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# PROPERTIES OF INTEGRALS WHICH HAVE THE TYPE OF DERIVATIVES OF VOLUME POTENTIALS FOR DEGENERATED $\overrightarrow{2b}$-PARABOLIC EQUATION OF KOLMOGOROV TYPE

In weighted Hölder spaces it is studied the smoothness of integrals, which have the structure and properties of derivatives of volume potentials which generated by fundamental solution of the Cauchy problem for degenerated $\overrightarrow{2b}$-parabolic equation of Kolmogorov type. The coefficients in this equation depend only on the time variable. Special distances and norms are used for constructing of the weighted Hölder spaces.

The results of the paper can be used for establishing of the correct solvability of the Cauchy problem and estimates of solutions of the given non-homogeneous equation in corresponding weighted Hölder spaces.

*Key words and phrases:* $\overrightarrow{2b}$-parabolic equation of Kolmogorov type, an integral which have the type of derivatives of the volume potential, weighted Hölder norm, Hölder space of increasing functions.

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## Introduction

While the fundamental solution is being constructed and investigated, correct solvability of the Cauchy problem is being established and estimates of solutions for parabolic equations are being obtained, properties of the corresponding volume potentials are very important. These properties have been established for parabolic equations in the sense of Petrovsky and for $\overrightarrow{2b}$-parabolic equations in the sense of Eidelman without any degenerations in [1, 2, 10] and for equations with degenerations on the initial hyperplane in works [2, 9, 11, 13, 14]. Volume potentials for the degenerated arbitrary order parabolic equations of the Kolmogorov type (ultraparabolic equations of the Kolmogorov type) were studied in [2, 5, 6, 7, 8].

It is convenient to obtain such properties if the statements about properties of integrals which have the type of derivatives of volume potentials are proved before. These properties

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are described by belonging such integrals to corresponding functional spaces according to the type of spaces which density and kernel of the integral belong to. The densities of these volume potentials belong to Hölder spaces of bounded functions or of functions which increasing as $|x| \to \infty$. Statements of such type are proved in works [2, 3, 4, 10, 12] for parabolic equations in the sense of Petrovsky, for parabolic equations in the sense of Eidelman and for degenerated arbitrary order parabolic equations of the Kolmogorov type.

In this paper we proved some corresponding statements in case of the Kolmogorov type equations, where the major part of equations are parabolic in the sense of Eidelman ($\overrightarrow{2b}$-parabolic) with respect to basic independent variables.

## 1 NOTATIONS AND ASSUMPTIONS

Let $n_1$, $n_2$, $n_3$ be given positive integer numbers such that $0 \le n_3 \le n_2 \le n_1$, $n := n_1 + n_2 + n_3$; $b_1$, ..., $b_{n_1}$ are some numbers in $\mathbb{N}$; $x := (x_1, x_2, x_3) \in \mathbb{R}^n$, $x_l := (x_{l1}, ..., x_{ln_j}) \in \mathbb{R}^{n_l}$, $l \in L := \{1, 2, 3\}$; $T$ is a positive number; if $k_1 := (k_{11}, ..., k_{1n_1}) \in \mathbb{Z}_+^{n_1}$ is a $n_1$-dimensional index, than $\partial_{x_1}^{k_1} := \partial_{x_{11}}^{k_{11}} \cdot ... \cdot \partial_{x_{1n_1}}^{k_{1n_1}}$.

Denote by $\vec{2b}$ the vector $(2b_1, ..., 2b_{n_1})$, by $b$ the least common multiple of the numbers $b_1$, ..., $b_{n_1}$, by $m_j$ a number $b/b_j$, $j \in \{1, ..., n_1\}$.

The paper is concerned with the study of properties of integrals of the type

$$u(t,x) := \int_0^t d\tau \int_{\mathbb{R}^n} M(t,x;\tau,\xi) f(\tau,\xi) d\xi, \quad (t,x) \in \Pi_{(0,T]} := (0,T] \times \mathbb{R}^n. \tag{1}$$

The kernel $M$ is a complex-valued function which has properties of the derivatives of the fundamental solution $G$ of the Cauchy problem for the equation

$$(\partial_t - \sum_{j=1}^{n_2} x_{1j}\partial_{x_{2j}} - \sum_{j=1}^{n_3} x_{2j}\partial_{x_{3j}} - \sum_{\|k_1\| \le 2b} a_{k_1}(t)\partial_{x_1}^{k_1}) u(t,x) = f(t,x), \quad (t,x) \in \Pi_{(0,T]}, \tag{2}$$

where $\|k_1\| := \sum_{j=1}^{n_1} m_j k_{1j}$ for $k_1 \in \mathbb{Z}_+^{n_1}$. In the equation (2) coefficients $a_{k_1}$ are continuous on $[0,T]$ functions and differential expression $\partial_t - \sum_{\|k_1\| \le 2b} a_{k_1}(t)\partial_{x_1}^{k_1}$ is uniformly $\vec{2b}$-parabolic on $[0,T] \times \mathbb{R}^{n_1}$, that there exists a constant $\delta > 0$ such that for all $t \in [0,T]$ and $\sigma_1 \in \mathbb{R}^{n_1}$ the inequality

$$\operatorname{Re} \sum_{\|k_1\| = 2b} a_{k_1}(t)(i\sigma_1)^{k_1} \le -\delta \sum_{j=1}^{n_1} \sigma_{1j}^{2b_j}$$

is valid. In the expression $i$ is an imaginary unit.

If $n_3 \ge 1$ then the equation (2) degenerates with respect to two groups of variables $x_2$ and $x_3$. When $n_3 = 0$ and $n_2 \ge 1$ then in the equations (2) the second sum is missing and degeneration is present with respect to one group of variables $x_2$. In the case $n_2 = n_3 = 0$ the equation (2) doesn't have the first two sums and it isn't degenerated.

The equation (2) with $n_2 > 0$ is degenerated equation of Kolmogorov type with $2\vec{b}$-parabolic part with respect to main variables. It generalizes the known equation of A.N.Kolmogorov of diffusion with inertia. There is a fundamental solution of Cauchy problem (FSCP) $G$ for it, detail properties of which is given in [2].

In [2] it was established a structure and properties of the function $G$ and its derivatives.

