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RELATIVE GROWTH OF ENTIRE DIRICHLET SERIES WITH DIFFERENT GENERALIZED ORDERS

For entire functions F and G defined by Dirichlet series with exponents increasing to $+\infty$ formulas are found for the finding the generalized order $\varrho_{\alpha,\beta}[F]_G = \varlimsup\limits_{\sigma\to=\infty} \dfrac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}$ and the generalized lower order $\lambda_{\alpha,\beta}[F]_G = \varliminf\limits_{\sigma\to+\infty} \dfrac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}$ of F with respect to G, where $M_F(\sigma) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$ and α and β are positive increasing to $+\infty$ functions.

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Introduction

Let f and g be entire transcendental functions and $M_f(r) = \max\{|f(z)| : |z| = r\}$. For the study of relative growth of the functions f and g Ch. Roy [1] used the order $\varrho_g[f] = \varlimsup\limits_{r\to+\infty} \ln M_g^{-1}(M_f(r))/\ln r$ and the lower order $\lambda_g[f] = \varliminf\limits_{r\to+\infty} \ln M_g^{-1}(M_f(r))/\ln r$ of the function f with respect to the function g. Researches of relative growth of entire functions was continued by S.K. Data, T. Biswas and other mathematicians (see, for example, [2, 3, 4, 5]) in terms of maximal terms, Nevanlinna characteristic function and k-logarithmic orders. In [6] it is considered a relative growth of entire functions of two complex variables and in [7] the relative growth of entire Dirichlet series is studied in terms of R-orders.

Suppose that $\Lambda = (\lambda_n)$ is an increasing to $+\infty$ sequence of non-negative numbers, and by $S(\Lambda)$ we denote a class of entire Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it. \tag{1}$$

For $\sigma < +\infty$ we put $M_F(\sigma) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$. We remark that the function $M_F(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, +\infty)$ and, therefore, there exists the function $M_F^{-1}(x)$ inverse to $M_F(\sigma)$, which increase to $+\infty$ on $(x_0, +\infty)$.

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By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, +\infty)$ then the quantities

$$\varrho_{\alpha,\beta}[F] = \varlimsup_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma), \quad \lambda_{\alpha,\beta}[F] = \varliminf_{\sigma\to+\infty} \alpha(\ln M_F(\sigma))/\beta(\sigma)$$

are called [8] the generalized (α, β)-order and the generalized lower (α, β)-order of F accordingly. We say that F has the generalized regular (α, β)-growth, if $0 < \lambda_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[F] < +\infty$.

We define the generalized (α, β)-order $\varrho_{\alpha,\beta}[F]_G$ and the generalized lower (α, β)-order $\lambda_{\alpha,\beta}[F]_G$ of the function $F \in S(\Lambda)$ with respect to a function $G \in S(\Lambda)$, given by Dirichlet series $G(s) = \sum\limits_{n=1}^{\infty} g_n \exp\{s\lambda_n\}$, as follows

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \alpha(M_G^{-1}(M_F(\sigma)))/\beta(\sigma), \quad \lambda_{\alpha,\beta}[F]_G = \varliminf_{\sigma\to+\infty} \alpha(M_G^{-1}(M_F(\sigma)))/\beta(\sigma).$$

The following theorems are proved in [9].

Theorem (A). *Let $\beta \in L$ and $\gamma \in L$. Except for the cases, when $\varrho_{\gamma,\beta}[F] = \varrho_{\gamma,\beta}[G] = 0$ or $\varrho_{\gamma,\beta}[F] = \varrho_{\gamma,\beta}[G] = +\infty$, the inequality $\varrho_{\beta,\beta}[F]_G \geq \varrho_{\gamma,\beta}[F]/\varrho_{\gamma,\beta}[G]$ is true and subject to the condition of the generalized regular (γ, β)-growth of G this inequality converts into an equality.*

Except for the cases, when $\lambda_{\gamma,\beta}[F] = \lambda_{\gamma,\beta}[G] = 0$ or $\lambda_{\gamma,\beta}[F] = \lambda_{\gamma,\beta}[G] = +\infty$, the inequality $\lambda_{\beta,\beta}[F]_G \leq \lambda_{\gamma,\beta}[F]/\lambda_{\gamma,\beta}[G]$ is true and subject to the condition of the generalized regular (γ, β)-growth of G this inequality converts into an equality.

Theorem (B). *Let $0 < p < +\infty$ and one of conditions is executed:*

a) $\gamma \in L^0$, $\beta(\ln x) \in L^0$, $\dfrac{d\beta^{-1}(c\gamma(x))}{d\ln x} \to \dfrac{1}{p}$ $(x \to +\infty)$ for each $c \in (0, +\infty)$ and $\ln n = o(\lambda_n)$ $(n \to \infty)$;

b) $\gamma \in L_{si}$, $\beta \in L^0$, $\varrho_{\gamma,\beta}[F] < +\infty$, $\dfrac{d\beta^{-1}(c\gamma(x))}{d\ln x} = O(1)$ $(x \to +\infty)$ and $\ln n = = o(\lambda_n\beta^{-1}(c\gamma(\lambda_n)))$ $(n \to \infty)$ for each $c \in (0, +\infty)$.

Suppose that $\gamma(\lambda_{n+1}/p) = (1 + o(1))\gamma(\lambda_n/p)$ as $n \to \infty$.

If the function G has generalized regular (γ, β)-growth and $\kappa_n[G] := \dfrac{\ln |g_n| - \ln |g_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\varrho_{\beta,\beta}[F]_G = \varlimsup_{n\to\infty} \beta\left(\frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|g_n|}\right) / \beta\left(\frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|f_n|}\right)$$

except for the cases, when $\varrho_{\gamma,\beta}[F] = \varrho_{\gamma,\beta}[G] = 0$ or $\varrho_{\gamma,\beta}[F] = \varrho_{\gamma,\beta}[G] = +\infty$.

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\lambda_{\beta,\beta}[F]_G = \varliminf_{n\to\infty} \beta\left(\frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|g_n|}\right) / \beta\left(\frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|f_n|}\right)$$

except for the cases, when $\lambda_{\gamma,\beta}[F]=\lambda_{\gamma,\beta}[G]=0$ *or* $\lambda_{\gamma,\beta}[F]=\lambda_{\gamma,\beta}[G]=+\infty$.

