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ON PSEUDOSTARLIKE AND PSEUDOCONVEX DIRICHLET SERIES

The concepts of the pseudostarlikeness of order $\alpha \in [0,\, 1)$ and type $\beta \in (0,\, 1]$ and the pseudoconvexity of the order α and type β are introduced for Dirichlet series of the form $F(s) = e^{-sh} + \sum_{j=1}^{n} a_j \exp\{-sh_j\} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}$, where $h > h_n > \cdots > h_1 \geq 1$ and (λ_k) is an increasing to $+\infty$ sequence of positive numbers. Criteria for pseudostarlikeness and pseudoconvexity in terms of coefficients are proved. The obtained results are applied to the study of meromorphic starlikeness and convexity of the Laurent series $f(s) = 1/z^p + \sum_{j=1}^{p-1} a_j/z^j + \sum_{k=1}^{\infty} f_k z^k$. Conditions, under which the differential equation $w'' + \gamma w' + (\delta e^{2sh} + \tau)w = 0$ has a pseudostarlike or pseudoconvex solution of the order α and the type $\beta = 1$ are investigated.

Key words and phrases: Dirichlet series, pseudostarlikeness, pseudoconvexity, differential equation.

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Dedicated to the memory of V.K. Maslyuchenko

1 Introduction

Let S be a class of functions

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n \tag{1}$$

analytic univalent in $\mathbb{D} = \{z :\ |z| < 1\}$. Function $f \in S$ is said to be starlike if $f(\mathbb{D})$ is starlike domain concerning of the origin. It is well known [1, p. 202] that the condition $\mathrm{Re}\,\{zf'(z)/f(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the starlikeness of f. A.W. Goodman [2] (see also [3, p. 9]) proved that if $\sum_{n=2}^{\infty} n|f_n| \leq 1$ then function (1) is starlike.

The concept of the starlikeness of function (1) got the series of generalizations. I.S. Jack [4] studied starlike functions of order $\alpha \in [0,\, 1)$, i. e. such functions (1), for which $\mathrm{Re}\,\{zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D})$. It is proved [4], [3, p. 13] that if $\sum_{n=2}^{\infty} (n-\alpha)|f_n| \leq 1-\alpha$ then function (1) is starlike function of order α. V.P. Gupta [5] introduced the concept of

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starlike function of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. We point out that the concept of p-valent starlike function $f(z) = z^p + \sum_{n=p+1}^{\infty} f_n z^n$ have appeared comparatively recently (see, for example, [6], [7] and [3, p. 14]).

Let Σ be a class of functions

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n \tag{2}$$

analytic in $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$. The function $f \in \Sigma$ is said to be meromorphically starlike of the order $\alpha \in [0, 1)$ if $\mathrm{Re}\{-zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D}_0)$. O.P. Juneja and T.R. Reddy [8] proved (see also [3, p. 14]) that if $\sum_{n=2}^{\infty}(n+\alpha)|f_n| \le 1-\alpha$ then function (2) is meromorphically starlike function of order α. According to B.A. Uralegaddi [9] the function (2) is said meromorphically starlike function of order $\beta \in (0, 1]$ if $|zf'(z) + f(z)| < \beta|zf'(z) - f(z)|$ for all $z \in \mathbb{D}_0$. Finally, combining these definitions, M.L. Mogra, T.R. Reddy and O.P. Juneja [10] call a function $f \in \Sigma$ to be meromorphically starlike of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if

$$|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|, \quad z \in \mathbb{D}_0,$$

and prove that if

$$\sum_{n=1}^{\infty}((1+\beta)n + \beta(2\alpha-1)+1)|f_n| \le 2\beta(1-\alpha),$$

then the function (2) is meromorphic starlike of the order α and the type β. Using this statement, O.M. Mulyava and Yu.S. Trukhan [11] indicated conditions on the parameters $a_1^{(1)}, a_2^{(1)}, a^{(0)}, a_2^{(0)}, a_3^{(0)}$ of the differential equation of S. Shah $z^2 w'' + (a_1^{(1)} z^2 + a_2^{(1)} z)w' + {}$ $+(a_1^{(0)} z^2 + a_2^{(0)} z + a_3^{(0)})w = 0$, under which this equation has a meromorphically starlike solution of the order α and the type β.

W.C. Royster [12] began to study the meromorphically starlikiness of the $f(z) = 1/z^p + \sum\limits_{n=1}^{\infty} f_n z^n$ $(p \in \mathbb{N})$ type species. Studies were continued by various authors (see, for example, the bibliography in [13]).

Since Dirichlet series with positive increasing to $+\infty$ exponents are direct generalizations of power series, here was a necessity of a construction of the geometrical theory for the class of Dirichlet series, absolutely convergent in half-plane $\Pi_0 = \{s : \mathrm{Re}\, s < 0\}$. To the decision this problem the article [14] is sanctified to (see also [3, p. 135-154]).