Let us describe properties of the kernel $M$ of the integral (1). For this purpose we denote: $q_j := 2b_j/(2b_j - 1)$, $j \in \{1, ..., n_1\}$; $m'$ and $m''$ are the most and the least of the numbers $m_j$, $j \in \{1, ..., n_1\}$; $N_s := \sum\limits_{l=1}^{s}\sum\limits_{j=1}^{n_l}(2b(l-1) + m_j)/(2b)$, $s \in L$, $N := N_3$. For any $\{x, x', \xi\} \subset \mathbb{R}^n$: $\Delta_x^{x'} f(t, x) := f(t, x) - f(t, x')$; $\overline{x}_{1j}(t) := x_{1j}$, $j \in \{1, ..., n_1\}$; $\overline{x}_{2j}(t) := x_{2j} + tx_{1j}$, $j \in \{1, ..., n_2\}$; $\overline{x}_{3j}(t) := x_{3j} + tx_{2j} + (t^2/2)x_{1j}$, $j \in \{1, ..., n_3\}$; $\overline{x}_l(t) := (\overline{x}_{l1}(t), ..., \overline{x}_{ln_l}(t))$, $l \in M$; $X_1(t) := (\overline{x}_1(t), \overline{x}_2(t), \overline{x}_3(t))$; $X_2(t) := (\xi_1, \overline{x}_2(t), \overline{x}_3(t))$; $X_3(t) := (\xi_1, \xi_2, \overline{x}_3(t))$; $\rho(t, x, \xi) := \sum\limits_{l=1}^{3}\sum\limits_{j=1}^{n_l} t^{1-lq_j}|\overline{x}_{lj}(t) - \xi_{lj}|^{q_j}$; $d(x; \xi; \lambda) := \sum\limits_{l=1}^{3}\sum\limits_{j=1}^{n_l}|x_{lj} - \xi_{lj}|^{\lambda/(2b(l-1)+m_j)}$, $d_1(x; \xi; \lambda) := \sum\limits_{j=1}^{n_1}|x_{1j} - \xi_{1j}|^{\lambda/m_j} + \sum\limits_{l=1}^{2}\sum\limits_{j=1}^{n_l}|x_{lj} - \xi_{lj}|^{(\lambda+m')/(2b(l-1)+m_j)}$, $d_2(x; \xi; \lambda) := \sum\limits_{j=1}^{n_1}|x_{1j} - \xi_{1j}|^{\lambda/m_j} + \sum\limits_{j=1}^{n_2}|x_{2j} - \xi_{2j}|^{(\lambda+m')/(2b+m_j)} + \sum\limits_{j=1}^{n_3}|x_{3j} - \xi_{3j}|^{(\lambda+2b+m')/(4b+m_j)}$, if $\lambda \in (0, 1]$; $d(x; \xi) := d(x; \xi; 1)$.

Note, that if $d(x; x') < 1$, then with some $c > 0$ the next inequalities

$$d_2(x; x'; \lambda) \le d_1(x; x'; \lambda) \le d(x; x'; \lambda) \le cd(x; x')^{\lambda},\ \{x, x'\} \subset \mathbb{R}^n, \lambda \in (0, 1]$$

are hold.

As the kernel of the integral (1), let us take the function $M$, which can be represented in the form

$$M(t, x; \tau, \xi) := (t - \tau)^{-\nu-N}\Omega(t, x; \tau, \xi),\quad 0 \le \tau < t \le T,\ \{x, \xi\} \subset \mathbb{R}^n, \tag{3}$$

where $\nu \in (0, 2 + m'/(2b)]$, and the function $\Omega$, with the values in $\mathbb{C}$, is continuous and it satisfies the conditions below with some numbers $c > 0$ and $\gamma \in (0, 1]$

$A_1$. $\forall\{t, \tau\} \subset (0, T],\ \tau < t,\ \forall x \in \mathbb{R}^n$ :

$$\int\limits_{\mathbb{R}^n} \Omega(t, x; \tau, \xi)d\xi = 0 \ \text{ for } \nu \in (1 - 1/(2b), 1],$$

$$\int\limits_{\mathbb{R}^{n_2+n_3}} \Omega(t, x; \tau, \xi)d\xi_2 d\xi_3 = 0 \ \text{ for } \nu \in (1, 1 + m'/(2b)], \tag{4}$$

$$\int\limits_{\mathbb{R}^{n_3}} \Omega(t, x; \tau, \xi)d\xi_3 = 0 \quad \text{for } \nu \in (1 + m'/(2b), 2 + m'/(2b)];$$

$A_2$. $\exists C > 0\ \forall\{t, \tau\} \subset (0, T],\ \tau < t,\ \forall\{x, \xi\} \subset \mathbb{R}^n$ :

$$|\Omega(t, x; \tau, \xi)| \le C\exp\{-c\rho(t - \tau, x, \xi\}; \tag{5}$$

$A_3$. $\exists C > 0\ \forall\{t, \tau\} \subset (0, T],\ \tau < t,\ \forall\{x, x', \xi\} \subset \mathbb{R}^n,\ d(x; x') < (t - \tau)^{1/(2b)}$ :

$$|\Delta_x^{x'}\Omega(t, x; \tau, \xi)| \le C(d(x; x'))^{\gamma}(t - \tau)^{-\gamma/(2b)}\exp\{-c\rho(t - \tau, x, \xi)\} \tag{6}$$

are true. In the conditions $A_2$, $A_3$ and below, we denote by the letter $C$ all positive constants the values of which are unimportant.

The definition of the function $M$ contains the number $\nu$, $c$ and $\gamma$, which assume are considered to be given. By $\mathcal{M}(\nu, c, \gamma)$ we denote a set of all functions $M$ determined by the formula (3), in which the function $\Omega$ satisfies conditions $A_1 - A_3$ with given $\gamma \in (0,1]$, $\nu \in (0, 2+m'/(2b)]$, $c \in \mathbb{R}_+$.

It should be noted that for $\nu \in [1, 2+m'/(2b)]$ integral (1) with the function $M \in \mathcal{M}(\nu, c, \gamma)$ is treated as the limit

$$\lim_{h\to 0} \int\limits_0^{t-h} d\tau \int\limits_{\mathbb{R}^n} M(t,x;\tau,\xi) f(\tau,\xi) d\xi,$$

which exists for suitable $f$, because of condition $A_1$.

