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ASYMPTOTIC BEHAVIOR OF THE LOGARITHMIC DERIVATIVE OF ENTIRE FUNCTION OF IMPROVED REGULAR GROWTH IN THE METRIC OF $L^Q[0, 2\pi]$

Let f be an entire function with $f(0) = 1$, $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of its zeros, $n(t) = \sum_{|\lambda_n|\le t} 1$, $N(r) = \int_0^r t^{-1}n(t)\,dt$, $r > 0$, $h(\varphi)$ be the indicator of f, and $F(z) = zf'(z)/f(z)$, $z = re^{i\varphi}$. An entire function f is called a function of improved regular growth if for some $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that

$$\log |f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

In this paper, we prove that an entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for some $\rho_3 \in (0, \rho)$

$$N(r) = c_0 r^\rho + o(r^{\rho_3}), \quad r \to +\infty, \quad c_0 \in [0, +\infty),$$

and for some $\rho_2 \in (0, \rho)$ and any $q \in [1, +\infty)$, one has

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\operatorname{Im} F(re^{i\varphi})}{r^\rho} + h'(\varphi) \right|^q d\varphi \right\}^{1/q} = o(r^{\rho_2 - \rho}), \quad r \to +\infty.$$

Key words and phrases: entire function of improved regular growth, logarithmic derivative, Fourier coefficients, finite system of rays, indicator, Hausdorff-Young theorem.

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1 Introduction and main result

It is well known that ([14, p. 24]) an entire function f of order $\rho \in (0, +\infty)$ can be represented in the form

$$f(z) = z^\lambda e^{Q(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{\lambda_n}, p\right),$$

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where λ_n are all nonzero roots of the function $f(z)$, $\lambda \in \mathbb{Z}_+$ is the multiplicity of the root at the origin, $Q(z) = \sum_{k=1}^{\nu} Q_k z^k$ is a polynomial of degree $\nu \le \rho$, $p \le \rho$ is the smallest integer for which $\sum_{n=1}^{\infty} |\lambda_n|^{-p-1} < +\infty$ and $E(w,p) = (1-w)\exp(w + w^2/2 + \cdots + w^p/p)$ is the Weierstrass primary factor.

Let f be an entire function of order $\rho \in (0,+\infty)$. The function

$$h(\varphi) = \limsup_{r\to+\infty} \frac{\log|f(re^{i\varphi})|}{r^\rho}, \quad \varphi \in [0, 2\pi],$$

is called the *indicator* of f ([14, p. 51]). The indicator is continuous 2π-periodic ρ-trigonometrically convex function that has a derivative at all points except possibly of a countable set (see [14, pp. 52–55]). A set $C \subset \mathbb{C}$ is called a *C^0-set* ([14, p. 90]) if it can be covered by a system of disks $\{z : |z - a_k| < s_k\}$, $k \in \mathbb{N}$, satisfying $\sum_{|a_k|\le r} s_k = o(r)$ as $r \to +\infty$. A set $E \subset [0,+\infty)$ is called an *E_η-set* ([14, p. 96]) if $\limsup_{r\to+\infty} r^{-1}\mathrm{mes}(E \cap [0,r]) \le \eta$, $\eta \in (0,1]$.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of zeros of an entire function f, $f(0) = 1$, let $F(z) := zf'(z)/f(z)$, $z = re^{i\varphi}$, and let

$$n(r) := \sum_{|\lambda_n|\le r} 1, \quad N(r) := \int_0^r \frac{n(t)}{t}\, dt, \quad r > 0.$$

An entire function f of order $\rho \in (0,+\infty)$ with the indicator $h(\varphi)$ is said to be of *completely regular growth* in the sense of Levin and Pfluger (see [2], [14, pp. 139–167]) if there exists a C^0-set such that

$$\log|f(re^{i\varphi})| = r^\rho h(\varphi) + o(r^\rho), \quad C^0 \not\ni re^{i\varphi} \to \infty,$$

uniformly in $\varphi \in [0, 2\pi)$.

The asymptotic behavior of the logarithms and logarithmic derivatives of entire and meromorphic functions of positive order of completely regular growth in the metric of $L^q[0,2\pi]$ have been described in [13, 16, 17]. Similar results for entire functions of zero order of slowly regular growth were obtained in [1, 15]. In particular, [17, Theorem 3, p. 140] implies the following statement.