Similar results in terms of R-types are obtained in [10].

Here we consider the general case when $\alpha\neq\beta$.

1 ANALOGUES OF THEOREM (A)

We begin from the following general theorem.

Theorem 1. *If* $\alpha\in L$ *and* $\beta\in L$ *then:*

1) the inequalities

$$\frac{\varrho_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]}\le\varrho_{\alpha,\beta}[F]_G\le\frac{\varrho_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]}\tag{2}$$

are true for each function $\gamma\in L$ *except for the cases* $\varrho_{\gamma,\beta}[F]=\varrho_{\gamma,\alpha}[G]=0$, $\varrho_{\gamma,\beta}[F]=\lambda_{\gamma,\alpha}[G]=0$, $\varrho_{\gamma,\beta}[F]=\varrho_{\gamma,\alpha}[G]=+\infty$, $\varrho_{\gamma,\beta}[F]=\lambda_{\gamma,\alpha}[G]=+\infty$;

2) the inequalities

$$\frac{\lambda_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]}\le\lambda_{\alpha,\beta}[F]_G\le\frac{\lambda_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]}\tag{3}$$

are true for each function $\gamma\in L$ *except for the cases* $\lambda_{\gamma,\beta}[F]=\lambda_{\gamma,\alpha}[G]=0$, $\lambda_{\gamma,\beta}[F]=\varrho_{\gamma,\alpha}[G]=0$, $\lambda_{\gamma,\beta}[F]=\lambda_{\gamma,\alpha}[G]=+\infty$, $\lambda_{\gamma,\beta}[F]=\varrho_{\gamma,\alpha}[G]=+\infty$.

Proof. Indeed,

$$\varrho_{\alpha,\beta}[F]_G=\varlimsup_{x\to+\infty}\frac{\alpha(M_G^{-1}(x))}{\beta(M_F^{-1}(x))}=\varlimsup_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(M_F^{-1}(x))}\frac{\alpha(M_G^{-1}(x))}{\gamma(\ln x)}\ge$$

$$\ge\varlimsup_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(M_F^{-1}(x))}\varliminf_{x\to+\infty}\frac{\alpha(M_G^{-1}(x))}{\gamma(\ln x)}=\varlimsup_{\sigma\to+\infty}\frac{\gamma(\ln M_F(\sigma)))}{\beta(\sigma)}\varliminf_{\sigma\to+\infty}\frac{\alpha(\sigma)}{\gamma(\ln M_G(\sigma))}=\frac{\varrho_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]}$$

and

$$\varrho_{\alpha,\beta}[F]_G\le\varlimsup_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(M_F^{-1}(x))}\varlimsup_{x\to+\infty}\frac{\alpha(M_G^{-1}(x))}{\gamma(\ln x)}=\varlimsup_{\sigma\to+\infty}\frac{\gamma(\ln M_F(\sigma)))}{\beta(\sigma)}\varlimsup_{\sigma\to+\infty}\frac{\alpha(\sigma)}{\gamma(\ln M_G(\sigma))}=$$

$$=\frac{\varrho_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]},$$

i. e. inequalities (2) are proved.

The proof of (3) is similar. Indeed,

$$\lambda_{\alpha,\beta}[F]_G=\varliminf_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(M_F^{-1}(x))}\frac{\alpha(M_G^{-1}(x))}{\gamma(\ln x)}\le\varliminf_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(M_F^{-1}(x))}\varlimsup_{x\to+\infty}\frac{\alpha(M_G^{-1}(x))}{\gamma(\ln x)}=\frac{\lambda_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]}$$

and

$$\lambda_{\alpha,\beta}[F]_G\ge\varliminf_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(M_F^{-1}(x))}\varliminf_{x\to+\infty}\frac{\alpha(M_G^{-1}(x))}{\gamma(\ln x)}=\frac{\lambda_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]},$$

whence (3) follows. Theorem 1 is proved. $\square$

Remark 1. *In the statements 1) and 2) of Theorem 1 the conditions for the function γ hold if $0 < \lambda_{\gamma,\alpha}[G] \le \varrho_{\gamma,\alpha}[G] < +\infty$. From (2) and (3) it follows that if G has the generalized regular (γ,α)-growth then $\varrho_{\alpha,\beta}[F]_G = \varrho_{\gamma,\beta}[F]/\varrho_{\gamma,\alpha}[G]$.*

If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \ge 3$ then from the definition of $\varrho_{\gamma,\beta}[F]$ and $\lambda_{\gamma,\beta}[F]$ we obtain the definition of the R-order $\varrho_R[G]$ and the lower R-order $\lambda_R[G]$ introduced by J. Ritt [11], and if we choose $\alpha(x) = \beta(x) = \ln x$ for $x \ge 3$ then we obtain the definition of the logarithmic order $\varrho_l[G]$ and the lower logarithmic order $\lambda_l[G]$.

For the characteristic of the relative growth of the function F with respect to a function G in Ritt's scale we use

$$\varrho_{R,R}[F]_G = \varlimsup_{\sigma\to+\infty} M_G^{-1}(M_F(\sigma))/\sigma, \quad \lambda_{R,R}[F]_G = \varliminf_{\sigma\to+\infty} M_G^{-1}(M_F(\sigma))/\sigma,$$

in the logarithmic scale we use

$$\varrho_{l,l}[F]_G = \varlimsup_{\sigma\to+\infty} \ln M_G^{-1}(M_F(\sigma))/\ln\sigma, \quad \lambda_{l,l}[F]_G = \varliminf_{\sigma\to+\infty} \ln M_G^{-1}(M_F(\sigma))/\ln\sigma$$

and in the mixed scale we use

$$\varrho_{R,l}[F]_G = \varlimsup_{\sigma\to+\infty} \ln M_G^{-1}(M_F(\sigma))/\sigma, \quad \lambda_{R,l}[F]_G = \varliminf_{\sigma\to+\infty} \ln M_G^{-1}(M_F(\sigma))/\sigma.$$

Then Theorem 1 implies the following statement.