So, let $h \ge 1$, $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers $(\lambda_1 > h)$ and $SD(\Lambda, 0)$ be a class of Dirichlet series

$$F(s) = e^{sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{3}$$

with the exponents Λ and the abscissa of absolute convergence $\sigma_a[F] = 0$. It is known [14] that each function $F \in SD(\Lambda, 0)$ is non-univalent in Π_0, but there exist conformal in Π_0 functions (3), and if $\sum\limits_{k=2}^{\infty} \lambda_k|f_k| \le \lambda_1$ then function (3) is conformal in Π_0. A conformal function (3) in Π_0 is said to be pseudostarlike if $\mathrm{Re}\{F'(s)/F(s)\} > 0$ for $s \in \Pi_0$. In [14] (see also [3, p. 139]) it is proved that if $\sum\limits_{k=2}^{\infty} \lambda_k |f_k| \le \lambda_1$ then function (3) is pseudostarlike.

A conformal function (3) in Π_0 is said to be pseudostarlike of the order $\alpha \in [0, 1)$ if [15] $\mathrm{Re}\{F'(s)/F(s)\} > \alpha$ for all $s \in \Pi_0$ and is said to be pseudostarlike of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if $|F'(s)/F(s) - h| < \beta |F'(s)/F(s) - (2\alpha - h)|$ for all $s \in \Pi_0$.

Proposition 1 [15]. *In order for function* (3) *to be pseudostarlike of the order* α *and the type* β *it is sufficient and in the case when* $f_k \le 0$ *for all* $k \ge 1$ *it is necessary that* $\sum\limits_{k=1}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\}|f_k| \le 2\beta(h - \alpha)$.

In [15] the properties of absolutely convergent in Π_0 Dirichlet series of the form

$$F(s) = e^{-sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{4}$$

were also studied, where $h \ge 1$ and (λ_k) is an increasing to $+\infty$ sequence of positive numbers. Dirichlet series (4) is called Σ-pseudostarlike of order $\alpha \in [0, 1)$ if

$$\mathrm{Re}\{F'(s)/F(s)\} < -\alpha, \quad s \in \Pi_0 \tag{5}$$

and is called Σ-pseudostarlike of order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if

$$\left|\frac{F'(s)}{F(s)} + h\right| < \beta \left|\frac{F'(s)}{F(s)} + (2\alpha - h)\right|, \quad s \in \Pi_0. \tag{6}$$

Proposition 2 [15]. *In order for function* (4) *to be* Σ*-pseudostarlike of the order* α *and the type* β *it is sufficient and in the case when* $f_k \ge 0$ *for all* $k \ge 1$ *it is necessary that* $\sum\limits_{k=1}^{\infty} \{(1+\beta)\lambda_k + 2\beta\alpha + h(1-\beta)\} \le 2\beta(h - \alpha)$.

In proposed article we consider Dirichlet series of the form

$$F(s) = e^{-sh} + \sum_{j=1}^{n} a_j \exp\{-sh_j\} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{7}$$

where $h > h_n > \dots > h_1 \ge 1$, (λ_k) is an increasing to $+\infty$ sequence of positive numbers and $\sigma_a[F] = 0$.

2 Pseudostarlikeness

Dirichlet series (7) is called Σ-pseudostarlike of order $\alpha \in [0, 1)$ if (5) holds and is called Σ-pseudostarlike of order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if (6) holds.

Theorem 1. *If*

$$\sum_{j=1}^{n} \{\beta(h + h_j - 2\alpha) + h - h_j\}|a_j| + \sum_{k=1}^{\infty} \{(1+\beta)\lambda_k + 2\beta\alpha + h(1-\beta)\} \le 2\beta(h - \alpha) \tag{8}$$

then function (7) *is* Σ*-pseudostarlike of the order* α *and the type* β.

Proof. Clearly, (6) holds if and only if

$$|F'(s)+hF(s)|-\beta|F'(s)+(2\alpha-h)F(s)|<0,\quad s\in\Pi_0. \tag{9}$$

On the other hand, in view of (8)

$$|F'(s)+hF(s)|-\beta|F'(s)+(2\alpha-h)F(s)|=$$

$$=\left|\sum_{j=1}^{n}a_j(h-h_j)e^{-sh_j}+\sum_{k=1}^{\infty}(\lambda_k+h)f_ke^{s\lambda_k}\right|-$$

$$-\beta\left|2(\alpha-h)e^{-sh}+\sum_{j=1}^{n}(2\alpha-h-h_j)a_je^{-sh_j}+\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_ke^{s\lambda_k}\right|=$$

$$=e^{-\sigma h}\left(\left|\sum_{j=1}^{n}(h-h_j)a_je^{s(h-h_j)}+\sum_{k=1}^{\infty}(\lambda_k+h)f_ke^{s(\lambda_k+h)}\right|-\right.$$