Theorem A. *Let f be an entire function of order $\rho \in (0,+\infty)$ with the indicator $h(\varphi)$ and $f(0) = 1$. Then the following assertions are equivalent:*

1) f is of completely regular growth;

2) for some $q \in [1,+\infty)$ and $\widetilde{h} : [0,2\pi] \to \mathbb{R}$, $\widetilde{h} \in L^q[0,2\pi]$, one has

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\mathrm{Im}\, F(re^{i\varphi})}{r^\rho} - \widetilde{h}(\varphi) \right|^q d\theta \right\}^{1/q} \to 0, \quad r \to +\infty, \quad r \notin E \in E_\eta, \quad \eta \in (0,1),$$

and

$$N(r) = c_0 r^\rho + o(r^\rho), \quad r \to +\infty, \quad c_0 \in [0,+\infty).$$

In this case, $\widetilde{h}(\varphi) = -h'(\varphi)$ for almost all $\varphi \in [0,2\pi]$.

The aim of the present paper is to obtain an analog of Theorem A for entire functions of improved regular growth (for details, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 19, 20]) with zeros on a finite system of rays.

An entire function f is called a function of *improved regular growth* ([5, 19]) if for some $\rho\in(0,+\infty)$ and $\rho_1\in(0,\rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi)\not\equiv-\infty$ there exists a set $U\subset\mathbb{C}$ contained in the union of disks with finite sum of radii and such that

$$\log|f(z)|=|z|^\rho h(\varphi)+o(|z|^{\rho_1}),\quad U\not\ni z=re^{i\varphi}\to\infty.$$

If an entire function f is of improved regular growth, then it has the order ρ and indicator $h(\varphi)$ ([19]). In the case when zeros of an entire function f of improved regular growth are situated on a finite system of rays $\{z:\arg z=\psi_j\}$, $j\in\{1,\dots,m\}$, $0\le\psi_1<\psi_2<\ldots<\psi_m<2\pi$, the indicator h has the form ([19])

$$h(\varphi)=\sum_{j=1}^m h_j(\varphi),\quad \rho\in(0,+\infty)\setminus\mathbb{N},\tag{1}$$

where $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j,\psi_j+2\pi)$

$$h_j(\varphi)=\frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi-\psi_j-\pi),\quad \Delta_j\in[0,+\infty).$$

In the case $\rho\in\mathbb{N}$, the indicator h is defined by the formula ([5])

$$h(\varphi)=\begin{cases}\tau_f\cos(\rho\varphi+\theta_f)+\displaystyle\sum_{j=1}^m h_j(\varphi), & p=\rho,\\ Q_\rho\cos\rho\varphi, & p=\rho-1,\end{cases}\tag{2}$$

where $\delta_f\in\mathbb{C}$, $\tau_f=|\delta_f/\rho+Q_\rho|$, $\theta_f=\arg(\delta_f/\rho+Q_\rho)$ and $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j,\psi_j+2\pi)$

$$h_j(\varphi)=\Delta_j(\pi-\varphi+\psi_j)\sin\rho(\varphi-\psi_j)-\frac{\Delta_j}{\rho}\cos\rho(\varphi-\psi_j).$$

Our main result is the following theorem.

Theorem 1. *Let f be an entire function of order $\rho\in(0,+\infty)$ with zeros on a finite system of rays $\{z:\arg z=\psi_j\}$, $j\in\{1,\dots,m\}$, $0\le\psi_1<\psi_2<\ldots<\psi_m<2\pi$, $f(0)=1$, and $h(\varphi)$ be the indicator of f. If f is a function of improved regular growth, then for some $\rho_2\in(0,\rho)$ and any $q\in[1,+\infty)$, one has*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\operatorname{Im}F(re^{i\varphi})}{r^\rho}+h'(\varphi)\right|^q d\varphi\right\}^{1/q}=o(r^{\rho_2-\rho}),\quad r\to+\infty,\tag{3}$$

where $h(\varphi)$ is defined by formulas (1) and (2). Conversely, if for some $\rho_3\in(0,\rho)$

$$N(r)=c_0r^\rho+o(r^{\rho_3}),\quad r\to+\infty,\quad c_0:=\frac{1}{\rho}\sum_{j=1}^m\Delta_j,\quad \Delta_j\in[0,+\infty),\tag{4}$$

and for some $\rho_2\in(0,\rho)$ and any $q\in[1,+\infty)$ relation (3) is true, then f is an entire function of improved regular growth.