Corollary 1. *If $0 < \lambda_R[G] \le \varrho_R[G] < +\infty$ then*

$$\frac{\varrho_R[F]}{\varrho_R[G]} \le \varrho_{R,R}[F]_G \le \frac{\varrho_R[F]}{\lambda_R[G]} \quad \text{and} \quad \frac{\lambda_R[F]}{\varrho_R[G]} \le \lambda_{R,R}[F]_G \le \frac{\lambda_R[F]}{\lambda_R[G]}.$$

If $0 < \lambda_l[G] \le \varrho_l[G] < +\infty$ then

$$\frac{\varrho_l[F]}{\varrho_l[G]} \le \varrho_{l,l}[F]_G \le \frac{\varrho_l[F]}{\lambda_l[G]} \quad \text{and} \quad \frac{\lambda_l[F]}{\varrho_l[G]} \le \lambda_{l,l}[F]_G \le \frac{\lambda_l[F]}{\lambda_l[G]}.$$

If $0 < \lambda_l[G] \le \varrho_l[G] < +\infty$ then

$$\frac{\varrho_R[F]}{\varrho_l[G]} \le \varrho_{R,l}[F]_G \le \frac{\varrho_R[F]}{\lambda_l[G]} \quad \text{and} \quad \frac{\lambda_R[F]}{\varrho_l[G]} \le \lambda_{R,l}[F]_G \le \frac{\lambda_R[F]}{\lambda_l[G]}.$$

For a more detailed description of the growth of Dirichlet series of finite nonzero order use the type. If $0 < \varrho_R[F] < +\infty$ then the quantities

$$T_R[F] = \varlimsup_{\sigma\to+\infty} \frac{\ln M_F(\sigma)}{\exp\{\sigma\varrho_R[F]\}} \quad \text{and} \quad t_R[F] = \varliminf_{\sigma\to+\infty} \frac{\ln M_F(\sigma)}{\exp\{\sigma\varrho_R[F]\}}$$

are called the R-type and the lower R-type of function F. Similarly, the quantities

$$T_l[F] = \varlimsup_{\sigma\to+\infty} \frac{\ln M_F(\sigma)}{\sigma^{\varrho_l[F]}} \quad \text{and} \quad t_l[F] = \varliminf_{\sigma\to+\infty} \frac{\ln M_F(\sigma)}{\sigma^{\varrho_l[F]}}$$

are called the logarithmic type and the lower logarithmic type of function F. Therefore, by analogy, if $0<\varrho_{\alpha,\beta}[F]<+\infty$ then we define the generalized (α,β)-type and the lower generalized (α,β)-type of F as follows

$$T_{\alpha,\beta}[F]=\varlimsup_{\sigma\to+\infty}\frac{\exp\{\alpha(\ln M_F(\sigma))\}}{\exp\{\beta(\sigma)\varrho_{\alpha,\beta}[F]\}},\quad t_{\alpha,\beta}[F]=\varliminf_{\sigma\to+\infty}\frac{\exp\{\alpha(\ln M_F(\sigma))\}}{\exp\{\beta(\sigma)\varrho_{\alpha,\beta}[F]\}}.$$

Similarly, if $0<\varrho_{\alpha,\beta}[F]_G<+\infty$ then we define the generalized (α,β)-type and the lower generalized (α,β)-type of the function F with respect to the function G as follows

$$T_{\alpha,\beta}[F]_G=\varlimsup_{\sigma\to+\infty}\frac{\exp\{\alpha(M_G^{-1}(M_F(\sigma)))\}}{\exp\{\beta(\sigma)\varrho_{\alpha,\beta}[F]_G\}},\quad t_{\alpha,\beta}[F]_G=\varliminf_{\sigma\to+\infty}\frac{\exp\{\alpha(M_G^{-1}(M_F(\sigma)))\}}{\exp\{\beta(\sigma)\varrho_{\alpha,\beta}[F]_G\}}.$$

Theorem 2. *Let $\alpha\in L$, $\beta\in L$ and $\gamma\in L$. If the function G has the regular generalized (γ,α)-growth and $0<t_{\gamma,\alpha}[G]\le T_{\gamma,\alpha}[G]<+\infty$ then*

$$\frac{T_{\gamma,\beta}[F]}{T_{\gamma,\alpha}[G]}\le (T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]}\le\frac{T_{\gamma,\beta}[F]}{t_{\gamma,\alpha}[G]}\tag{4}$$

and

$$\frac{t_{\gamma,\beta}[F]}{T_{\gamma,\alpha}[G]}\le (t_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]}\le\frac{t_{\gamma,\beta}[F]}{t_{\gamma,\alpha}[G]}.\tag{5}$$

Proof. Since G has the regular generalized (γ,α)-growth, by Theorem 1 (see Remark 1) we have $\varrho_{\alpha,\beta}[F]_G=\dfrac{\varrho_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]}$. Therefore,

$$(T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]}=\varlimsup_{\sigma\to+\infty}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(M_G^{-1}(M_F(\sigma)))\}}{\exp\{\beta(\sigma)\varrho_{\gamma,\beta}[F]\}}=\varlimsup_{x\to+\infty}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(M_G^{-1}(x))\}}{\exp\{\varrho_{\gamma,\beta}[F]\beta(M_F^{-1}(x))\}}=$$

$$=\varlimsup_{x\to+\infty}\frac{\exp\{\gamma(\ln x)\}}{\exp\{\varrho_{\gamma,\beta}[F]\beta(M_F^{-1}(x))\}}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(M_G^{-1}(x))\}}{\exp\{\gamma(\ln x)\}}\ge$$

$$\ge\varlimsup_{x\to+\infty}\frac{\exp\{\gamma(\ln x)\}}{\exp\{\varrho_{\gamma,\beta}[F]\beta(M_F^{-1}(x))\}}\varliminf_{x\to+\infty}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(M_G^{-1}(x))\}}{\exp\{\gamma(\ln x)\}}=$$