$$\left.-\beta\left|2(\alpha-h)+\sum_{j=1}^{n}(2\alpha-h-h_j)a_je^{s(h-h_j)}+\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_ke^{s(\lambda_k+h)}\right|\right)\le$$

$$=e^{-\sigma h}\left(\left|\sum_{j=1}^{n}(h-h_j)a_je^{s(h-h_j)}+\sum_{k=1}^{\infty}(\lambda_k+h)f_ke^{s(\lambda_k+h)}\right|-\right.$$

$$\left.-\beta\left(2(h-\alpha)-\left|\sum_{j=1}^{n}(2\alpha-h-h_j)a_je^{s(h-h_j)}+\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_ke^{s(\lambda_k+h)}\right|\right)\right)\le$$

$$=e^{-\sigma h}\left(\left|\sum_{j=1}^{n}(h-h_j)a_je^{s(h-h_j)}+\sum_{k=1}^{\infty}(\lambda_k+h)f_ke^{s(\lambda_k+h)}\right|+\right.$$

$$\left.+\beta\left|\sum_{j=1}^{n}(2\alpha-h-h_j)a_je^{s(h-h_j)}+\sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_ke^{s(\lambda_k+h)}\right|-2\beta(h-\alpha)\right)\le$$

$$\le e^{-\sigma h}\left(\sum_{j=1}^{n}\{\beta(h+h_j-2\alpha)+h-h_j\}|a_j|e^{\sigma(h-h_j)}+\right.$$

$$\left.+\sum_{k=1}^{\infty}\{(\beta+1)\lambda_k+2\alpha\beta+(1-\beta)h)|f_k|e^{\sigma(\lambda_k+h)}-2\beta(h-\alpha)\right)<$$

$$<e^{-\sigma h}\left(\sum_{j=1}^{n}\{\beta(h+h_j-2\alpha)+h-h_j\}|a_j|+\right.$$

$$\left.+\sum_{k=1}^{\infty}\{(\beta+1)\lambda_k+2\alpha\beta+(1-\beta)h)|f_k|-2\beta(h-\alpha)\right)\le 0,$$

i. e. (9) holds, and Theorem 1 is proved.

Theorem 2. *Let $f_k\ge 0$ for all $k\ge 1$ and $a_j\ge 0$ for all $1\le j\le n$. If function* (7) *is Σ-pseudostarlike of the order α and the type β then* (8) *holds.*

Proof. Since function (7) is Σ-pseudostarlike of the order α and the type β, $f_k = |f_k|$ for all $k \geq 1$ and $a_j = |a_j|$ for all $1 \leq j \leq n$, in view of (9) we have for all $s \in \Pi_0$

$$\mathrm{Re}\,\frac{\sum\limits_{j=1}^{n} a_j(h-h_j)e^{-sh_j} + \sum\limits_{k=1}^{\infty}(\lambda_k+h)f_k e^{s\lambda_k}}{2(h-\alpha)e^{-sh} - \sum\limits_{j=1}^{n}(2\alpha-h-h_j)a_j e^{-sh_j} - \sum\limits_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k e^{s\lambda_k}} \leq$$

$$\leq \left|\frac{\sum\limits_{j=1}^{n} a_j(h-h_j)e^{-sh_j} + \sum\limits_{k=1}^{\infty}(\lambda_k+h)f_k e^{s\lambda_k}}{2(h-\alpha)e^{-sh} + \sum\limits_{j=1}^{n}(2\alpha-h-h_j)a_j e^{-sh_j} + \sum\limits_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k e^{s\lambda_k}}\right| =$$

$$= \left|\frac{F'(s)+hF(s)}{F'(s)+(2\alpha-h)F(s)}\right| < \beta,$$

and, therefore, since $f_k = |f_k|$ and $a_j = |a_j|$, for all $\sigma < 0$ we obtain

$$\frac{\sum\limits_{j=1}^{n} a_j(h-h_j)e^{-\sigma h_j} + \sum\limits_{k=1}^{\infty}(\lambda_k+h)f_k e^{\sigma\lambda_k}}{2(h-\alpha)e^{-\sigma h} - \sum\limits_{j=1}^{n}(h+h_j-2\alpha)a_j e^{-\sigma h_j} - \sum\limits_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k e^{\sigma\lambda_k}} < \beta.$$

Letting $\sigma \to 0$ from here we get

$$\frac{\sum\limits_{j=1}^{n} a_j(h-h_j) + \sum\limits_{k=1}^{\infty}(\lambda_k+h)f_k}{2(h-\alpha) - \sum\limits_{j=1}^{n}(h+h_j-2\alpha)a_j - \sum\limits_{k=1}^{\infty}(\lambda_k+2\alpha-h)f_k} \leq \beta.$$

whence we get

$$\sum_{j=1}^{n}|a_j|(h-h_j) + \sum_{k=1}^{\infty}(\lambda_k+h)|f_k| \leq$$

$$\leq \beta\left(2(h-\alpha) - \sum_{j=1}^{n}(h+h_j-2\alpha)|a_j| - \sum_{k=1}^{\infty}(\lambda_k+2\alpha-h)|f_k|\right),$$

i. e. (8) follows. Theorem 2 is proved.