2 Preliminaries

Let f be an entire function with $f(0)=1$, let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of its zeros, let $\Omega=\{|\lambda_n|: n\in\mathbb{N}\}$, and let ([16, p. 42])

$$c_k(r,\log|f|):=\frac{1}{2\pi}\int_0^{2\pi}e^{-ik\varphi}\log|f(re^{i\varphi})|\,d\varphi,\quad k\in\mathbb{Z},\quad r>0,$$

$$c_k(r,\operatorname{Im}F):=\frac{1}{2\pi}\int_0^{2\pi}e^{-ik\varphi}\operatorname{Im}F(re^{i\varphi})\,d\varphi,\quad k\in\mathbb{Z},\quad r>0,\quad r\notin\Omega,$$

be a *Fourier coefficients* of the functions $\log|f(re^{i\varphi})|$ and $\operatorname{Im}F(re^{i\varphi})$, respectively.

Lemma 1. *If an entire function f of order $\rho\in(0,+\infty)$ with zeros on a finite system of rays $\{z:\arg z=\psi_j\}$, $j\in\{1,\ldots,m\}$, $0\le\psi_1<\psi_2<\ldots<\psi_m<2\pi$, is of improved regular growth, then for some $\rho_4\in(0,\rho)$*

$$c_k(r,\operatorname{Im}f)=-ikc_kr^{\rho}+\frac{k}{k^2+1}o(r^{\rho_4}),\quad r\to+\infty,\tag{5}$$

holds uniformly in $k\in\mathbb{Z}$, where

$$c_k:=\frac{1}{2\pi}\int_0^{2\pi}e^{-ik\varphi}h(\varphi)\,d\varphi=\frac{\rho}{\rho^2-k^2}\sum_{j=1}^{m}\Delta_je^{-ik\psi_j},\quad \Delta_j\in[0,+\infty),\tag{6}$$

if $\rho\in(0,+\infty)\setminus\mathbb{N}$, and

$$c_k=\begin{cases}\dfrac{\rho}{\rho^2-k^2}\displaystyle\sum_{j=1}^{m}\Delta_je^{-ik\psi_j}, & |k|\ne\rho=p,\\ \dfrac{\tau_fe^{i\theta_f}}{2}-\dfrac{1}{4\rho}\displaystyle\sum_{j=1}^{m}\Delta_je^{-i\rho\psi_j}, & k=\rho=p,\\ 0, & |k|\ne\rho=p+1,\\ \dfrac{Q_\rho}{2}, & k=\rho=p+1,\end{cases}\tag{7}$$

if $\rho\in\mathbb{N}$.

Proof. If an entire function f of order $\rho\in(0,+\infty)$ satisfies the assumptions of Lemma 1, then ([6, Lemma 1, p. 10]) (see also [9]) for some $\rho_4\in(0,\rho)$ the following relation holds

$$c_k(r,\log|f|)=c_kr^{\rho}+\frac{o(r^{\rho_4})}{k^2+1},\quad r\to+\infty,\tag{8}$$

uniformly in $k\in\mathbb{Z}$, where c_k are defined by formulas (6) and (7). Since ([16, p. 43])

$$c_k(r,\operatorname{Im}f)=-ikc_k(r,\log|f|),\quad k\in\mathbb{Z},\tag{9}$$

using (8), we obtain (5). Lemma 1 is proved. □

Remark that, from relations (6)–(8) and the Fischer–Riesz theorem ([13, p. 5]) it follows the existence of an indicator function $h \in L^2[0,2\pi]$ defined by the equality $h(\varphi) := \sum_{k\in\mathbb{Z}} c_k e^{ik\varphi}$ (see also [13, p. 77]).

Lemma 2 ([7]). *An entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for some $\rho_5 \in (0,\rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0+1,\dots,k_0+m-1\}$, one has*

$$c_k(r,\log|f|) = c_k r^\rho + o(r^{\rho_5}), \quad r \to +\infty,$$

where c_k are defined by formulas (6) *and* (7).

3 Proof of Theorem 1

Let f be an entire function of improved regular growth of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, and $h(\varphi)$ be the indicator of f defined by formulas (1) and (2). By virtue of Lemma 1, for some constant $C > 0$ and all $r \ge r_0 > 0$, we have

$$\left|\frac{c_k(r,\operatorname{Im} F)}{r^\rho} + ikc_k\right| \le C\frac{k}{k^2+1}, \quad k \in \mathbb{Z}. \tag{10}$$

In view of this, the sequence $(r^{-\rho}c_k(r,\operatorname{Im} F) + ikc_k)_{k\in\mathbb{Z}}$ belongs to the space $l_{\widetilde{q}}$ for all $\widetilde{q} > 1$ and $r \ge r_0$. Moreover, applying the Hausdorff-Young theorem ([13, p. 5]), for $q \ge 2$, $q^{-1} + \widetilde{q}^{-1} = 1$, we get

