n)-8ah^3n)/(2\alpha h^2)$, $r=[\alpha^3(h-1)-2\alpha^2bh^2-\alpha\beta^2h+\alpha\beta h(2a-f)+2\alpha h^2(2b^2+n)+\beta^2bh(2h-1)-4\beta abh^2-4bh^3n)/(2\alpha h^2)$, $\alpha^3g(1-h)+\alpha^2h(2bg+hn-3n)+\alpha g(h\beta^2-2h^2n-3\beta^2)-2g^2h\beta^2=0$, $\alpha=2b-c$;

(vii) $b=c=g=k=l=p=q=0$, $n=[a_{01}^3(3-2h)+4a_{01}^2f(h-1)+4a_{01}(af-2m)+8fm)/(4a_{01})$, $r=[a_{01}(2f-a_{01})]/4$, $s=[a_{01}^3(h-1)+a_{01}^2a(2h-1)+2a_{01}(a^2+m)+4am)/(2a_{01})$, $3a_{01}^3(h-1)+2a_{01}^2(ah-2fh+2f)+4a_{01}(3m-af-hm)-8fm=0$, $a_{01}=(d-2a)/h$;

(viii) $b = c = l = p = 0$, $k = (-a_{01}g)/2$, $n = (8fm - a_{01}^3 + 4fa_{01}^2 + 4(af - 2m)a_{01})/(4a_{01})$, $r = (2fa_{01} - a_{01}^2)/4$, $s = (a_{01}^3 + 3aa_{01}^2 + 2(a^2 + m)a_{01} + 4am)/(2a_{01})$, $q = -k$, $3a_{01}^3 + 4(a - f)a_{01}^2 + 4(m - af)a_{01} - 8fm = 0$, $a_{01} = (d - 2a)/2$;

(ix) $c = 2b$, $d = (fg - ab + 2ag)/g$, $k = (ab - fg)/2$, $l = [b(fg - ab)]/(2g)$, $p = -l$, $q = k$, $m = (a^2b^2 - f^2g^2)/(4bg)$, $n = [(ab - 2ag - fg)(ab - fg)]/(4g^2)$, $r = [(ab + fg)(fg - ab)]/(4g^2)$, $s = [(ab - 2ag - fg)(fg - ab)]/(4bg)$, $h = 1$;

(x) $a = (d - a_{01}h)/2$, $b = c = g = k = l = p = q = 0$, $f = [(3 - h)(a_{01}^3 - da_{01}^2) + 2a_{01}(3a_{20} - 3a_{02} - a_{20}h) + 6da_{02}]/(4a_{20})$, $n = [(h - 1)(da_{01}^2 - a_{01}^3 - 2a_{01}a_{20}) + 2da_{02}]/(2a_{01})$, $r = [(h - 2)(da_{01}^2a_{02} - a_{01}^3a_{02}) + 2a_{01}a_{02}(2a_{20} - a_{02} - a_{20}h) + 2da_{02}^2]/(2a_{01}a_{20})$, $m = (a_{01}^3 - da_{01}^2 + 2a_{01}(a_{20} - a_{02}h) + 2da_{02})/(2a_{01})$, $s = [a_{01}a_{20}(1 - h) + da_{20}]/a_{01}$;

(xi) $a = [-\alpha^2(f + \beta)]/\gamma$, $d = 2a - \beta$, $g = [-\alpha(b\alpha + \gamma)]/\gamma$, $l = (\alpha\beta - b\beta + f\alpha)/2$, $k = (\alpha^2l)/\gamma$, $m = (r\alpha^2)/\gamma$, $n = (2b\alpha\gamma + 2f\alpha^2\beta + \gamma^2 + \beta^2\gamma + 2\alpha^2\beta^2)/(4\gamma)$, $p = l$, $q = k$, $r = (\alpha\gamma - 2b\gamma - 2f\alpha\beta - \alpha\beta^2)/(4\alpha)$, $s = (n\alpha^2)/\gamma$, $c = 2b - \alpha$, $\gamma = 4a_{02} - \beta^2$, $h = 1$;