Since the inequality $|w+h| < |w+(2\alpha-h)|$ holds if and only if $\mathrm{Re}\,w < -\alpha$, function (7) is Σ-pseudostarlike of the order α if and only if $\left|\frac{F'(s)}{F(s)}+h\right| < \left|\frac{F'(s)}{F(s)}+(2\alpha-h)\right|$ for all $s \in \Pi_0$, i. e. (6) holds with $\beta = 1$. Therefore, Theorems 1 and 2 imply the following statement.

Corollary 1. *In order for function* (7) *to be Σ-pseudostarlike of the order $\alpha \in [0, 1)$ it is sufficient and in the case, when $f_k \geq 0$ for all $k \geq 1$ and $a_j \geq 0$ for all $1 \leq j \leq n$, it is necessary that*

$$\sum_{j=1}^{n}(h-\alpha)|a_j| + \sum_{k=1}^{\infty}(\lambda_k+\alpha)|f_k| \leq h-\alpha. \tag{10}$$

3 PSEUDOCONVEXITY

A conformal function (3) in Π_0 is said to be pseudoconvex if $\operatorname{Re}\{F''(s)/F'(s)\} > > 0$ for $s \in \Pi_0$. In [14] and [3, p. 139] it is proved that if $\sum\limits_{k=2}^{\infty} \lambda_k^2 |f_k| \le \lambda_1^2$ then function (3) is pseudoconvex. Here we call the function (7) Σ-pseudoconvex of the order $\alpha \in [0, 1)$ if $\operatorname{Re}\{F''(s)/F'(s)\} < -\alpha$, and Σ-pseudoconvex of the order α and the type $\beta \in (0, 1]$ if

$$\left|\frac{F''(s)}{F'(s)} + h\right| < \beta \left|\frac{F''(s)}{F'(s)} + 2\alpha - h\right|, \quad s \in \Pi_0. \tag{11}$$

Since $F''(s)/F'(s) = G'(s)/G(s)$, where

$$G(s) = e^{-sh} + \sum_{j=1}^{n} \frac{h_j}{h} a_j \exp\{sh\} - \sum_{k=1}^{\infty} \frac{\lambda_k}{h} f_k \exp\{s\lambda_k\},$$

the function F is Σ-pseudoconvex of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if and only if the function G is Σ-pseudostarlike of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$. Therefore, from Theorems 1 and 2 one can easily obtain the corresponding results for Σ-pseudoconvex functions.

Proposition 3. *In order for function* (7) *to be Σ-pseudoconvex of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ it is sufficient and in the case, when $f_k \le 0$ for all $k \le 1$ and $a_j \ge 0$ for all $1 \le j \le n$, it is necessary that*

$$\sum_{j=1}^{n} h_j\{(\beta(h + h_j - 2\alpha) + h - h_j)\}|a_j| +$$

$$+ \sum_{k=1}^{\infty} \lambda_k\{(1+\beta)\lambda_k + 2\beta\alpha + h(1-\beta)\}|f_k| \le 2h\beta(h-\alpha).$$

For $\beta = 1$ hence we obtain the following statement.

Proposition 4. *In order for function* (7) *to be Σ-pseudoconvex of the order $\alpha \in [0, 1)$ it is sufficient and in the case, when $f_k \le 0$ for all $k \le 1$ and $a_j \ge 0$ for all $1 \le j \le n$, it is necessary that*

$$\sum_{j=1}^{n} h_j(h-\alpha)|a_j| + \sum_{k=1}^{\infty} \lambda_k\{\lambda_k + \alpha\}|f_k| \le h(h-\alpha).$$

4 EROMORPHICAL STARLIKENESS AND CONVEXITY OF LAURENT SERIES.

Let the function f analytic in $\mathbb{D}_0 = \{0 < |z| < 1\}$ represented by the Laurent series

$$f(s) = \frac{1}{z^p} + \sum_{j=1}^{p-1} \frac{a_j}{z^j} + \sum_{k=1}^{\infty} f_k z^k \tag{12}$$

Function (12) is called meromorphically starlike of the order $\alpha \in [0, 1)$ if $\mathrm{Re}\{zf'(z)/f(z)\} < -\alpha$ for all $z \in \mathbb{D}_0$, and is called meromorphically starlike of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if

$$\left|\frac{zf'(z)}{f(z)} + p\right| < \beta \left|\frac{zf'(z)}{f(z)} + 2\alpha - p)\right|, \quad z \in \mathbb{D}_0, \tag{13}$$

Making in (12) a replacement $z = e^s$, we obtain Dirichlet series

$$F(s) = f(e^s) = e^{-ps} + \sum_{j=1}^{p-1} a_j e^{-js} + \sum_{k=1}^{\infty} f_k e^{ks} \tag{14}$$

i. e. we obtain Dirichlet series (7) with $\lambda_k = k$, $h_j = j$, $h = p$ and $n = p - 1$.