Let us define spaces to which the functions $f$ and $u$ belong. They are the spaces of functions which are continuous or satisfy Hölder condition and which have certain restrictions as $|x| \to \infty$. Their behavior as $|x| \to \infty$ will be described by the functions

$$\varphi(t,x) := \exp \sum_{l=1}^{3} \sum_{j=1}^{n_l} k_{lj}(t, a_{lj}) |x_{lj}|^{q_j}$$

or

$$\psi(t,x) := \exp \sum_{l=1}^{3} \sum_{j=1}^{n_l} s_{lj}(t) |x_{lj}|^{q_j}, \ t \in [0,T], \ x \in \mathbb{R}^n.$$

Here for a fixed number $c_0$ from the interval $(0, c)$, where $c$ is the constant from the conditions $A_2$ and $A_3$, and for a set $a = (a_1, a_2, a_3) \in \mathbb{R}^n$, $a_l := (a_{l1}, ..., a_{ln_l})$, $l \in M$, of non-negative numbers $a_{lj}$, $j \in \{1, ..., n_l\}$, $l \in M$, such that $T < \min\limits_{l\in M, j\in\{1,...,n_l\}} (c_0/a_{lj})^{(2b_j-1)/(2b_j(l-1)+1)}$:

$$k_{lj}(t, a_{lj}) := c_0 a_{lj} (c_0^{2b_j-1} - a_{lj}^{2b_j-1} t^{2b_j(l-1)+1})^{1-q_j}, \ j \in \{1, ..., n_l\}, \ l \in L;$$

$$s_{1j}(t) := k_{1j}(t, a_{1j}) + 2^{q_j-1}\theta(n_2 - j) t^{q_j} k_{2j}(t, a_{2j}) + 2^{q_j-2}\theta(n_3 - j) t^{2q_j} k_{3j}(t, a_{3j}), j \in \{1, ..., n_1\};$$

$$s_{2j}(t) := 2^{q_j-1} k_{2j}(t, a_{2j}) + 4^{q_j-1}\theta(n_3 - j) t^{q_j} k_{3j}(t, a_{3j}), \ j \in \{1, ..., n_2\};$$

$$s_{3j}(t) := 4^{q_j-1} k_{3j}(t, a_{3j}), \ j \in \{1, ..., n_3\}; \ t \in [0,T],$$

where $\theta(\tau) = 1$ for $\tau \geq 0$ and $\theta(\tau) = 0$ for $\tau < 0$.

The functions $k(t) := (k_1(t, a_1), k_2(t, a_2), k_3(t, a_3))$, $k_l(t, a_l) := (k_{l1}(t, a_{l1}), ..., k_{ln_l}(t, a_{ln_l}))$, $l \in L$, and $s(t) := (s_1(t), s_2(t), s_3(t))$, where $s_l(t) := (s_{l1}(t), ..., s_{ln_l}(t))$, $l \in L$, $t \in [0,T]$, have the following properties [2, 8]:

$$k(0) = a, a_{lj} \leq k_{lj}(\tau, a_{lj}) < k_{lj}(t, a_{lj}) < s_{lj}(t), \quad 0 \leq \tau < t \leq T, j \in \{1, ..., n_l\}, l \in L; \tag{7}$$

$$k_{lj}(t-\tau, k_{lj}(\tau, a_{lj})) \leq k_{lj}(t, a_{lj}), 0 \leq \tau \leq t \leq T, j \in \{1, ..., n_l\}, l \in L; \tag{8}$$

$$-c_0 \rho(t-\tau, x, \xi) + \sum_{l=1}^{3} \sum_{j=1}^{n_j} a_{lj} |x_{lj}|^{q_j} \leq \sum_{l=1}^{3} \sum_{j=1}^{n_j} k_{lj}(t, a_{lj}) |\bar{x}_{lj}(t)|^{q_j} \leq$$

$$\leq \sum_{l=1}^{3}\sum_{j=1}^{n_j} s_{lj}(t)|x_{lj}|^{q_j}, \quad 0 \leq \tau < t \leq T, \ \{x,\xi\} \subset \mathbb{R}^n, \tag{9}$$

$$-c_0\rho(t-\tau,x,\xi)\sum_{l=1}^{3}\sum_{j=1}^{n_j} k_{lj}(\tau,a_{lj})|\xi_{lj}|^{q_j} \leq \sum_{l=1}^{3}\sum_{j=1}^{n_j} k_{lj}(t,a_{lj})|\bar{x}_{lj}|^{q_j} \leq$$

$$\leq \sum_{l=1}^{3}\sum_{j=1}^{n_j} s_{lj}(t)|x_{lj}|^{q_j}, \quad 0 \leq \tau < t \leq T, \ \{x,\xi\} \subset \mathbb{R}^n. \tag{10}$$

From these properties it is follows that

$$\varphi(\tau, X_1(t-\tau)) \leq \varphi(t, X_1(t)) \leq \psi(t,x),$$

$$\exp\{-c_0\rho(t-\tau,x,\xi)\}\varphi(\tau,\xi) \leq \psi(t,x), \quad 0 \leq \tau < t \leq T, \quad \{x,\xi\} \subset \mathbb{R}^n. \tag{11}$$

For a given number $\lambda \in (0,1]$ we denote by $C^0$, $C^\lambda_\varphi$, $C^\lambda_{1,\varphi}$ and $C^\lambda_{2,\varphi}$ the spaces of continuous functions $u: \ \Pi_{[0,T]} \to \mathbb{C}$, for which the corresponding norms $||u||^0_\varphi$, $||u||^\lambda_\varphi := ||u||^0_\varphi + [u]^\lambda_\varphi$, $||u||^\lambda_{1,\varphi} := ||u||^0_\varphi + [u]^\lambda_{1,\varphi}$ and $||u||^\lambda_{2,\varphi} := ||u||^0_\varphi + [u]^\lambda_{2,\varphi}$ are finite, where

$$||u||^0_\varphi := \sup_{(t,x)\in\Pi_{[0,T]}} \frac{|u(t,x)|}{\varphi(t,x)},$$

$$[u]^\lambda_\varphi := \sup_{\substack{\{(t,x),(t,x')\}\subset\Pi_{[0,T]}\\ (t,x)\neq(t,x')}} \frac{|\Delta^{x'}_x u(t,x)|}{(d(x;x'))^\lambda(\varphi(t,x)+\varphi(t,x'))},$$

$$[u]^\lambda_{1,\varphi} := \sup_{\substack{\{(t,x),(t,x')\}\subset\Pi_{[0,T]}\\ (t,x)\neq(t,x')}} \frac{|\Delta^{x'}_x u(t,x)|}{d_1(x;x';\lambda)(\varphi(t,x)+\varphi(t,x'))},$$

$$[u]^\lambda_{2,\varphi} := \sup_{\substack{\{(t,x),(t,x')\}\subset\Pi_{[0,T]}\\ (t,x)\neq(t,x')}} \frac{|\Delta^{x'}_x u(t,x)|}{d_2(x;x';\lambda)(\varphi(t,x)+\varphi(t,x'))}.$$

Except these spaces we will use the space $C^\lambda_\psi$. The definition of this space is obtained if in the definition of the space $C^\lambda_\varphi$ the function $\varphi$ replace by the function $\psi$.

## 2 Main theorem

Let us formulate the main results of this paper.