(xii) $a = [\beta(\alpha^2 - 4b\alpha + \gamma)]/(4\gamma)$, $d = 2a - \beta$, $g = [\alpha(4b\alpha\gamma + 4b\alpha\beta^2 + 4f\alpha^2\beta - \gamma^2 - \gamma\alpha^2)]/(4\gamma^2)$, $n = (4b\gamma\alpha + 8b\alpha\beta^2 - 4f\alpha^2\beta + \gamma^2 - 2\gamma\alpha^2 + \gamma\beta^2 - 4\alpha^2\beta^2)/(16\gamma)$, $m = (2\alpha^2r - n\gamma)/\gamma$, $p = -l$, $q = -k$, $r = (\gamma\alpha - 4b\gamma - 4f\alpha\beta - \alpha\beta^2)/(16\alpha)$, $s = (r\alpha^4)/\gamma^2$, $c = 2b - \alpha$, $\gamma = 16a_{02} - \beta^2$, $h = 2$;

(xiii) $k = (\alpha\beta - ah\alpha + gh\beta)/(2h^2)$, $l = (\alpha\beta - bh\beta + fh\alpha)/(2h^2)$, $m = [\alpha^3(2h\gamma - \gamma + \beta^2) - 4bh\alpha^2\gamma + 2\alpha\gamma(ah\beta - \gamma - h\beta^2 + \beta^2) - 2gh\gamma^2]/(4h^2\alpha\gamma)$, $n = [2\alpha^2(\gamma - fh\beta - \gamma h - \beta^2) + 2bh\alpha\gamma + \gamma(\gamma - 4ah\beta + 2h\beta^2 - \beta^2)]/(4h^2\gamma)$, $p = [(bh\beta - fh\alpha - \alpha\beta)(2h - 3)]/(2h^2)$, $q = [\alpha^3\beta(h - 1) - 2bh\alpha^2\beta + \alpha\gamma(h\beta - ah - 2\beta) - gh\beta\gamma]/(2h^2\gamma)$, $r = [\alpha(\gamma - 2fh\beta - \beta^2) - 2bh\gamma]/(4\alpha h^2)$, $s = [\alpha^2(\beta^2 - 2ah\beta - \gamma) - 2gh\alpha\gamma)/(4h^2\gamma)$, $a = [\alpha^2(2fh^2 - 4fh + 3h\beta - 5\beta) + (h - 1)(\beta\gamma - 2bh\alpha\beta)]/(2h\gamma)$, $g = [-\alpha(6\alpha^2\beta^2(h - 1)(h - 2) - (h - 3)\gamma^2 + 2fh(2h - 5)(h - 1)\alpha^2\beta + (2h - 3)(h - 1)\alpha^2\gamma - 2((h - 1)\beta^2 + \gamma)(2h - 3)bh\alpha)]/(2h\gamma^2)$, $\beta^3(2bh^2 - 3bh - 3h\alpha + 6\alpha) + fh\alpha\beta^2(5 - 2h) + h\beta\gamma(3b - \alpha) - fh\alpha\gamma = 0$, $\alpha = 2b - c$, $\beta = 2a - d$, $\gamma = 4h^2a_{02} - \beta^2$;

(xiv) $a = (8a_{01}b^2 - 27a_{01}^3 - 18a_{01}a_{20} + 72fa_{20} - 32b^2f)/(36a_{01}^2)$, $c = (-4b)/3$, $d = (5a_{01} + 4a)/2$, $g = [b(54a_{01}a_{20} - 9a_{01}^3 - 8b^2a_{01} - 216fa_{20} + 32b^2f)]/(27a_{01}^3)$, $k = (4la_{20})/a_{01}^2$, $p = -2l$, $l = [b(4f - a_{01})]/6$, $r = [a_{01}(2f - a_{01})]/4$, $m = (30a_{01}a_{20} - 9a_{01}^3 - 8b^2a_{01} - 24fa_{20} + 32b^2f)/(24a_{01})$, $n = [5a_{20}(2f - a_{01}) - 2ma_{01}]/(3a_{01})$, $s = [a_{20}(32b^2f - 9a_{01}^3 - 18a_{01}a_{20} - 8b^2a_{01} + 72fa_{20})]/(18a_{01}^3)$, $q = (-8la_{20})/a_{01}^2$, $h = 5/2$;