Since $\dfrac{F'(s)}{F(s)} = \dfrac{e^s f'(e^s)}{f(e^s)} = \dfrac{zf'(z)}{f(z)}$, conditions (13) and (6) are equivalent. Therefore, Theorems 1 and 2 imply the following statement.

Proposition 5. *In order for function* (12) *to be meromorphically starlike of the order* $\alpha \in [0, 1)$ *and of the type* $\beta \in (0, 1]$ *it is sufficient and in the case, when* $f_k \geq 0$ *for all* $k \leq 1$ *and* $a_j \geq 0$ *for all* $1 \leq j \leq n$*, it is necessary that*

$$\sum_{j=1}^{p-1} \{(1+\beta)p - 2\beta\alpha - j(1-\beta)\}|a_j| + \sum_{k=1}^{\infty} \{(1+\beta)\lambda_k + 2\beta\alpha + p(1-\beta)\}|f_k| \leq 2\beta(p - \alpha).$$

The following proposition also is true.

Proposition 6. *In order for function* (12) *to be meromorphically starlike of the order* $\alpha \in [0, 1)$ *it is sufficient and in the case when* $f_k \geq 0$ *for all* $k \leq 1$ *and* $a_j \geq 0$ *for all* $1 \leq j \leq n$ *it is necessary that*

$$\sum_{j=1}^{p-1} (p - \alpha)|a_j| + \sum_{k=1}^{\infty} (\lambda_k + \alpha)|f_k| \leq p - \alpha.$$

Finally, function (12) is called meromorphically convex of the order $\alpha \in [0, 1)$ if $\mathrm{Re}\{zf''(z)/f'(z)\} < -\alpha$ for all $z \in \mathbb{D}_0$, and is called meromorphically convex of the order $\alpha \in [0, 1)$ and the type $\beta \in (0, 1]$ if

$$\left|\frac{zf''(z)}{f'(z)} + p\right| < \beta \left|\frac{zf''(z)}{f'(z)} + 2\alpha - p)\right|, \quad z \in \mathbb{D}_0, \tag{15}$$

Since conditions (15) and (11) are equivalent, from Propositions 3 and 4 we get the following statements.

Proposition 7. *In order for function* (12) *to be meromorphically convex of the order* $\alpha \in [0, 1)$ *and the type* $\beta \in (0, 1]$ *it is sufficient and in the case, when* $f_k \leq 0$ *for all* $k \leq 1$ *and* $a_j \geq 0$ *for all* $1 \leq j \leq n$*, it is necessary that*

$$\sum_{j=1}^{p-1} \{(\beta(p + j - 2\alpha) + p - j)\}|a_j| +$$

$$+\sum_{k=1}^{\infty}\lambda_k\{(1+\beta)\lambda_k+2\beta\alpha+p(1-\beta)\}|f_k|\le 2p\beta(p-\alpha).$$

Proposition 8. *In order for function* (12) *to be meromorphically convex of the order* $\alpha\in[0,\,1)$ *it is sufficient and in the case, when* $f_k\le 0$ *for all* $k\le 1$ *and* $a_j\ge 0$ *for all* $1\le j\le n$, *it is necessary that*

$$\sum_{j=1}^{p-1}p(p-\alpha)|a_j|+\sum_{k=1}^{\infty}\lambda_k\{\lambda_k+\alpha\}|f_k|\le p(p-\alpha),$$

5 Addition. Properties of some differential equation

In [14] the conditions were studied, under which the differential equation $w''+(\gamma_0 e^{2sh}+ +\gamma_1 e^{sh}+\gamma_2)w=0$ has an entire solution $F(s)=e^{-sh}+\sum\limits_{k=1}^{\infty}f_k\exp\{s\lambda_k\}$, pseudostarlike or pseudoconvex in Π_0.