**Theorem.** *Let $M \in \mathcal{M}(\nu,c,\gamma)$ and function $u$ is determined by the formula* (1). *Then the following statements are valid:*

**a)** *if $\nu \leq 1 - 1/(2b)$ and $f \in C^0$, then $u \in C^\gamma_\psi$ and*

$$||u||^\gamma_\psi \leq C||f||^0_\varphi; \tag{12}$$

**b)** *if $\nu \in (1-1/(2b),1]$ and $f \in C^\lambda_\varphi$, $\lambda \in (0,1]$, then with $\nu + (\gamma-\lambda)/(2b) < 1$ we have $u \in C^\gamma_\psi$ and*

$$||u||^\gamma_\psi \leq C||f||^\lambda_\varphi, \tag{13}$$

*and with* $\nu+(\gamma-\lambda)/(2b)>1$ *we have* $u\in C^{\lambda}_{\psi}$ *and*

$$||u||^{\lambda}_{\psi}\le C||f||^{\lambda}_{\varphi}; \tag{14}$$

**c)** *if* $\nu\in(1,1+m'/(2b)]$ *and* $f\in C^{\lambda}_{1,\varphi}$, $\lambda\in(0,1]$, *then with* $\nu+(\gamma-m'-\lambda)/(2b)<1$ *we have* $u\in C^{\gamma}_{\psi}$ *and*

$$||u||^{\gamma}_{\psi}\le C||f||^{\lambda}_{1,\varphi}, \tag{15}$$

*and with* $\nu+(\gamma-m'-\lambda)/(2b)>1$ *we have* $u\in C^{\lambda}_{\psi}$ *and*

$$||u||^{\lambda}_{\psi}\le C||f||^{\lambda}_{1,\varphi}; \tag{16}$$

**d)** *if* $\nu\in(1+m'/(2b),2+m'/(2b)]$ *and* $f\in C^{\lambda}_{2,\varphi}$, $\lambda\in(0,1]$, *then with* $\nu-1+(\gamma-m'-\lambda)/(2b)<1$ *we have* $u\in C^{\gamma}_{\psi}$ *and*

$$||u||^{\gamma}_{\psi}\le C||f||^{\lambda}_{2,\varphi}, \tag{17}$$

*and with* $\nu-1+(\gamma-m'-\lambda)/(2b)>1$ *we have* $u\in C^{\lambda}_{\psi}$ *and*

$$||u||^{\lambda}_{\psi}\le C||f||^{\lambda}_{2,\varphi}. \tag{18}$$

*The constants* $C$ *in the inequalities* (12)–(18) *depend only on the constant* $C$ *from the conditions* $A_2$ *and* $A_3$, *and also they depend on the numbers* $n_1$, $n_2$, $n_3$, $b$, $\nu$, $c$, $\gamma$ *and* $\lambda$.

*Proof.* We will denote by the same letters the below various constants if we have no interest in constant's values.

**a)** Using the equality [2]

$$\int\limits_{\mathbb{R}^n}(t-\tau)^{-N}\exp\{-c'\rho(t-\tau,x,\xi)\}d\xi=C,\quad 0<\tau<t\le T,\ x\in\mathbb{R}^n,\ c'>0, \tag{19}$$

with the help of (3), (5), (11) and of the definition of the norm $||f||^{0}_{\varphi}$ we have

$$|u(t,x)|\le C\int\limits_0^t(t-\tau)^{-\nu-N}d\tau\int\limits_{\mathbb{R}^n}\exp\{-c\rho(t-\tau,x,\xi)\}|f(\tau,\xi)|d\xi$$

$$=C\int\limits_0^t(t-\tau)^{-\nu-N}d\tau\int\limits_{\mathbb{R}^n}\exp\{-c_0\rho(t-\tau,x,\xi)\}\varphi(\tau,\xi)\frac{|f(\tau,\xi)|}{\varphi(\tau,\xi)}\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}d\xi$$

$$\le C\psi(t,x)\int\limits_0^t(t-\tau)^{-\nu}d\tau||f||^{0}_{\varphi}=C\psi(t,x)t^{1-\nu}||f||^{0}_{\varphi},\quad (t,x)\in\Pi_{(0,T]}. \tag{20}$$

Let $x$ and $x'$ be arbitrary fixed points from $\mathbb{R}^n$ and $d:=d(x;x')$. Let us estimate the difference $\Delta^{x'}_{x}u$.

When $d^{2b}\ge t$, with the help of estimate (20) we obtain

$$|\Delta^{x'}_{x}u(t,x)|\le|u(t,x)|+|u(t,x')|\le C(\psi(t,x)+\psi(t,x'))t^{1-\nu}||f||^{0}_{\varphi}$$

$$\leq C(\psi(t,x)+\psi(t,x'))(d(x;x'))^{\gamma}t^{1-\nu-\gamma/(2b)}||f||^0_{\varphi},\quad t\in(0,T],\ \{x,x'\}\subset\mathbb{R}^n,\ \gamma\in(0,1]. \quad (21)$$

Let us consider the case when $d^{2b}<t$. We have

$$|\Delta_x^{x'}u(t,x)|\leq\int_0^t d\tau\int_{\mathbb{R}^n}|\Delta_x^{x'}M(t,x;\tau,\xi)|\,|f(\tau,\xi)|d\xi,\quad t\in(0,T],\ \{x,x'\}\subset\mathbb{R}^n. \quad (22)$$

Let us prove for the difference $\Delta M:=\Delta_x^{x'}M(t,x;\tau,\xi)$ the inequality

$$|\Delta M|\leq Cd^{\gamma}(t-\tau)^{-\gamma/(2b)-\nu-N}\exp\{-c\rho(t-\tau,x,\xi)\}. \quad (23)$$

We shall distinguish the following cases: 1) $d^{2b}\geq t-\tau$, 2) $d^{2b}<t-\tau$.

In the first case, we obtain estimate (23) immediately from (3), (5) and from the inequality $|\Delta M|\leq|M(t,x;\tau,\xi)|+|M(t,x';\tau,\xi)|$. In the case 2) note that

$$\Delta M=(t-\tau)^{-\nu-N}\Delta_x^{x'}\Omega(t,x;\tau,\xi).$$

Because of (6) we have estimate (23) in case 2).

With the help of (11), (19), (22) and (23) we get

$$|\Delta_x^{x'}u(t,x)|\leq C(\psi(t,x)+\psi(t,x'))d^{\gamma}t^{1-\nu-\gamma/(2b)}||f||^0_{\varphi},$$
$$t\in(0,T],\ \{x,x'\}\subset\mathbb{R}^n,\gamma\in(0,1]. \quad (24)$$

From (21) and (24) the estimate

$$[u]^{\gamma}_{\psi}\leq C||f||^0_{\varphi}$$

follows and by this result and (20) the estimate (12) holds.