(xv) $a = [6\alpha^3 - 2h^4a_{01}^2\alpha + h^3(7a_{01}^2\alpha - 8ba_{20} + 4\alpha a_{20}) + h^2\alpha(2\alpha^2 - 5a_{01}^2 - 4a_{20} - 4b\alpha) + 4h\alpha^2(3b - 2\alpha)]/[2(2h - 5)a_{01}h^2\alpha]$, $d = 2a + ha_{01}$, $f = (2h^2a_{01}b - 3ha_{01}(b + \alpha) + 6\alpha a_{01})/[(5 - 2h)\alpha]$, $g = [\alpha(5 - 2h)(a_{01}^2 + aa_{01}) + 4(2bh - h\alpha + \alpha)a_{20}]/(h(2h - 5)a_{01}^2)$, $k = (4la_{20})/a_{01}^2$, $l = [ha_{01}(\alpha - 2b) - a_{01}\alpha]/[2h(2h - 5)]$, $m = [2\alpha^3 - 2h^4a_{01}^2\alpha + h^3(7a_{01}^2\alpha + 8a_{20}b + 4a_{20}\alpha) + h^2\alpha(2\alpha^2 - 5a_{01}^2 - 16a_{20} - 4b\alpha) + 4h\alpha^2(b - \alpha)]/[4\alpha(2h - 5)h^2]$, $n = [2\alpha^3 - 16h^4ba_{20} + 2h^3(a_{01}^2\alpha + 4ba_{20} + 8\alpha a_{20}) - h^2\alpha(5a_{01}^2 + 16a_{20}) + 2h\alpha^2(2b - \alpha)]/[4h^2\alpha(2h - 5)]$, $p = [(2bh - h\alpha + \alpha)(2h - 3)a_{01}]/[2h(2h - 5)]$, $q = (4pa_{20})/a_{01}^2$, $r = [(4bh^2 + 6bh +$

$4h\alpha - 7\alpha)a_{01}^2]/[4\alpha(5-2h)]$, $s = [4\alpha^3 a_{20} - 2h^4\alpha a_{01}^2 a_{20} + h^3\alpha a_{20}(9a_{01}^2 - 8ba_{20} + 4\alpha a_{20}) + 2\alpha h^2 a_{20}(\alpha^2 - 5a_{01}^2 - 2a_{20} - 2b\alpha) + 2ha_{20}\alpha^2(4b-3\alpha)]/[(2h-5)a_{01}^2 h^2\alpha]$, $\alpha = 2b - c$;