Here we consider a differential equation

$$w''+\gamma w'+(\delta e^{2sh}+\tau)w=0, \tag{16}$$

where the parameters γ, δ, τ are real numbers, and study the conditions, under which this equation has Σ-pseudostarlike or Σ-pseudoconvex solution of the form

$$F(s)=e^{-sh}+a_1e^{-sh_1}+\sum_{k=1}^{\infty}f_k\exp\{s\lambda_k\},\quad s=\sigma+it, \tag{17}$$

where $h>h_1\ge 1$. Substituting (17) in (16) we get

$$h^2e^{-sh}+h_1^2a_1e^{-sh_1}+\sum_{k=1}^{\infty}\lambda_k^2f_ke^{s\lambda_k}+\gamma\left(-he^{-sh}-h_1a_1e^{-sh_j}+\sum_{k=1}^{\infty}\lambda_kf_ke^{s\lambda_k}\right)+$$
$$+(\delta e^{2sh}+\tau)\left(e^{-sh}+a_1e^{-sh_1}+\sum_{k=1}^{\infty}f_ke^{s\lambda_k}\right)\equiv 0,$$

i. e.

$$(h^2-\gamma h+\tau)e^{-sh}+(h_1^2-\gamma h_1+\tau)e^{-sh_1}+\delta a_1e^{s(2h-h_1)}+\delta e^{sh}+$$
$$+\sum_{k=1}^{\infty}(\lambda_k^2+\gamma\lambda_k+\tau)f_ke^{s\lambda_k}+\sum_{k=1}^{\infty}\delta f_ke^{s(\lambda_k+2h)}\equiv 0. \tag{18}$$

Suppose that

$$\tau=\gamma h-h^2,\quad h+1\le\gamma\le 2h,\quad h_1=\gamma-h. \tag{19}$$

Then $h^2-\gamma h+\tau=0$, $h_1^2-\gamma h_1+\tau=0$, $h>h_1\ge 1$ and from (18) we get

$$\delta a_1e^{s(2h-h_1)}+\delta e^{sh}+\sum_{k=1}^{\infty}(\lambda_k^2-h^2+\gamma(\lambda_k+h))f_ke^{s\lambda_k}+\sum_{k=1}^{\infty}\delta f_ke^{s(\lambda_k+2h)}\equiv 0. \tag{20}$$

Since $e^{as} = o(e^{bs})$ as $\sigma \to -\infty$ for $a > b$ and $2h - h_1 > h$, from (20) we get

$$(1 + o(1))\delta e^{sh} + +(\lambda_1^2 - h^2 + \gamma(\lambda_1 + h))f_1 e^{s\lambda_1} = 0, \quad \sigma \to -\infty,$$

whence

$$\lambda_1 = h, \quad f_1 = -\frac{\delta}{2\gamma h} \tag{21}$$

and

$$\sum_{k=2}^{\infty}(\lambda_k^2 - h^2 + \gamma(\lambda_k + h))f_k e^{s\lambda_k} + \sum_{k=1}^{\infty}\delta f_k e^{s(\lambda_k+2h)} \equiv 0. \tag{22}$$

From (22) we get

$$(\lambda_2^2 - h^2 + \gamma(\lambda_2 + h))f_2 e^{s\lambda_2} + (1 + o(1))\delta f_1 e^{s(\lambda_1+2h)} = 0, \quad \sigma \to -\infty,$$

whence

$$\lambda_2 = 3h, \quad f_2 = -\frac{\delta}{\lambda_2^2 - h^2 + \gamma(\lambda_2 + h)} f_1 = \frac{\delta^2}{8\gamma h^3(2h + \gamma)}.$$

Continuing this process, we will come to the equality

$$\sum_{k=n}^{\infty}(\lambda_k^2 - h^2 + \gamma(\lambda_k + h))f_k e^{s\lambda_k} + \sum_{k=n-1}^{\infty}\delta f_k e^{s(\lambda_k+2h)} \equiv 0,$$

whence we obtain

$$(\lambda_n^2 - h^2 + \gamma(\lambda_n + h))f_n e^{s\lambda_n} + (1 + o(1))\delta f_{n-1} e^{s(\lambda_{n-1}+2h)} = 0, \quad \sigma \to -\infty,$$

and, thus, $\lambda_n = \lambda_{n-1} + 2h = (2n - 1)h$ and

$$f_n = -\frac{\delta}{\lambda_n^2 - h^2 + \gamma(\lambda_n + h)} f_{n-1} = -\frac{\delta}{2nh(2(n - 1)h + \gamma)} f_{n-1} \tag{23}$$

for $n \geq 2$. Therefore, the following lemma is valid.

Lemma 1. *Let the parameters γ, τ and h_1 satisfy conditions* (19). *Then function* (17) *is a solution to equation* (16) *if and only if $\lambda_k = (2k - 1)h$ and for the coefficients f_k formulas* (21) *and* (23) *are correct.*

Corollary 1 and Proposition 4 imply the following lemma.

Lemma 2. *Let $h \geq 1$, $\lambda_k = (2k - 1)h$ and $\alpha \in [0, 1)$. If*

$$(h - \alpha)|a_1| + \sum_{k=1}^{\infty}((2k - 1)h + \alpha)|f_k| \leq h - \alpha. \tag{24}$$

then function (17) *is Σ-pseudostarlike of the order α, and if*

$$h_1(h - \alpha)|a_1| + \sum_{k=1}^{\infty}(2k - 1)h\{(2k - 1)h + \alpha\}|f_k| \leq h(h - \alpha). \tag{25}$$

then function (17) *is Σ-pseudoconvex of the order α.*

Using these lemmas now prove the following theorem.