**b)** Let $\nu\in(1-1/(2b),1]$. Because of the first condition from (4) we represent the integral (1) in the form

$$u(t,x)=\int_0^t d\tau\int_{\mathbb{R}^n}M(t,x;\tau,\xi)\Delta_{\xi}^{X_1(t-\tau)}f(\tau,\xi)d\xi,\quad (t,x)\in\Pi_{(0,T]}. \quad (25)$$

With the help of (3), (5) and (7)–(11) we get

$$|u(t,x)|\leq C\int_0^t(t-\tau)^{-\nu-N}d\tau\int_{\mathbb{R}^n}\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}\exp\{-c_0\rho(t-\tau,x,\xi)\}$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau)))\frac{|\Delta_{\xi}^{X_1(t-\tau)}f(\tau,\xi)|}{\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau))}d\xi\leq C\int_0^t(t-\tau)^{-\nu-N}d\tau$$

$$\times\int_{\mathbb{R}^n}\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}(d(\xi,X_1(t-\tau)))^{\lambda}d\xi\psi(t,x)[f]^{\lambda}_{\varphi}.$$

Now let us use the inequality [2]

$$(d(\xi, X_1(t-\tau)))^{\lambda} \exp\{-\bar{c}\rho(t-\tau, x, \xi)\} \leq C(t-\tau)^{\lambda/(2b)} \exp\{-\bar{c}_1\rho(t-\tau, x, \xi)\},$$

$$0 \leq \tau < t \leq T,\ \{x, \xi\} \subset \mathbb{R}^n,\ 0 < \bar{c}_1 < \bar{c},\ \lambda \in (0,1]. \tag{26}$$

For $\bar{c} = c - c_0$ with the help of (19) we have

$$|u(t,x)| \leq C \int\limits_0^t (t-\tau)^{-\nu-N+\lambda/(2b)} d\tau \int\limits_{\mathbb{R}^n} \exp\{-\bar{c}_1\rho(t-\tau, x, \xi)\} d\xi \psi(t,x)[f]^{\lambda}_{\varphi}$$

$$= C\psi(t,x)[f]^{\lambda}_{\varphi} \int\limits_0^t (t-\tau)^{-\nu+\lambda/(2b)} d\tau = C\psi(t,x)[f]^{\lambda}_{\varphi} t^{1-\nu+\lambda/(2b)},\quad (t,x) \in \Pi_{(0,T]}. \tag{27}$$

Then

$$||u||^0_{\psi} \leq C[f]^{\lambda}_{\varphi}. \tag{28}$$

Let us estimate the difference $\Delta^{x'}_x u$. If $d^{2b} \geq t$, where $d := d(x; x')$, then under condition (27) we have the estimate

$$|\Delta^{x'}_x u(t,x)| \leq C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{\varphi} t^{1-\nu+\lambda/(2b)},\quad t \in (0,T],\ \{x, \xi\} \subset \mathbb{R}^n.$$

We obtain

$$|\Delta^{x'}_x u(t,x)| \leq C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{\varphi} d^{\lambda} t^{1-\nu}$$

$$\leq C(\psi(t,x) + \psi(t,x')) d^{\lambda} [f]^{\lambda}_{\varphi},\ t \in (0,T],\ \{x, \xi\} \subset \mathbb{R}^n, \tag{29}$$

for any $\{\gamma, \lambda\} \subset (0,1]$, including where $\nu + (\gamma - \lambda)/(2b) > 1$; and with $\nu + (\gamma - \lambda)/(2b) < 1$ we receive from (27)

$$|\Delta^{x'}_x u(t,x)| \leq C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{\varphi} t^{1-\nu-(\gamma-\lambda)/(2b)} t^{\gamma/(2b)}$$

$$\leq C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{\varphi} t^{1-\nu-(\gamma-\lambda)/(2b)} d^{\gamma}$$

$$\leq C(\psi(t,x) + \psi(t,x')) d^{\gamma} [f]^{\lambda}_{\varphi},\ t \in (0,T],\ \{x, \xi\} \subset \mathbb{R}^n. \tag{30}$$

It is sufficient to consider case, where $d^{2b} < t$. By the first condition from (4) like (25) we write

$$\Delta^{x'}_x u(t,x) = \int\limits_0^{t-d^{2b}} d\tau \int\limits_{\mathbb{R}^n} \Delta^{x'}_x M(t,x;\tau,\xi) \Delta^{X_1(t-\tau)}_{\xi} f(\tau,\xi) d\xi$$

$$+ \int\limits_{t-d^{2b}}^t d\tau \int\limits_{\mathbb{R}^n} M(t,x;\tau,\xi) \Delta^{X_1(t-\tau)}_{\xi} f(\tau,\xi) d\xi$$

$$- \int\limits_{t-d^{2b}}^t d\tau \int\limits_{\mathbb{R}^n} M(t,x';\tau,\xi) \Delta^{X'_1(t-\tau)}_{\xi} f(\tau,\xi) d\xi =: \sum_{l=1}^3 K_l, \tag{31}$$

where $X'_1(t) := X_1(t)|_{x=x'}$.

Using (3), (6), the second inequality from (9), (11), we get

$$|K_1| \le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} (d(x;x'))^{\gamma} (t-\tau)^{-\gamma/(2b)} \exp\{-c\rho(t-\tau,x,\xi)\} |\Delta_{\xi}^{X_1(t-\tau)} f(\tau,\xi)| d\xi$$

$$\le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} (d(x;x'))^{\gamma} (t-\tau)^{-\gamma/(2b)} \exp\{-c\rho(t-\tau,x,\xi)\}$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau))) \frac{|\Delta_{\xi}^{X_1(t-\tau)} f(\tau,\xi)|}{\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau))} d\xi \le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N-\gamma/(2b)} d\tau$$

$$\times \int\limits_{\mathbb{R}^n} \psi(t,x) \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} (d(\xi;X_1(t-\tau)))^{\lambda} d\xi d^{\gamma} [f]_{\varphi}^{\lambda}.$$

Now let us use the inequality (26) and equality (19). We get

$$|K_1| \le C d^{\gamma} \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-(\gamma-\lambda)/(2b)} d\tau \psi(t,x) [f]_{\varphi}^{\lambda}. \tag{32}$$