(xvi) $a = [a_{10}^3 a_{01}a_{02}(a_{20} - a_{02}) + a_{10}^2 a_{01}a_{02}(ga_{02} - ca_{20}) + a_{10}a_{01}^2 a_{20}(a_{01}a_{20} - a_{01}a_{02} + da_{02} - fa_{20}) + ba_{01}^3 a_{20}^2]/(a_{10}^3 a_{02}^2)$, $k = [a_{10}^2 a_{20}a_{02}a_{01}(a_{10} - c) + a_{10}a_{01}^2 a_{20}(a_{01}a_{20} - a_{01}a_{02} + da_{02} - fa_{20}) + ba_{01}^3 a_{20}^2]/(2a_{10}^2 a_{02}^2)$, $m = [2ca_{10}^4 a_{02} - 2a_{10}^5 a_{02} + a_{10}^3(2a_{01}^2 a_{02} - a_{01}^2 a_{20} - 2da_{01}a_{02} + fa_{01}a_{20} - 4a_{02}^2 + 6a_{02}a_{20}) + a_{10}^2(4ga_{02}^2 - ba_{01}^2 a_{20} - 4ca_{02}a_{20}) + 4a_{10}a_{01}a_{20}(a_{01}a_{20} - a_{01}a_{02} + da_{02} - fa_{20}) + 4ba_{01}^2 a_{20}^2]/(2a_{10}^3 a_{02})$, $n = [a_{10}^3 a_{02} - ca_{10}^2 a_{02} + a_{10}(2a_{02}^2 - a_{01}^2 a_{02} - a_{01}^2 a_{20} + da_{01}a_{02} + fa_{01}a_{20}) + ba_{20}(4a_{02} - a_{01}^2)]/(2a_{10}a_{02})$, $p = [a_{10}^3 a_{02}(2f - 3a_{01}) + a_{10}^2 a_{01}a_{02}(c - 2b) + a_{10}(a_{01}^3 a_{02} - a_{01}^3 a_{20} - da_{01}^2 a_{02} + fa_{01}^2 a_{20} - 4a_{01}a_{02}^2 + 4a_{01}a_{02}a_{20} + 4da_{02}^2 - 4fa_{02}a_{20}) + ba_{01}a_{20}(4a_{02} - a_{01}^2)]/(2a_{10}^2 a_{02})$, $q = [a_{10}^3 a_{02}a_{20}(f - 3a_{01}) + a_{10}^2 a_{01}a_{02}a_{20}(2c - b) + 2a_{10}a_{20}(a_{01}^3 a_{02} - a_{01}^3 a_{20} - da_{01}^2 a_{02} + fa_{01}^2 a_{20} - 2a_{01}a_{02}^2 + 2a_{01}a_{02}a_{20} + 2da_{02}^2 - 2fa_{02}a_{20}) + 2ba_{01}a_{20}^2(2a_{02} - a_{01}^2)]/(2a_{10}^2 a_{02}^2)$, $r = [a_{10}(2a_{02} - a_{01}^2 + fa_{01}) + b(4a_{02} - a_{01}^2)]/(2a_{10})$, $l = (a_{01}a_{10} + ba_{01} - fa_{10})/2$, $s = [a_{10}^4 a_{02}a_{20}(c - a_{10}) + a_{10}^3 a_{20}(a_{01}^2 a_{02} - a_{01}^2 a_{20} - da_{01}a_{02} + fa_{01}a_{20} - 2a_{02}^2 + 4a_{02}a_{20}) + a_{10}^2 a_{20}(4ga_{02}^2 - ba_{01}^2 a_{20} - 4ca_{02}a_{20}) + 4a_{10}a_{01}a_{20}^2(a_{01}a_{20} - a_{01}a_{02} + da_{02} - fa_{20}) + 4ba_{01}^2 a_{20}^3]/(2a_{10}^3 a_{02}^2)$, $g = (-4bh^5 a_{01}^2 a_{20}^2 + 4\alpha h^4 a_{01}a_{20}(a_{01}a_{20} - a_{01}a_{02} + a_{02}d - a_{20}f) + h^3 a_{02}\alpha^2(2a_{02}\alpha - a_{01}^2\alpha + a_{01}d\alpha + 12a_{20}b - 6a_{20}\alpha) + h^2 a_{02}\alpha^3(3a_{01}^2 - 3a_{01}d - 6a_{02} + 6a_{20} + 2b\alpha - \alpha^2) + 2ha_{02}\alpha^4(2\alpha - 3b) - 3a_{02}\alpha^5)/(4h^3 a_{02}^2\alpha^2)$, $d = [a_{01}^3 h^2(a_{02}\alpha + a_{20}bh - a_{20}\alpha) + a_{01}^2 a_{20}fh^2\alpha + a_{01}a_{02}(3h\alpha^3 - 4a_{02}h^2\alpha - 4a_{20}bh^3 + 4a_{20}h^2\alpha - 2bh^2\alpha^2 + 3bh\alpha^2 - 6\alpha^3) + a_{02}f\alpha(5\alpha^2 - 4a_{20}h^2 - 2h\alpha^2)]/[(a_{01}^2 - 4a_{02})a_{02}h^2\alpha]$, $((h-3)a_{01}^2 + 2a_{02})f\alpha + 2(3b - \alpha)a_{01}a_{02}h + (bh - \alpha)(h-3)a_{01}^3 = 0$, $a_{10} = (c - 2b)/h$, $\alpha = 2b - c$.

Proof. Let the cubic system (6) have and an invariant conic $\Phi = 0$ of the form (7). In this case at least one set of the conditions (c_1)–(c_6) from Theorem 1 holds. The system (6) will have an integrating factor of the form (10) if and only if the identity (11) holds. Identifying the coefficients of the monomials $x^i y^j$ in (11), we obtain a system of five equations

$$\{F_{ij} = 0, \quad i + j = 1,\, 2\} \tag{12}$$

for the unknowns $a_{10}, a_{01}, a_{20}, a_{02}, h$ and the coefficients of system (6). We denote $\alpha = 2b - c$, $\beta = 2a - d$ and study the consistency of system (12) in each of the cases (c_1)–(c_6).