$$=\varlimsup_{\sigma\to+\infty}\frac{\exp\{\gamma(\ln M_F(\sigma))\}}{\exp\{\varrho_{\gamma,\beta}[F]\beta(\sigma)\}}\varliminf_{\sigma\to+\infty}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(\sigma)\}}{\exp\{\gamma(\ln M_G(\sigma))\}}=\frac{T_{\gamma,\beta}[F]}{T_{\gamma,\alpha}[G]}$$

and

$$(T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]}\le\varlimsup_{x\to+\infty}\frac{\exp\{\gamma(\ln x)\}}{\exp\{\varrho_{\gamma,\beta}[F]\beta(M_F^{-1}(x))\}}\varlimsup_{x\to+\infty}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(M_G^{-1}(x))\}}{\exp\{\gamma(\ln x)\}}=$$

$$=\varlimsup_{\sigma\to+\infty}\frac{\exp\{\gamma(\ln M_F(\sigma))\}}{\exp\{\varrho_{\gamma,\beta}[F]\beta(\sigma)\}}\varlimsup_{\sigma\to+\infty}\frac{\exp\{\varrho_{\gamma,\alpha}[G]\alpha(\sigma)\}}{\exp\{\gamma(\ln M_G(\sigma))\}}=\frac{T_{\gamma,\beta}[F]}{t_{\gamma,\alpha}[G]}.$$

Estimates (4) are proved. The proof of (5) is similar and we will omit it. □

Theorem 2 implies the following statement.

Corollary 2. *If the function G has the regular growth and $0 < t_R[G] \le T_R[G] < +\infty$ then*

$$\frac{T_R[F]}{T_R[G]} \le (T_{R,R}[F]_G)^{\varrho_R[G]} \le \frac{T_R[F]}{t_R[G]}, \quad \frac{t_R[F]}{T_R[G]} \le (t_{R,R}[F]_G)^{\varrho_R[G]} \le \frac{t_R[F]}{t_R[G]},$$

where

$$T_{R,R}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\sigma\varrho_{R,R}[F]_G\}}, \quad t_{R,R}[F]_G = \varliminf_{\sigma\to+\infty} \frac{\exp\{M_G^{-1}(M_F(\sigma))\}}{\exp\{\sigma\varrho_{R,R}[F]_G\}}.$$

If the function G has the regular logarithmic growth and $0 < t_l[G] \le T_l[G] < +\infty$ then

$$\frac{T_l[F]}{T_l[G]} \le (T_{l,l}[F]_G)^{\varrho_l[G]} \le \frac{T_l[F]}{t_l[G]}, \quad \frac{t_l[F]}{T_l[G]} \le (t_{l,l}[F]_G)^{\varrho_l[G]} \le \frac{t_l[F]}{t_l[G]},$$

where

$$T_{l,l}[F]_G = \varlimsup_{\sigma\to+\infty} \sigma^{-\varrho_{l,l}[F]_G} M_G^{-1}(M_F(\sigma)), \quad t_{l,l}[F]_G = \varliminf_{\sigma\to+\infty} \sigma^{-\varrho_{l,l}[F]_G} M_G^{-1}(M_F(\sigma)).$$

If the function G has the regular logarithmic growth and $0 < t_l[G] \le T_l[G] < +\infty$ then

$$\frac{T_R[F]}{T_l[G]} \le (T_{R,l}[F]_G)^{\varrho_l[G]} \le \frac{T_R[F]}{t_l[G]}, \quad \frac{t_R[F]}{T_l[G]} \le (t_{R,l}[F]_G)^{\varrho_l[G]} \le \frac{t_R[F]}{t_l[G]},$$

where

$$T_{R,l}[F]_G := \varlimsup_{\sigma\to+\infty} \frac{M_G^{-1}(M_F(\sigma))}{\exp\{\sigma\varrho_{R,l}[F]_G\}}, \quad t_{R,l}[F]_G := \varliminf_{\sigma\to+\infty} \frac{M_G^{-1}(M_F(\sigma))}{\exp\{\sigma\varrho_{R,l}[F]_G\}}.$$

2 Analogues of Theorem (B)

We need the following lemma.

Lemma 1 ([8]). *Let $\alpha \in L_{si}$, $\beta \in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for each $c \in (0,+\infty)$. If $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n \to \infty$ for each $c \in (0,+\infty)$ and $G \in S(\Lambda)$ then*

$$\varrho_{\alpha,\beta}[G] = \varlimsup_{n\to\infty} \alpha(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right). \tag{6}$$

If, moreover, $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$\lambda_{\alpha,\beta}[G] = \varliminf_{n\to\infty} \alpha(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right). \tag{7}$$

Now we prove the following theorem.

Theorem 3. *Let $\alpha \in L^0$, $\beta \in L^0$, $\gamma \in L_{si}$, $\dfrac{d\alpha^{-1}(c\gamma(x))}{d\ln x} = O(1)$ and $\dfrac{d\beta^{-1}(c\gamma(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for each $c \in (0,+\infty)$. Suppose that $\ln n = o(\lambda_n\alpha^{-1}(c\gamma(\lambda_n)))$ and $\ln n = = o(\lambda_n\beta^{-1}(c\gamma(\lambda_n)))$ as $n \to \infty$ for each $c \in (0,+\infty)$.*

If the function G has generalized regular (γ,α)-growth, $\gamma(\lambda_{n+1})\sim\gamma(\lambda_n)$ and $\kappa_n[G]\nearrow+\infty$ as $n_0\le n\to\infty$ then

$$\varrho_{\alpha,\beta}[F]_G=P_{\alpha,\beta}:=\varlimsup_{n\to\infty}\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right). \tag{8}$$

If, moreover, $\kappa_n[F]\nearrow+\infty$ as $n_0\le n\to\infty$ then

$$\lambda_{\alpha,\beta}[F]_G=p_{\alpha,\beta}:=\varliminf_{n\to\infty}\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right). \tag{9}$$

Proof. Since G has generalized regular (γ,α)-growth, by Theorem 1 $\varrho_{\alpha,\beta}[F]_G=\dfrac{\varrho_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]}$, $\lambda_{\alpha,\beta}[F]_G=\dfrac{\lambda_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]}$ and by Lemma 1

$$\varrho_{\gamma,\beta}[F]=\varlimsup_{n\to\infty}\gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right),\quad \varrho_{\gamma,\alpha}[G]=\varlimsup_{n\to\infty}\gamma(\lambda_n)/\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right).$$