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\operatorname{Im} F(re^{i\varphi})}{r^\rho} + h'(\varphi)\right|^q d\varphi\right\}^{1/q} \le \left\{\sum_{k\in\mathbb{Z}}\left|\frac{c_k(r,\operatorname{Im} F)}{r^\rho} + ikc_k\right|^{\widetilde{q}}\right\}^{1/\widetilde{q}}.$$

According to (10), the obtained series is uniformly convergent on $[r_0,+\infty)$. Passing termwise to the limit as $r \to +\infty$ in this series and using Lemma 1, we obtain relation (3) for $q \ge 2$. From this and Hölder's inequality it follows that (3) holds for $1 \le q < 2$.

Let us prove the second part of the theorem. Let relations (3) and (4) be hold. Then for some $\rho_2 \in (0,\rho)$ and each $k \in \mathbb{Z}\setminus\{0\}$

$$\begin{aligned}\left|\frac{c_k(r,\operatorname{Im} F)}{r^\rho} + ikc_k\right| &\le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\operatorname{Im} F(re^{i\varphi})}{r^\rho} + h'(\varphi)\right| d\varphi \\ &\le \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\operatorname{Im} F(re^{i\varphi})}{r^\rho} + h'(\varphi)\right|^q d\varphi\right\}^{1/q} = o(r^{\rho_2-\rho}), \quad r \to +\infty,\end{aligned}$$

whence it follows

$$c_k(r,\operatorname{Im} F) = -ikc_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty.$$

From this, using relations (9), for some $\rho_2 \in (0,\rho)$, we obtain

$$c_k(r,\log|f|) = c_k r^\rho + o(r^{\rho_2}), \quad r \to +\infty, \quad k \in \mathbb{Z}\setminus\{0\},$$

and by using (4) and the equality ([16, p. 42]) $c_0(r, \log|f|) = N(r)$, for some $\rho_3 \in (0, \rho)$, we get

$$c_0(r, \log|f|) = c_0 r^\rho + o(r^{\rho_3}), \quad r \to +\infty.$$

Thus, according to Lemma 2, the entire function f is a function of improved regular growth. This concludes the proof of the theorem.

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Хаць Р.В. *Асимптотична поведінка логарифмічної похідної цілої функції покращеного регулярного зростання в $L^q[0,2\pi]$-метриці* // Буковинський матем. журнал — 2021. — Т.9, №1. — С. 49–55.

Нехай f — ціла функція, $f(0)=1$, $(\lambda_n)_{n\in\mathbb{N}}$ — послідовність її нулів, $n(t)=\sum_{|\lambda_n|\le t}1$, $N(r)=\int_0^r t^{-1}n(t)\,dt$, $r>0$, $h(\varphi)$ — індикатор функції f, і $F(z)=zf'(z)/f(z)$, $z=re^{i\varphi}$. Ціла функція f називається функцією покращеного регулярного зростання, якщо для деяких $\rho\in(0,+\infty)$, $\rho_1\in(0,\rho)$ і 2π-періодичної ρ-тригонометрично опуклої функції $h(\varphi)\not\equiv-\infty$ існує множина $U\subset\mathbb{C}$, яка міститься в об'єднанні кругів із скінченною сумою радіусів така, що

$$\log|f(z)|=|z|^\rho h(\varphi)+o(|z|^{\rho_1}),\quad U\not\ni z=re^{i\varphi}\to\infty.$$

В цій роботі доведено, що ціла функція f порядку $\rho\in(0,+\infty)$ з нулями на скінченній системі променів $\{z:\arg z=\psi_j\}$, $j\in\{1,\dots,m\}$, $0\le\psi_1<\psi_2<\ldots<\psi_m<2\pi$, є функцією покращеного регулярного зростання тоді і тільки тоді, коли для деякого $\rho_3\in(0,\rho)$

$$N(r)=c_0r^\rho+o(r^{\rho_3}),\quad r\to+\infty,\quad c_0\in[0,+\infty),$$

і для деякого $\rho_2\in(0,\rho)$ і кожного $q\in[1,+\infty)$ виконується

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\operatorname{Im}F(re^{i\varphi})}{r^\rho}+h'(\varphi)\right|^q d\varphi\right\}^{1/q}=o(r^{\rho_2-\rho}),\quad r\to+\infty.$$