1. Let the set of conditions (c_1) hold. Then the equations $F_{10} = 0, F_{01} = 0$ of (12) yield $a_{01} = (-\beta)/h$, $a_{10} = (-\alpha)/h$ and $F_{02} \equiv f_1 f_2 = 0$, where $f_1 = 2h - 7$, $f_2 = (\alpha f - b\beta)h + \alpha\beta$.

If $f_1 = 0$, then $h = 7/2$. We express g from the equation $F_{20} = 0$ and obtain the set of conditions (i) for the existence of the integrating factor (10) with $h = 7/2$ and

$$\Phi = (14a\beta - 8b^2 - 6bc + 5c^2 - 49m - 5\beta^2)x^2 + 14\alpha x + 14\beta y - 49.$$

Assume that $f_1 \neq 0$ and let $f_2 = 0$. Then $f_2 = 0$ yields $b = [(fh + \beta)\alpha]/(h\beta)$. If $h = 5/2$, then we get the set of conditions (ii) for the existence of the integrating factor (10), where

$$\Phi = 5a_{20}x^2 - 2\alpha x - 2\beta y + 5.$$

If $h \neq 5/2$, then express g from the equation $F_{11} = 0$. We have the set of conditions (iii) for the existence of the integrating factor (10) with $\Phi = a_{20}hx^2 - \alpha x - \beta y + h$.

2. Let the set of conditions (c_2) hold. Then the equations $F_{10} = 0, F_{01} = 0$ of (12) yield $a_{01} = (-\beta)/h$, $a_{10} = (-\alpha)/h$ and $F_{20} \equiv g_1 g_2 = 0$, where $g_1 = 2h - 7$, $g_2 = (g\beta - a\alpha)h + \alpha\beta$.

If $g_1 = 0$, then $h = 7/2$. We express f from the equation $F_{02} = 0$ and obtain the set of conditions (iv) for the existence of the integrating factor (10) with $h = 7/2$ and

$$\Phi = (14a\beta - 14b\alpha + 49n + 5\alpha^2 - 5\beta^2)y^2 - 14\alpha x - 14\beta y + 49.$$

Assume that $g_1 \neq 0$ and let $g_2 = 0$. Then $g_2 = 0$ yields $a = [\beta(gh + \alpha)]/(h\alpha)$. If $h = 5/2$, then we get the set of conditions (v) for the existence of the integrating factor (10) with

$$\Phi = (3\alpha^3 + \alpha\beta^2 - 10b\alpha^2 + 10g\beta^2 + 25n\alpha)y^2 - 10\alpha^2 x - 10\alpha\beta y + 25\alpha.$$

If $h \neq 5/2$, then express f from the equation $F_{02} = 0$. In this case we have the set of conditions (vi) for the existence of the integrating factor (10), where

$$\Phi = (3\alpha\beta^2 - 2bh\alpha^2 + 2gh\beta^2 + 2h^2 n\alpha + h\alpha^3 - h\alpha\beta^2 - \alpha^3)y^2 - 2h\alpha^2 x - 2h\alpha\beta y + 2h^2\alpha.$$

3. Let the set of conditions (c_3) hold. Then $F_{01} = 0$, $F_{10} = 0$ yield $c = 2b$, $d = 2a + ha_{01}$.

If $b = g = 0$, then we obtain the set of conditions (vii) for the existence of the integrating factor (10) with $\Phi = (2aa_{01} + 2ha_{01}^2 - 2a_{01}^2 + 4m)x^2 + (a_{01}y + 2)^2$.

If $b = 0$, $h = 2$ and $g \neq 0$, then we get the set of conditions (viii) for the existence of the integrating factor (10), where $\Phi = (2aa_{01} + 2a_{01}^2 + 4m)x^2 + (a_{01}y + 2)^2$.

If $b \neq 0$ and $h = 1$, then express m and a_{01} from the equations $F_{20} = 0$ and $F_{11} = 0$. In this case we have the set of conditions (ix) for the existence of the integrating factor (10) with $\Phi = (ab - fg)^2(gx^2 - by^2) + 4bg(ab - fg)y - 4bg^2$.