Therefore,

$$\varrho_{\alpha,\beta}[F]_G=\varlimsup_{n\to\infty}\gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\varliminf_{n\to\infty}\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\gamma(\lambda_n)\le$$

$$\le\varlimsup_{n\to\infty}\left(\gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right)\left(\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\gamma(\lambda_n)\right)=P_{\alpha,\beta}.$$

On the other hand, let $P_{\alpha,\beta}>0$. Then for every $\varepsilon\in(0,\,P_{\alpha,\beta})$ there exists an increasing to $+\infty$ sequence (n_k) of integers such that

$$\alpha\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|g_{n_k}|}\right)>(P_{\alpha,\beta}-\varepsilon)\beta\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|f_{n_k}|}\right)$$

i. e.

$$\gamma(\lambda_{n_k})/\beta\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|f_{n_k}|}\right)>(P_{\alpha,\beta}-\varepsilon)\gamma(\lambda_{n_k})/\alpha\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|g_{n_k}|}\right)$$

and, thus,

$$\varrho_{\gamma,\beta}[F]=\varlimsup_{n\to\infty}\gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\ge(P_{\alpha,\beta}-\varepsilon)\varliminf_{n\to\infty}\gamma(\lambda_n)/\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)=$$

$$=(P_{\alpha,\beta}-\varepsilon)\lambda_{\gamma,\alpha}[G]=(P_{\alpha,\beta}-\varepsilon)\varrho_{\gamma,\alpha}[G],$$

whence in view of the arbitrariness of ε we get $\varrho_{\alpha,\beta}[F]_G\ge P_{\alpha,\beta}$. For $P_{\alpha,\beta}=0$ the last inequality is obvious. Equality (8) is proved.

For the proof of (9) we remark that since G has generalized regular (α,β)-growth, by Theorem 1 and Lemma 1

$$\lambda_{\alpha,\beta}[F]_G=\varliminf_{n\to\infty}\gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\varlimsup_{n\to\infty}\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\gamma(\lambda_n)\ge$$

$$\geq \varliminf_{n\to\infty} \left(\gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right)\left(\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\gamma(\lambda_n)\right) = p_{\alpha,\beta}.$$

On the other hand, let $p_{\alpha,\beta} < +\infty$. Then for every $\varepsilon > 0$ there exists an increasing to $+\infty$ sequence (n_k) of integers such that

$$\alpha\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|g_{n_k}|}\right) < (p_{\alpha,\beta}+\varepsilon)\beta\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|f_{n_k}|}\right)$$

and, as above,

$$\lambda_{\gamma,\beta}[F] = \varliminf_{n\to\infty} \gamma(\lambda_n)/\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right) \leq (p_{\alpha,\beta}+\varepsilon)\varlimsup_{n\to\infty}\gamma(\lambda_n)/\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right) =$$
$$= (p_{\alpha,\beta}+\varepsilon)\varrho_{\gamma,\alpha}[G] = (p_{\alpha,\beta}+\varepsilon)\lambda_{\gamma,\alpha}[G],$$

whence in view of the arbitrariness of ε we get $\lambda_{\alpha,\beta}[F]_G \leq p_{\alpha,\beta}$. For $p_{\alpha,\beta} = +\infty$ the last inequality is obvious. Equality (9) is proved, and the proof of Theorem 3 is complete. □

For the study of the relative growth in classical scales we need the following lemmas.

Lemma 2 ([11], [12], [13], [14]). *If* $\ln n = o(\lambda_n \ln \lambda_n)$ *as* $n\to\infty$ *then*

$$\varrho_R[F] = \varlimsup_{n\to\infty} \lambda_n \ln \lambda_n/(-\ln|f_n|)$$

and if, moreover, $\ln\lambda_{n+1} \sim \ln\lambda_n$ *and* $\kappa_n[G] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then*

$$\lambda_R[F] = \varliminf_{n\to\infty} \lambda_n \ln \lambda_n/(-\ln|f_n|).$$

If $\ln n = o(\lambda_n)$ *as* $n\to\infty$ *then*

$$T_R[F] = (1/(e\varrho_R[F]))\varlimsup_{n\to\infty} \lambda_n |f_n|^{\varrho_R[F]/\lambda_n}$$

and if, moreover, $\lambda_{n+1} \sim \lambda_n$ *and* $\kappa_n[G] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then*

$$t_R[F] = (1/(e\varrho_R[F]))\varliminf_{n\to\infty} \lambda_n |f_n|^{\varrho_R[F]/\lambda_n}.$$

Lemma 3 ([8]). *If* $\varlimsup\limits_{n\to\infty} \ln\ln n/\ln\lambda_n \leq 1$ *then*

$$\varrho_l[F] = 1 + \varlimsup_{n\to\infty} \ln\lambda_n/\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)$$

and if, moreover, $\ln\lambda_{n+1} \sim \ln\lambda_n$ *and* $\kappa_n[G] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then*

$$\lambda_l[F] = 1 + \varliminf_{n\to\infty} \ln\lambda_n/\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right).$$

If $1 < \varrho_l[F] < +\infty$ *and* $\ln n = o(\lambda_n^{\varrho_l[F]/(\varrho_l[F]-1)})$ *as* $n\to\infty$ *then*

$$T_l[F] = A(\varrho_l[F])\varlimsup_{n\to\infty} \lambda_n^{\varrho_l[F]}\left(\ln\frac{1}{|f_n|}\right)^{1-\varrho_l[F]}, \quad A(\varrho) = (\varrho-1)^{\varrho-1}\varrho^{\varrho},$$

and if, moreover, $\lambda_{n+1} \sim \lambda_n$ *and* $\kappa_n[G] \nearrow +\infty$ *as* $n_0 \leq n \to \infty$ *then*

$$t_l[F] = A(\varrho_l[F])\varliminf_{n\to\infty} \lambda_n^{\varrho_l[F]}\left(\ln\frac{1}{|f_n|}\right)^{1-\varrho_l[F]},$$

Choosing $\alpha(x)=\beta(x)=x$ and $\gamma(x)=\ln x$, from Theorem 3 we obtain the following statement.