Theorem 3. *Let $\alpha \in [0,\, 1)$, $\lambda_k = (2k-1)h$ and conditions* (19) *hold. Suppose that*

$$4h(2h+\gamma)(h+\alpha) > (3h+\alpha)|\delta|. \tag{26}$$

Then differential equation (16) *has solution* (17), *which by condition*

$$(h-\alpha)|a_1| + \frac{2(2h+\gamma)(h+\alpha)^2|\delta|}{4h(2h+\gamma)(h+\alpha)-(3h+\alpha)|\delta|} \le h-\alpha \tag{27}$$

is Σ-pseudostarlike of the order α, and by condition

$$h_1(h-\alpha)|a_1| + \frac{2h(2h+\gamma)(h+\alpha)^2|\delta|}{\gamma(4h(2h+\gamma)(h+\alpha)-(3h+\alpha)|\delta|)} \le h(h-\alpha) \tag{28}$$

is Σ-pseudoconvex of the order α.

Proof. From (21) and (23) we get

$$\sum_{k=1}^{\infty}((2k-1)h+\alpha)|f_k| = (h+\alpha)|f_1| + \sum_{k=2}^{\infty}((2k-1)h+\alpha)|f_k| \le$$

$$\le \frac{(h+\alpha)|\delta|}{2\gamma h} + \sum_{k=2}^{\infty}\frac{((2k-1)h+\alpha)|\delta|}{2kh(2(k-1)h+\gamma)}|f_{k-1}| =$$

$$= \frac{(h+\alpha)|\delta|}{2\gamma h} + \sum_{k=1}^{\infty}\frac{((2k+1)h+\alpha)|\delta|}{2(k+1)h(2kh+\gamma)}|f_k| =$$

$$= \frac{(h+\alpha)|\delta|}{2\gamma h} + \sum_{k=1}^{\infty}\frac{((2k+1)h+\alpha)|\delta|}{2(k+1)h(2kh+\gamma)((2k-1)h+\alpha)}((2k-1)h+\alpha)|f_k| <$$

$$= \frac{(h+\alpha)|\delta|}{2\gamma h} + \sum_{k=1}^{\infty}\frac{(3h+\alpha)|\delta|}{4h(2h+\gamma)(h+\alpha)}((2k-1)h+\alpha)|f_k|,$$

because the sequence $\left(\dfrac{((2k+1)h+\alpha)|\delta|}{2(k+1)h(2kh+\gamma)((2k-1)h+\alpha)}\right)$ is decreasing. Hence it follows that

$$\sum_{k=1}^{\infty}\left(1-\frac{(3h+\alpha)|\delta|}{4h(2h+\gamma)(h+\alpha)}\right)((2k-1)h+\alpha)f_k| < \frac{(h+\alpha)|\delta|}{2\gamma h}.$$

Condition (26) implies $1-\dfrac{(3h+\alpha)|\delta|}{4h(2h+\gamma)(h+\alpha)} > 0$. Therefore,

$$\left(1-\frac{(3h+\alpha)|\delta|}{4h(2h+\gamma)(h+\alpha)}\right)\sum_{k=1}^{\infty}((2k-1)h+\alpha)f_k| < \frac{(h+\alpha)|\delta|}{2\gamma h}$$

and by condition (27)

$$(h-\alpha)|a_1| + \sum_{k=1}^{\infty}((2k-1)h+\alpha)f_k| < (h-\alpha)|a_1| + \frac{h(h+\alpha)|\delta|}{2\gamma h\left(1-\dfrac{(3h+\alpha)|\delta|}{4h(2h+\gamma)(h+\alpha)}\right)} =$$

$$= (h-\alpha)|a_1| + \frac{2(2h+\gamma)(h+\alpha)^2|\delta|}{\gamma(4h(2h+\gamma)(h+\alpha)-(3h+\alpha)|\delta|)} \le h-\alpha$$

In view of Lemma 2 function (17) is Σ-pseudostarlike of the order α.