4. Let the set of conditions (c_4) hold. In this case the equations $F_{02} = 0$ and $F_{01} = 0$ yield $p = g = 0$. We express r from $F_{11} = 0$ and m from $F_{10} = 0$. We obtain the set of conditions (x) for the existence of the integrating factor (10) with $\Phi = a_{20}x^2 + a_{02}y^2 + a_{01}y + 1$.

5. Let the set of conditions (c_5) hold. Then the equations $F_{10} = 0$, $F_{01} = 0$ of (6) imply $a_{01} = (-\beta)/h$, $a_{10} = (-\alpha)/h$. We express g and a from the equations $F_{11} = 0$ and $F_{02} = 0$.

If $h = 1$, then we obtain the set of conditions (xi) for the existence of the integrating factor (10) with $\Phi = (\beta^2 + \gamma)(\alpha^2 x^2 + \gamma y^2) - 4\gamma(\alpha x + \beta y - 1)$.

If $h = 2$, then we get the set of conditions (xii) for the existence of the integrating factor (10), where $\Phi = (\beta^2 + \gamma)(\alpha^2 x^2 + \gamma y^2) - 8\gamma(\alpha x + \beta y - 2)$.

Assume that $(h - 1)(h - 2) \neq 0$. In this case we have the set of conditions (xiii) for the existence of the integrating factor (10) with $\Phi = (\beta^2 + \gamma)(\alpha^2 x^2 + \gamma y^2) + 4h\gamma(h - \alpha x - \beta y)$.

6. Let the set of conditions (c_6) hold. Then from the equation $F_{01} = 0$ of (12) we find $a_{10} = (-\alpha)/h$ and express g from the equation $F_{11} = 0$.

If $a_{02} = (a_{01}^2)/4$ and $h = 5/2$, then $u = (10b)/3$. In this case we obtain the set of conditions (xiv) for the existence of the integrating factor (10) with

$$\Phi = 12a_{20}x^2 + 3a_{01}^2 y^2 - 16bx + 12a_{01}y + 12.$$

If $a_{02} = (a_{01}^2)/4$ and $h \neq 5/2$, then we express f and d from the equations $F_{02} = 0$ and $F_{10} = 0$ of (12). In this case we get the set of conditions (xv) for the existence of the integrating factor (10), where $\Phi = 4ha_{20}x^2 + ha_{01}^2 y^2 - 4ux + 4ha_{01}y + 4h$.

Assume that $a_{02} \neq (a_{01}^2)/4$. We express d from the equation $F_{02} = 0$ and obtain the set of conditions (xvi) for the existence of the integrating factor (10) with

$$\Phi = h(a_{20}^2 x^2 + a_{02}y^2) - \alpha x + h(a_{01}y + 1).$$

Theorem 2 is proved. $\square$

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Розглядається двовимірна кубічна диференціальна система вигляду

$$\dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3,$$
$$\dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3),$$

в якій всі змінні та коефіцієнти передбачаються дійсними. Початок координат $O(0;0)$ є особливою точкою з чисто уявними коренями характеристичного рівняння ($\lambda_{1,2} = \pm i$) типу центр або фокус. Для даної системи вивчається проблема розрізнення центра та фокуса за наявності однієї дійсної непривідної алгебраїчної інваріантної кривої другого порядку. У роботі отримано шість необхідних і достатніх умов існування інваріантної кривої другого порядку $\Phi \equiv 1+a_{10}x+a_{01}y+a_{20}x^2+a_{02}y^2 = 0$, де $(a_{20}, a_{02}) \neq 0,\ (a_{20}, a_{10}) \neq 0,\ (a_{02}, a_{01}) \neq 0$.

Початок координат є центром для кубічної диференціальної системи тоді і тільки тоді, коли система має в околі особливої точки $O(0;0)$ має аналітичний інтегруючий множник $\mu(x,y) = 1+\sum_{k=1}^{\infty} \mu_k(x,y)$, де μ_k однорідні многочлени степеня k. Отримано шістнадцять серій умов існування аналітичного інтегруючого множника вигляду $\mu^{-1} = \Phi^h$, де $\Phi = 0$ інваріантна крива другого порядку, а h - дійсний параметр. Для кубічної диференціальної системи з особливою точкою типу центр або фокус та з інваріантною кривою другого порядку отримано шістнадцять нових умов існування центру.