If $\nu+(\gamma-\lambda)/(2b)<1$, then from (32) we obtain

$$|K_1| \le C d^{\gamma} \psi(t,x) [f]_{\varphi}^{\lambda} (t-\tau)^{1-\nu-(\gamma-\lambda)/(2b)} |_{\tau=t-d^{2b}}^{0}$$

$$= C d^{\gamma} \psi(t,x) [f]_{\varphi}^{\lambda} (t^{1-\nu-(\gamma-\lambda)/(2b)} - d^{2b(1-\nu)-\gamma+\lambda}) \le C d^{\gamma} \psi(t,x) [f]_{\varphi}^{\lambda}.$$

If $\nu+(\gamma-\lambda)/(2b)>1$, then from (32) we obtain

$$|K_1| \le C d^{\gamma} \psi(t,x) [f]_{\varphi}^{\lambda} (t-\tau)^{1-\nu-(\gamma-\lambda)/(2b)} |_{\tau=0}^{t-d^{2b}} = C d^{\gamma} \psi(t,x) [f]_{\varphi}^{\lambda} (d^{2b(1-\nu)-\gamma+\lambda}$$

$$-t^{1-\nu-(\gamma-\lambda)/(2b)}) \le C d^{2b(1-\nu)+\lambda} \psi(t,x) [f]_{\varphi}^{\lambda} = C d^{\lambda} \psi(t,x) [f]_{\varphi}^{\lambda}.$$

Let us estimate $K_2$. With the help of (3), (9), (11) and (26) we obtain

$$|K_2| \le C \int\limits_{t-d^{2b}}^{t} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} (d(\xi;X_1(t-\tau)))^{\lambda} \exp\{-c\rho(t-\tau,x,\xi)\}$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau))) d\xi [f]_{\varphi}^{\lambda} \le C \int\limits_{t-d^{2b}}^{t} (t-\tau)^{-\nu-N} d\tau$$

$$\times \int\limits_{\mathbb{R}^n} (d(\xi;X_1(t-\tau)))^{\lambda} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} \psi(t,x) d\xi [f]_{\varphi}^{\lambda}$$

$$\le C \int\limits_{t-d^{2b}}^{t} (t-\tau)^{-\nu-N+\lambda/(2b)} d\tau \int\limits_{\mathbb{R}^n} \exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\} \psi(t,x) d\xi [f]_{\varphi}^{\lambda}.$$

Using (19) with $c' = \bar{c}_1$, we have

$$|K_2| \le C \int\limits_{t-d^{2b}}^{t} (t-\tau)^{-\nu+\lambda/(2b)} d\tau \psi(t,x)[f]^{\lambda}_{\varphi}.$$

Since $-\nu + \lambda/(2b) > -1$, we obtain

$$|K_2| \le C(t-\tau)^{1-\nu+\lambda/(2b)}|^{t-d^{2b}}_{\tau=t} \psi(t,x)[f]^{\lambda}_{\varphi} = Cd^{2b(1-\nu)+\lambda}\psi(t,x)[f]^{\lambda}_{\varphi} \tag{33}$$

and thus, we have

$$|K_2| \le Cd^{\lambda}d^{2b(1-\nu)}\psi(t,x)[f]^{\lambda}_{\varphi} \le Cd^{\lambda}\psi(t,x)[f]^{\lambda}_{\varphi},$$

if $\nu + (\gamma-\lambda)/(2b) > 1$. In case, where $\nu + (\gamma-\lambda)/(2b) < 1$, we receive from (33) the following inequality

$$|K_2| \le Cd^{\gamma}d^{2b(1-\nu)+\lambda-\gamma}\psi(t,x)[f]^{\lambda}_{\varphi} \le Cd^{\gamma}\psi(t,x)[f]^{\lambda}_{\varphi}.$$

By the similar way we obtain

$$|K_3| \le Cd^{\lambda}\psi(t,x')[f]^{\lambda}_{\varphi}$$

in case, where $\nu \in (1-1/(2b), 1]$, and

$$|K_3| \le Cd^{\gamma}\psi(t,x')[f]^{\lambda}_{\varphi}$$

in case, where $\nu \in (1-1/(2b), 1]$ and $\nu - (\gamma-\lambda)/(2b) < 1$.

From (28), (29), (30) and from the estimates for $K_l$, $l \in L$, the estimates (13) and (14) follow with $\nu \in (1-1/(2b), 1]$.

**c)** Let $\nu \in (1, 1+m'/(2b)]$. Because of the second condition from (4) we represent the integral (1) in the form

$$u(t,x) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n_1}} \Big( \int\limits_{\mathbb{R}^{n_2+n_3}} (t-\tau)^{-\nu-N}\Omega(t,x;\tau,\xi)\Delta_{\xi}^{X_2(t-\tau)} f(\tau,\xi)d\xi_2 d\xi_3 \Big) d\xi_1,$$

$$(t,x) \in \Pi_{(0,T]}. \tag{34}$$

With the help of (3), (5) and (7)–(11) we get

$$|u(t,x)| \le C \int\limits_0^t (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}$$

$$\times \exp\{-c_0\rho(t-\tau,x,\xi)\}(\varphi(\tau,\xi)+\varphi(\tau,X_2(t-\tau)))\frac{|\Delta_{\xi}^{X_2(t-\tau)} f(\tau,\xi)|}{\varphi(\tau,\xi)+\varphi(\tau,X_2(t-\tau))} d\xi$$

$$\le C \int\limits_0^t (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} d_1(\xi; X_2(t-\tau); \lambda) d\xi \psi(t,x)[f]^{\lambda}_{1,\varphi}.$$

The inequality below follows from definitions of $d$, $d_1$ and $X_2$:

$$d_1(\xi; X_2(t-\tau); \lambda) = \sum_{l=2}^{3} |\xi_l - \bar{x}_l(t-\tau)|^{(\lambda+m')/(2b(l-1)+1)}$$

$$\leq C \left( \sum_{l=2}^{3} |\xi_l - \bar{x}_l(t-\tau)|^{1/(2b(l-1)+1)} \right)^{\lambda+m'} = C \left( d(\xi; X_2(t-\tau)) \right)^{\lambda+m'},$$

$$0 \leq \tau < t \leq T, \quad \{x, \xi\} \subset \mathbb{R}^n, \quad \lambda \in (0,1].$$

Here $C > 0$ is some constant. Then take to account inequality (26) we have

$$d_1(\xi; X_2(t-\tau); \lambda) \exp\{-\bar{c}\rho(t-\tau, x, \xi)\} \leq C(d(\xi; X_2(t-\tau)))^{m'+\lambda} \exp\{-\bar{c}\rho(t-\tau, x, \xi)\}$$

$$\leq C(t-\tau)^{(m'+\lambda)/(2b)} \exp\{-\bar{c}_1\rho(t-\tau, x, \xi)\},$$

$$0 \leq \tau < t \leq T, \ \{x, \xi\} \subset \mathbb{R}^n, \ 0 < \bar{c}_1 < \bar{c}, \ \lambda \in (0,1]. \tag{35}$$