Similarly,

$$\sum_{k=1}^{\infty}(2k-1)h((2k-1)h+\alpha)|f_k| = (h+\alpha)|f_1| + \sum_{k=2}^{\infty}(2k-1)h((2k-1)h+\alpha)|f_k| \le$$

$$\le \frac{h(h+\alpha)|\delta|}{2\gamma h}+$$

$$+\sum_{k=1}^{\infty}\frac{(2k-1)h((2k+1)h+\alpha)|\delta|}{2(k+1)h(2kh+\gamma)(2k-1)h((2k-1)h+\alpha)}(2k-1)h((2k-1)h+\alpha)|f_k| <$$

$$< \frac{h(h+\alpha)|\delta|}{2\gamma h} + \sum_{k=1}^{\infty}\frac{h(3h+\alpha)|\delta|}{4h(2h+\gamma)(h+\alpha)}(2k-1)h((2k-1)h+\alpha)|f_k|,$$

whence in view of (27) as above we get

$$\left(1-\frac{(3h+\alpha)|\delta|}{4(2h+\gamma)(h+\alpha)}\right)\sum_{k=1}^{\infty}(2k-1)h((2k-1)h+\alpha)|f_k| < \frac{h(h+\alpha)|\delta|}{2\gamma}$$

and, thus, in view of (**??**)

$$h_1(h-\alpha)|a_1| + \sum_{k=1}^{\infty}(2k-1)h((2k-1)h+\alpha)|f_k| <$$

$$< h_1(h-\alpha)|a_1| + \frac{2h(2h+\gamma)(h+\alpha)^2|\delta|}{\gamma(4h(2h+\gamma)(h+\alpha)-(3h+\alpha)|\delta|)} \le h(h-\alpha).$$

By Lemma 2 function (17) is Σ-pseudoconvex of the order α. The proof of Theorem 3 is completed.

References

[1] Golusin G. M. Geometrical theory of functions of complex variables. M., Nauka, 1966 (in Russian); Engl. transl.: AMS: Translations of Mathematical monograph., 1969, **26**.

[2] Goodman A. W. *Univalent functions and nonanalytic curves.* Proc. Amer. Math. Soc., 1957, **8** (3), 597–601.

[3] Sheremeta M. M. Geometric properties of analytic solutions of differential equations. Lviv, Publisher I. E. Chyzhykov, 2019.

[4] Jack I. S. *Functions starlike and convex of order α.* J. London Math. Soc., 1971, **3**, 469–474.

[5] Gupta V. P. *Convex class of starlike functions.* Yokohama Math. J., 1984, **32**, 55–59.

[6] Owa S. *On certain classes of p-valent functions with negative coefficients.* Simon Stevin, 1985, **59**, 385–402.

[7] El-Ashwah R. M., Aouf M. K., Moustava A. O. *Starlike and convexity properties for p-valent hypergeometric functions.* Acta Math. Univ. Comenianae, 2010, **79** (1), 55–64.

[8] Juneja O. P., Reddy T. R. *Meromorphic starlike and univalent functions with positive coefficients.* Ann. Univ. Mariae Curie-Sklodowska, 1985, (39), 65–76.

[9] Uralegaddi B. A. *Meromorphic starlike functions with positive coefficients.* Kyungpook. Math. J., 1989, **29** (1), 64–68.

[10] Mogra M. L., Reddy T. R., Juneja O. P. *Meromorphic univalent functions with positive coefficients.* Bull. Austral. Math. Soc., 1985, **32** (2), 161–176.

[11] Mulyava O. M., Truhan Yu. S. *On meromorphically starlike functions of the order α and the type β, which satisfy Shah's differential equations.* Carpatian Math. Publ., 2017, **9** (2), 154–162.

[12] Royster W. C. *Meromorphic starlike multivalent functions.* Trans. Amer. Math. Soc., 1963, **107**, 300–308.

[13] Faisal Imran, Darus M., Shah F. A. *A critical studi of meromorphic starlike functions.* TWMS J. Appl. Eng. Math., 2018, **8** (1), 238–242.

[14] Holovata O. M., Mulyava O. M., Sheremeta M. M. *Pseudostarlike, pseudoconvex and close-to-pseudoconvex Dirichlet series satisfying differential equations with exponential coefficients.* Math. methods and physicomech. fields, 2018, **61** (1), 57–70 (in Ukrainian).

[15] Sheremeta M. M. *Pseudostarlike and pseudoconvex Dirichlet series of the order α and the type β.* Mat. Stud., 2020, **54** (1), 23–31.

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Введено поняття псевдозірковості порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$ і псевдоопуклості порядку α і типу β для рядів Діріхле вигляду $F(s) = e^{-sh} + \sum\limits_{j=1}^{n} a_j \exp\{-sh_j\} + \sum\limits_{k=1}^{\infty} f_k \exp\{s\lambda_k\}$, де $h > h_n > \cdots > h_1 \geq 1$ і (λ_k) – зростаюча до $+\infty$ послідовність додатних чисел. Доведено критерії псевдозірковості і псевдоопуклості у термінах коефіцієнтів. Отримані результати застосовано до вивчення мероморфної зірковсті та опуклості рядів Лорана $f(s) = 1/z^p + \sum_{j=1}^{p-1} a_j/z^j + \sum_{k=1}^{\infty} f_k z^k$. Досліджено умови, за яких диференціальне рівняння $w'' + \gamma w' + (\delta e^{2sh} + \tau)w = 0$ має псевдозірковий, або псевдоопуклий розв'язок порядку α типу $\beta = 1$.