Proposition 1. *If the function G has regular growth, $\kappa_n[G]\nearrow+\infty$, $\ln n=o(\lambda_n\ln\lambda_n)$ and $\ln\lambda_{n+1}\sim\ln\lambda_n$ as $n_0\le n\to\infty$ then $\varrho_{R,R}[F]_G=\varlimsup\limits_{n\to\infty}\ln|g_n|/\ln|f_n|$.*

If, moreover, $\kappa_n[F]\nearrow+\infty$ then $\lambda_{R,R}[F]_G=\varliminf\limits_{n\to\infty}\ln|g_n|/\ln|f_n|$.

This result can be directly obtained using Lemma 2. It is easy to see also that the functions $\alpha(x)=\beta(x)=\gamma(x)=\ln x$ do not satisfy the conditions of Theorem 3. However, the following statement is correct.

Proposition 2. *If the function G has regular logarithmic growth, $\varlimsup\limits_{n\to\infty}\ln\ln n/\ln\lambda_n\le1$, $\ln\lambda_{n+1}\sim\ln\lambda_n$ and $\kappa_n[G]\nearrow+\infty$ as $n_0\le n\to\infty$ then*

$$\varrho_{l,l}[F]_G=P_l:=\varlimsup_{n\to\infty}\frac{\ln\ln\dfrac{1}{|f_n|}\ln\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}{\ln\ln\dfrac{1}{|g_n|}\ln\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|f_n|}\right)}. \tag{10}$$

If, moreover, $\kappa_n[F]\nearrow+\infty$ as $n_0\le n\to\infty$ then

$$\lambda_{l,l}[F]_G=p_l:=\varliminf_{n\to\infty}\frac{\ln\ln\dfrac{1}{|f_n|}\ln\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|g_n|}\right)}{\ln\ln\dfrac{1}{|g_n|}\ln\left(\dfrac{1}{\lambda_n}\ln\dfrac{1}{|f_n|}\right)}. \tag{11}$$

Proof. Since G has generalized regular logarithmic growth, by Corollary 2 $\varrho_{l,l}[F]_G=\dfrac{\varrho_l[F]}{\varrho_l[G]}$, $\lambda_{l,l}[F]_G=\dfrac{\lambda_l[F]}{\lambda_l[G]}$ and by Lemma 3

$$\varrho_{l,l}[F]_G=\varlimsup_{n\to\infty}\ln\ln\frac{1}{|f_n|}/\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\varliminf_{n\to\infty}\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)/\ln\ln\frac{1}{|g_n|}\le P_l.$$

On the other hand, if $P_l>0$ then for every $\varepsilon\in(0,\,P_l)$ there exists an increasing to $+\infty$ sequence (n_k) of integers such that

$$\frac{\ln\ln\dfrac{1}{|f_{n_k}|}}{\ln\left(\dfrac{1}{\lambda_{n_k}}\ln\dfrac{1}{|f_{n_k}|}\right)}\ge(P_l-\varepsilon)\frac{\ln\ln\dfrac{1}{|g_{n_k}|}}{\ln\left(\dfrac{1}{\lambda_{n_k}}\ln\dfrac{1}{|g_{n_k}|}\right)},$$

i. e. by Lemma 3

$$\varrho_l[F]=\varlimsup_{n\to\infty}\ln\ln\frac{1}{|f_n|}/\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\ge(P_l-\varepsilon)\varliminf_{n\to\infty}\ln\ln\frac{1}{|g_n|}/\ln\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)=$$
$$=(P_l-\varepsilon)\lambda_l[G]=(P_l-\varepsilon)\varrho_l[G],$$

whence in view of the arbitrariness of ε we get $\varrho_l[F]_G\ge P_l$. For $P_l=0$ the last inequality is obvious. Equality (10) is proved.

The proof of (11) is similar. □

The condition $\ln n = o(\lambda_n \ln \lambda_n)$ as $n \to \infty$ implies the condition $\varlimsup\limits_{n\to\infty} \ln \ln n/ \ln \lambda_n \le 1$. Therefore, using Lemmas 2 and 3 it is easy to prove the following statement.

Proposition 3. *If the function G has regular logarithmic growth, $\ln n = o(\lambda_n \ln \lambda_n)$, $\ln \lambda_{n+1} \sim \ln \lambda_n$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then*

$$\varrho_{R,l}[F]_G = \varlimsup_{n\to\infty} \frac{\lambda_n \ln \lambda_n \ln (1/|g_n|)}{\ln (1/|f_n|) \ln \ln (1/|g_n|)}.$$

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$\lambda_{R,l}[F]_G = \varliminf_{n\to\infty} \frac{\lambda_n \ln \lambda_n \ln (1/|g_n|)}{\ln (1/|f_n|) \ln \ln (1/|g_n|)}.$$

Let us turn to the results about the relative growth of functions in terms of their types. For classic growth scales, we can use Lemmas 2 and 3, and for generalized orders we need such lemma.

Lemma 4. *Let $\alpha \in L$, $\beta \in L$, $x\alpha'(x) = o(1)$, $x\beta'(x) = O(1)$ and $\dfrac{d\beta^{-1}(c_1 + c_2\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for each $0 < c_1, c_2 < +\infty$). If $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n \to \infty$ for each $c \in (0, +\infty)$ and $G \in S(\Lambda)$ then*

$$\varlimsup_{n\to\infty} \exp\left\{\alpha(\lambda_n) - \varrho_{\alpha,\beta}[G]\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)\right\} = T_{\alpha,\beta}[G]. \tag{12}$$

If, moreover, $\alpha(\lambda_{n+1}) - \alpha(\lambda_n) \to 0$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$\varliminf_{n\to\infty} \exp\left\{\alpha(\lambda_n) - \varrho_{\alpha,\beta}[G]\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)\right\} = t_{\alpha,\beta}[G]. \tag{13}$$