For $\bar{c} = c - c_0$ with the help of (19) we have

$$|u(t,x)| \leq C \int_0^t (t-\tau)^{-\nu-N+(m'+\lambda)/(2b)} d\tau \int_{\mathbb{R}^n} \exp\{-\bar{c}_1\rho(t-\tau, x, \xi)\} d\xi \psi(t,x) [f]_{1,\varphi}^{\lambda}$$

$$= C\psi(t,x)[f]_{1,\varphi}^{\lambda} \int_0^t (t-\tau)^{-\nu+(m'+\lambda)/(2b)} d\tau = C\psi(t,x)[f]_{1,\varphi}^{\lambda} t^{1-\nu+(m'+\lambda)/(2b)}, \quad (t,x) \in \Pi_{(0,T]}. \tag{36}$$

Then

$$||u||_{\psi}^{0} \leq C[f]_{1,\varphi}^{\lambda}. \tag{37}$$

Let us estimate the difference $\Delta_x^{x'} u$. If $d^{2b} \geq t$, where $d := d(x; x')$, then under estimate (36) we have the inequality

$$|\Delta_x^{x'} u(t,x)| \leq C(\psi(t,x) + \psi(t,x'))[f]_{1,\varphi}^{\lambda} d^{\lambda} t^{1-\nu+m'/(2b)}$$

$$\leq C(\psi(t,x) + \psi(t,x')) d^{\lambda} [f]_{1,\varphi}^{\lambda}, \ t \in (0,T], \ \{x, \xi\} \subset \mathbb{R}^n, \tag{38}$$

and with $\nu + (\gamma - m' - \lambda)/(2b) < 1$ we receive

$$|\Delta_x^{x'} u(t,x)| \leq C(\psi(t,x) + \psi(t,x'))[f]_{1,\varphi}^{\lambda} t^{1-\nu-(\gamma-m'-\lambda)/(2b)} t^{\gamma/(2b)}$$

$$\leq C(\psi(t,x) + \psi(t,x'))[f]_{1,\varphi}^{\lambda} t^{1-\nu-(\gamma-m'-\lambda)/(2b)} d^{\gamma}$$

$$\leq C(\psi(t,x) + \psi(t,x')) d^{\gamma} [f]_{1,\varphi}^{\lambda}, \ t \in (0,T], \ \{x, \xi\} \subset \mathbb{R}^n. \tag{39}$$

It is sufficient to consider case, where $d^{2b} < t$. By the second condition from (4) like (34) we write

$$\Delta_x^{x'} u(t,x) = \int_0^{t-d^{2b}} d\tau \int_{\mathbb{R}^{n_1}} \Big( \int_{\mathbb{R}^{n_2+n_3}} \Delta_x^{x'} M(t,x;\tau,\xi) \Delta_{\xi}^{X_2(t-\tau)} f(\tau,\xi) d\xi_2 d\xi_3 \Big) d\xi_1$$

$$
+\int\limits_{t-d^{2b}}^{t} d\tau \int\limits_{\mathbb{R}^{n_1}} \Big( \int\limits_{\mathbb{R}^{n_2+n_3}} M(t,x;\tau,\xi)\Delta_\xi^{X_2(t-\tau)} f(\tau,\xi)d\xi_2 d\xi_3\Big) d\xi_1
$$

$$
-\int\limits_{t-d^{2b}}^{t} d\tau \int\limits_{\mathbb{R}^{n_1}} \Big( \int\limits_{\mathbb{R}^{n_2+n_3}} M(t,x';\tau,\xi)\Delta_\xi^{X_2'(t-\tau)} f(\tau,\xi)d\xi_2 d\xi_3\Big) d\xi_1 =: \sum_{l=1}^{3} K_l', \tag{40}
$$

where $X_2'(t) := X_2(t)|_{x=x'}$.

Using (3), (6), the second inequality from (9), (11), we get

$$
|K_1'| \le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} (d(x;x'))^\gamma (t-\tau)^{-\gamma/(2b)} \times
$$

$$
\times \exp\{-c\rho(t-\tau,x,\xi)\}(\varphi(\tau,\xi)+\varphi(\tau,X_2(t-\tau)))
$$

$$
\times \frac{|\Delta_\xi^{X_2(t-\tau)} f(\tau,\xi)|}{\varphi(\tau,\xi)+\varphi(\tau,X_2(t-\tau))} d\xi \le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N-\gamma/(2b)} d\tau
$$

$$
\times \int\limits_{\mathbb{R}^n} \psi(\tau,x)\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} d_1(\xi;X_2(t-\tau);\lambda) d\xi d^\gamma [f]^\lambda_{1,\varphi}.
$$

Now let us use the inequalities (35) and equality (19). We get

$$
|K_1'| \le C d^\gamma \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N-\gamma/(2b)+(m'+\lambda)/(2b)} d\tau \int\limits_{\mathbb{R}^n} \psi(\tau,x)\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\} d\xi
$$

$$
\times d^\gamma [f]^\lambda_{1,\varphi} = C d^\gamma \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-(\gamma-m'-\lambda)/(2b)} d\tau \psi(t,x)[f]^\lambda_{1,\varphi}.
$$

If $\nu+(\gamma-m'-\lambda)/(2b)>1$, then

$$
|K_1'| \le C d^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}(t-\tau)^{1-\nu-(\gamma-m'-\lambda)/(2b)}|_{\tau=0}^{t-d^{2b}} = C d^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}(d^{2b(1-\nu)-\gamma+m'+\lambda}
$$

$$
-t^{1-\nu-(\gamma-m'-\lambda)/(2b)}) \le C d^{2b(1-\nu)+m'+\lambda}\psi(t,x)[f]^\lambda_{1,\varphi} \le C d^\lambda \psi(t,x)[f]^\lambda_{1,\varphi}.
$$

If $\nu+(\gamma-m'-\lambda)/(2b)<1$, then

$$
|K_1'| \le C d^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}(t-\tau)^{1-\nu-(\gamma-m'-\lambda)/(2b)}|_{\tau=t-d^{2b}}^{0} = C d^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}
$$

$$
\times(t^{1-\nu-(\gamma-m'-\lambda)/(2b)} - d^{2b(1-\nu)-\gamma+m'+\lambda}) \le C d^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}.
$$

Let us estimate $K_2'$. With the help of (3), (9), (11) and (35) we obtain

$$
|K_2'| \le C \int\limits_{t-d^{2b}}^{t} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} d_1(\xi;X_2(t-\tau);\lambda)\exp\{-c\rho(t-\tau,x,\xi)\}
$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_2(t-\tau)))d\xi[f]^{\lambda}_{1,\varphi}\le C\int\limits_{t-d^{2b}}^{t}(t-\tau)^{-\nu-N}d\tau$$

$$\times\int\limits_{\mathbb{R}^n}d_1(\xi;X_2(t-\tau);\lambda)\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}\psi(t,x)d\xi[f]^{\lambda}_{1,\varphi}$$

$$\le C\int\limits_{t-d^{2b}}^{t}(t-\tau)^{-\nu-N+(m'+\lambda)/(2b)}d\tau\int\limits_{\mathbb{R}^n}\exp\{-\bar c_1\rho(t-\tau,x,\xi)\}\psi(t,x)d\xi[f]^{\lambda}_{1,\varphi}.$$

Using (19) with $c'=\bar c_1$, we have

$$|K'_2|\le C\int\limits_{t-d^{2b}}^{t}(t-\tau)^{-\nu+(m'+\lambda)/(2b)}d\tau\psi(t,x)[f]^{\lambda}_{1,\varphi}.$$

Since $\nu-(m'+\lambda)/(2b)<1$, we obtain

$$|K'_2|\le C(t-\tau)^{1-\nu+(m'+\lambda)/(2b)}|^{t-d^{2b}}_{\tau=t}\psi(t,x)[f]^{\lambda}_{1,\varphi}=Cd^{2b(1-\nu)+m'+\lambda}\psi(t,x)[f]^{\lambda}_{1,\varphi}. \qquad (41)$$

The estimate

$$|K'_2|\le Cd^{\gamma}d^{2b(1-\nu)+m'+\lambda-\gamma}\psi(t,x)[f]^{\lambda}_{1,\varphi}\le Cd^{\gamma}\psi(t,x)[f]^{\lambda}_{1,\varphi}$$

follow from (41) if $\nu+(\gamma-m'-\lambda)/(2b)<1$, and the estimate

$$|K'_2|\le Cd^{\lambda}d^{2b(1-\nu)+m'}\psi(t,x)[f]^{\lambda}_{1,\varphi}\le Cd^{\lambda}\psi(t,x)[f]^{\lambda}_{1,\varphi}$$

if $\nu+(\gamma-m'-\lambda)/(2b)>1$.

By the similar way we obtain

$$|K'_3|\le Cd^{\gamma}\psi(t,x')[f]^{\lambda}_{1,\varphi}$$

in case, where $\nu+(\gamma-m'-\lambda)/(2b)<1$, and

$$|K'_3|\le Cd^{\lambda}\psi(t,x')[f]^{\lambda}_{1,\varphi}$$

in the case, where $\nu+(\gamma-m'-\lambda)/(2b)>1$.

From (37), (39), (40) and from estimates for $K'_l$, $l\in L$, the estimates (15) and (16) follow.

**d)** This case can be proved by the similar way as the case **c)**. We must use the third equality from (4); representation of the integral (1) in the form

$$u(t,x)=\int\limits_0^t d\tau\int\limits_{\mathbb{R}^{n_1+n_2}}\Big(\int\limits_{\mathbb{R}^{n_3}}(t-\tau)^{-\nu-N}\Omega(t,x;\tau,\xi)\Delta_{\xi}^{X_3(t-\tau)}f(\tau,\xi)d\xi_3\Big)d\xi_1d\xi_2,\ (t,x)\in\Pi_{(0,T]},$$

and estimates

$$\begin{aligned}&d_2(\xi;X_3(t-\tau);\lambda)\exp\{-\bar c\rho(t-\tau,x,\xi)\}\\&\le C(d(\xi;X_3(t-\tau)))^{m'+2b+\lambda}\exp\{-\bar c\rho(t-\tau,x,\xi)\}\\&\le C(t-\tau)^{(m'+2b+\lambda)/(2b)}\exp\{-\bar c_1\rho(t-\tau,x,\xi)\},\end{aligned}$$

$$0\le\tau<t\le T,\qquad\{x,\xi\}\subset\mathbb{R}^n,\qquad 0<\bar c_1<\bar c,\qquad\lambda\in(0,1].$$

These estimates are obtained in the same way as estimates (35). □

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Дронь В.С., Мединський І.П. *Властивості інтегралів типу похідних від об'ємного потенціалу для виродженого $\overrightarrow{2b}$-параболічного рівняння типу Колмогорова* // Буковинський матем. журнал — 2021. — Т.9, №2. — С. 7–21.

При побудові і дослідженні фундаментального розв'язку, встановленні коректної розв'язності задачі Коші та одержанні оцінок розв'язків параболічних рівнянь важливе значення мають властивості відповідних об'ємних потенціалів. Такі властивості встановлено для параболічних за Петровським і $\overrightarrow{2b}$-параболічних за Ейдельманом рівнянь як без усяких вироджень, так і з виродженнями на початковій гіперплощині. Також вивчалися об'ємні потенціали для вироджених параболічних типу Колмогорова (ультрапараболічних типу Колмогорова) рівнянь довільного порядку. Проте лише для рівнянь другого порядку були встановлені властивості об'ємних потенціалів із густиною з просторів Гельдера обмежених і зростаючих при $|x| \to \infty$ функцій.

Такі властивості зручно отримувати, якщо попередньо довести твердження про властивості інтегралів типу похідних від об'ємних потенціалів. Ці властивості описуються належністю таких інтегралів до відповідних функціональних просторів залежно від того, до яких просторів належить густина та ядро інтеграла.

У статті розглядаються інтеграли, які мають структуру та властивості, подібні до похідних від об'ємних потенціалів, породжених фундаментальним розв'язком задачі Коші для виродженого $\overrightarrow{2b}$-параболічного рівняння типу Колмогорова. Коефіцієнти цього рівняння залежать тільки від часової змінної. Залежно від розмірності груп просторових змінних рівняння може вироджуватися за двома або однією групою просторових змінних, або навіть може бути невиродженим $\overrightarrow{2b}$-параболічним за Ейдельманом рівнянням.

Для побудови просторів Гельдера використовуються спеціальні відстані та вагові норми. Відстані враховують анізотропність за просторовими змінними рівняння, яке породжує інтеграли, що розглядаються. Ваговими функціями є експоненти, які необмежено зростають при $|x| \to \infty$ і тип їх зростання спеціальним способом залежить від змінної $t$.

Результати роботи можуть бути використані для встановлення коректної розв'язності задачі Коші та оцінок розв'язків даного неоднорідного рівняння у відповідних вагових просторах Гельдера.