Proof. Put $\alpha_1(x) = \exp\{\alpha(x)\}$ and $\beta_1(x) = \exp\{\varrho_{\alpha,\beta}[G]\beta(x)\}$. Then

$$T_{\alpha,\beta}[G] = \varlimsup_{\sigma\to+\infty} \frac{\alpha_1(\ln M_G(\sigma))}{\beta_1(\sigma)} = \varrho_{\alpha_1,\beta_1}[G].$$

If, for example, $c > 1$ then

$$\alpha(cx) - \alpha(x) = \alpha'(\xi)(c-1)x \le (c-1)\xi\alpha'(\xi)$$

for some $\xi \in [x, cx]$ and, since $x\alpha'(x) = o(1)$ as $x \to +\infty$, we have $\alpha(cx) - \alpha(x) \to 0$ as $x \to +\infty$, i. e. $\alpha_1 \in L_{si}$. Similarly, in view of condition $x\beta'(x) = O(1)$ as $x \to +\infty$, we have $\beta((1 + o(1))x) - \beta(x) = \beta'(\xi)o(\xi) \to 0$ as $x \to +\infty$, whence it follows that $\beta_1 \in L^0$.

Since $\beta_1^{-1}(x) = \beta^{-1}((\ln x)/\varrho_{\alpha,\beta}[G])$, we have

$$\frac{d\beta_1^{-1}(c\alpha_1(x))}{d\ln x} = \frac{d\beta^{-1}((\ln c + \alpha(x))/\varrho_{\alpha,\beta}[G])}{d\ln x} = \frac{d\beta^{-1}(c_1 + c_2\alpha(x))}{d\ln x} = O(1), \quad x \to +\infty.$$

Finally, the condition $\ln n = o(\lambda_n\beta_1^{-1}(c\alpha_1(\lambda_n)))$ as $n \to \infty$ holds if $\ln n = o(\lambda_n\beta^{-1}((\ln c + \alpha(\lambda_n))/\varrho_{\alpha,\beta}[G]))$ as $n \to \infty$. But $(\ln c+\alpha(\lambda_n))/\varrho_{\alpha,\beta}[G] \ge \alpha(\lambda_n)/(2\varrho_{\alpha,\beta}[G])$ for $n \ge n_0$. Therefore, the last condition holds if $\ln n = o(\lambda_n\beta^{-1}(c\alpha(\lambda_n)))$ as $n \to \infty$ for $c = 1/(2\varrho_{\alpha,\beta}[G])$. Thus, the functions α_1 and β_1 satisfy the conditions of Lemma 1 and in view of (8) formula (12) is proved. Also $\dfrac{\alpha_1(\lambda_{n+1})}{\alpha_1(\lambda_n)} = \exp\{\alpha(\lambda_{n+1}) - \alpha(\lambda_n)\} \to 1$ as $n \to \infty$ and, therefore, by Lemma 1 formulas (13) is correct. □

Using Lemma 4, we prove the following theorem.

Theorem 4. *Let* $\alpha \in L$, $\beta \in L$, $\gamma \in L$, $x\alpha'(x) = O(1)$, $x\beta'(x) = O(1)$, $x\gamma'(x) = o(1)$, $\dfrac{d\alpha^{-1}(c_1 + c_2\gamma(x))}{d\ln x} = O(1)$ *and* $\dfrac{d\beta^{-1}(c_1 + c_2\gamma(x))}{d\ln x} = O(1)$ *as* $x \to +\infty$ *for each* $0 < c_1, c_2 < +\infty$. *Suppose that* $\ln n = o(\lambda_n\alpha^{-1}(c\gamma(\lambda_n)))$ *and* $\ln n = o(\lambda_n\beta^{-1}(c\gamma(\lambda_n)))$ *as* $n \to \infty$ *for each* $c \in (0, +\infty)$.

If the function G *has strongly regular generalized* (γ, α)*-growth (i. e.* $0 < t_{\gamma,\alpha}[G] = T_{\gamma,\alpha}[G] < +\infty$*),* $\gamma(\lambda_{n+1}) - \gamma(\lambda_n) \to 0$ *and* $\varkappa_n[G] \nearrow +\infty$ *as* $n_0 \le n \to \infty$ *then*

$$(T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} = Q := \exp\{\varlimsup_{n\to\infty} Q_n(F,\, G)\},$$

where

$$Q_n(F,\, G) = \varrho_{\gamma,\alpha}[G]\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right) - \varrho_{\gamma,\beta}[F]\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right).$$

If, moreover, $\varkappa_n[F] \nearrow \infty$ *as* $n_0 \le n \to \infty$ *then*

$$(t_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} = q := \exp\{\varliminf_{n\to\infty} Q_n(F,\, G)\}.$$

Proof. Since the function G has strongly regular generalized (γ, α)-growth, by Theorem 2 $(T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} = T_{\gamma,\beta}[F]/T_{\gamma,\alpha}[G]$ and $(t_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} = t_{\gamma,\beta}[F]/t_{\gamma,\alpha}[G]$. Therefore, by Lemma 4

$$(T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} =$$
$$= \varlimsup_{n\to\infty} \exp\left\{\gamma(\lambda_n) - \varrho_{\gamma,\beta}[F]\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right\} \varliminf_{n\to\infty} \exp\left\{\varrho_{\gamma,\alpha}[G]\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right) - \gamma(\lambda_n)\right\} \le$$
$$\le \varlimsup_{n\to\infty} \exp\{Q_n(F,\, G)\} = Q$$

and

$$(t_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} =$$
$$= \varliminf_{n\to\infty} \exp\left\{\gamma(\lambda_n) - \varrho_{\gamma,\beta}[F]\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right\} \varlimsup_{n\to\infty} \exp\left\{\varrho_{\gamma,\alpha}[G]\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right) - \gamma(\lambda_n)\right\} \ge$$
$$\ge \varliminf_{n\to\infty} \exp\{Q_n(F,\, G)\} = q.$$

On the other hand, let $Q > 0$. Then for every $Q_1 \in (0,\, Q)$ there exists an increasing to ∞ sequence (n_k) of integers such that $\exp\{Q_{n_k}(F,\, G)\} \ge Q_1$, i. e.

$$\exp\left\{\gamma(\lambda_{n_k}) - \varrho_{\gamma,\beta}[F]\beta\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|f_{n_k}|}\right)\right\} > Q_1 \exp\left\{\gamma(\lambda_{n_k}) - \varrho_{\gamma,\alpha}[F]\alpha\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|g_{n_k}|}\right)\right\}$$

and, thus,

$$T_{\gamma,\beta}[F] = \varlimsup_{n\to\infty} \exp\left\{\gamma(\lambda_n) - \varrho_{\gamma,\beta}[F]\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)\right\} \ge$$
$$\ge Q_1 \varliminf_{n\to\infty} \exp\left\{\gamma(\lambda_n) - \varrho_{\gamma,\alpha}[F]\alpha\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)\right\} Q_1 t_{\gamma,\alpha}[G] = Q_1 T_{\gamma,\alpha}[G],$$

whence in view of the arbitrariness of Q_1 we get $T_{\alpha,\beta}[F]_G \ge Q$. For $Q = 0$ the last inequality is obvious. The equality $(T_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} = Q$ is proved.

Similar we prove the inequality $t_{\alpha,\beta}[F]_G \le q$, i. e. we get the equality $(t_{\alpha,\beta}[F]_G)^{\varrho_{\gamma,\alpha}[G]} = q$. The proof of Theorem 4 is complete. □

Next three statements are proved in general a way and we will drop their proofs. Using Corollary 2 and Lemma 2, we get the following statement.

Proposition 4. *If the function G has the strongly regular growth (i. e. $0 < t_R[G] = T_R[G] < +\infty$), $\ln n = o(\lambda_n)$, $\lambda_{n+1} \sim \lambda_n$ and $\varkappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then*

$$(T_{R,R}[F]_G)^{\varrho_R[G]} = \overline{\lim_{n\to+\infty}} \frac{\varrho_R[G]}{\varrho_R[F]} |f_n|^{\varrho_R[F]/\lambda_n} |g_n|^{-\varrho_R[G]/\lambda_n}.$$

If, moreover, $\varkappa_n[F] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$(t_{R,R}[F]_G)^{\varrho_R[G]} = \underline{\lim_{n\to+\infty}} \frac{\varrho_R[G]}{\varrho_R[F]} |f_n|^{\varrho_R[F]/\lambda_n} |g_n|^{-\varrho_R[G]/\lambda_n}.$$

Since the condition $\ln \ln n = o(\ln \lambda_n)$ as $n \to \infty$ implies the condition $\ln n = o(\lambda_n^{\varrho/(\varrho-1)})$ as $n \to \infty$ for every $\varrho > 1$, using Corollary 2 and Lemma 3, we get the next statement.

Proposition 5. *Let $1 < \varrho_l[F], \varrho_l[G] < +\infty$. If the function G has the strongly regular logarithmic growth (i. e. $0 < t_l[G] = T_l[G] < +\infty$), $\ln \ln n = o(\ln \lambda_n)$, $\lambda_{n+1} \sim \lambda_n$ and $\varkappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then*

$$\frac{A(\varrho_l[G])}{A(\varrho_l[F])} (T_{l,l}[F]_G)^{\varrho_R[G]} = \overline{\lim_{n\to\infty}} \lambda_n^{\varrho_l[F]-\varrho_l[G]} \left(\ln \frac{1}{|f_n|}\right)^{1-\varrho_l[F]} \left(\ln \frac{1}{|g_n|}\right)^{\varrho_l[G]-1}.$$

If, moreover, $\varkappa_n[F] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$\frac{A(\varrho_l[G])}{A(\varrho_l[F])} (t_{R,R}[F]_G)^{\varrho_R[G]} = \underline{\lim_{n\to\infty}} \lambda_n^{\varrho_l[F]-\varrho_l[G]} \left(\ln \frac{1}{|f_n|}\right)^{1-\varrho_l[F]} \left(\ln \frac{1}{|g_n|}\right)^{\varrho_l[G]-1}.$$

Finally, since the condition $\ln \ln n = o(\ln \lambda_n)$ as $n \to \infty$ implies the condition $\ln n = o(\lambda_n)$ as $n \to \infty$, using Corollary 2 and Lemmas 2 and 3, we get the next statement.

Proposition 6. *Let $1 < \varrho_l[G] < +\infty$. If the function G has the strongly regular logarithmic growth, $\ln \ln n = o(\ln \lambda_n)$, $\lambda_{n+1} \sim \lambda_n$ and $\varkappa_n[G] \nearrow +\infty$ as $n_0 \le n \to \infty$ then*

$$e\varrho_R[F] A(\varrho_l[G]) (T_{R,l}[F]_G)^{\varrho_R[G]} = \overline{\lim_{n\to\infty}} \lambda_n^{1-\varrho_l[G]} |f_n|^{\varrho_l[F]/\lambda_n} \left(\ln \frac{1}{|g_n|}\right)^{\varrho_l[G]-1}.$$

If, moreover, $\varkappa_n[F] \nearrow +\infty$ as $n_0 \le n \to \infty$ then

$$e\varrho_R[F] A(\varrho_l[G]) (t_{R,l}[F]_G)^{\varrho_R[G]} = \underline{\lim_{n\to\infty}} \lambda_n^{1-\varrho_l[G]} |f_n|^{\varrho_l[F]/\lambda_n} \left(\ln \frac{1}{|g_n|}\right)^{\varrho_l[G]-1}.$$

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Мулява О. М., Шеремета М. М. *Відносне зростання цілих рядів Діріхле з різними узагальненими порядками* // Буковинський матем. журнал — 2021. — Т.9, №2. — С. 22–34.

Для цілих функції F і G, зображених рядами Діріхле зі зростаючими до $+\infty$ показниками, знайдено формули для знаходження узагальненого порядку

$$\varrho_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}$$

і узагальненого нижнього порядку

$$\lambda_{\alpha,\beta}[F]_G = \varliminf_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}$$

функції F відносно функції G, де $M_F(\sigma) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$, а α і β - додатні зростаючі до $+\infty$ функції.

Ключові слова і фрази: ряд Діріхле, узагальнений порядок, відносне зростання.