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UNIQUENESS THEOREMS FOR ALMOST PERIODIC OBJECTS

Uniqueness theorems are considered for various types of almost periodic objects: functions, measures, distributions, multisets, holomorphic and meromorphic functions.

Key words and phrases: almost periodic function, almost periodic measure, almost periodic meromorphic function.

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Introduction

It easily follows from the definition of almost periodic functions that if values of two such functions converge at infinity, then these almost periodic functions coincide. This effect also manifested itself in [7] for the zeros of holomorphic almost periodic functions, and then in [1] and [2] for Fourier quasicrystals and some classes of transformable measures on LCA-groups.

In this note, we discuss this effect in detail, show how can it be strengthened, what form it takes for other almost periodic objects - almost periodic distributions, almost periodic measures, almost periodic multisets, a -points of holomorphic and meromorphic almost periodic functions.

1 Almost periodic functions

We start with the simplest almost periodic object - uniformly almost periodic functions on a finite-dimensional space and on tube sets. The definitions introduced in this section will also be used in subsequent sections.

Let $B_{\mathbb{C}}(z^0,R)$ be the open ball $\{z\in\mathbb{C}:\,|z-z^0|<R\}$ in the space $\mathbb{C}^d$, and $B_{\mathbb{R}}(x^0,R)$ be the open ball $\{x\in\mathbb{R}:\,|x-x^0|<R\}$ in the space $\mathbb{R}^d$. The tube set $T_K\subset\mathbb{C}^d$ means the set of the form

$$T_K=\left\{z=x+iy\in\mathbb{C}^d : x\in\mathbb{R}^d,\ y\in K\right\},$$

where K is a compact subset of $\mathbb{R}^d$. Clearly, $\mathbb{R}^d=T_{\{0\}}$. Then T_Ω means the domain

$$T_\Omega=\left\{z=x+iy : x\in\mathbb{R}^d,\ y\in\Omega\right\},$$

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where Ω is a domain in $\mathbb{R}^d$, maybe $\Omega = \mathbb{R}^d$. The set E is relatively dense in $\mathbb{R}^d$, if there exists $R < \infty$ such that each ball $B_{\mathbb{R}}(x, R)$ intersects with E. By $\#A$ denote the number of elements of a finite set A.

Definition 1. *A continuous complex-valued function $f(z)$ on a tube set T_K is called almost periodic if for all $\varepsilon > 0$ the set of its ε-almost periods*

$$E_{\varepsilon,K} = E_{\varepsilon,K}(f) = \{\tau \in \mathbb{R}^d : \sup_{z\in T_K} |f(z+\tau) - f(z)| < \varepsilon\}$$

is relatively dense in $\mathbb{R}^d$.

It easily follows from this definition that almost periodic functions on T_K are bounded. Less obvious is the following statement:

Theorem 1. *([9]) A continuous function $f(z)$ on T_K is almost periodic iff for any sequence $\{x_n\} \subset \mathbb{R}^d$ there is a subsequence $\{x_{n'}\}$ such that the functions $f_{n'}(z) = f(z + x_{n'})$ form the fundamental sequence with respect to the uniform convergence on T_K.*

Definition 2. *A function $f(z)$ on a tube domain T_Ω is called almost periodic if for every compact set $K \subset \Omega$ its restriction to T_K is almost periodic.*

Theorem 2. *([9]) A continuous function $f(z)$ is almost periodic on a tube domain T_Ω iff for any sequence $\{x_n\} \subset \mathbb{R}^d$ there is a subsequence $\{x_{n'}\}$ such that the functions $f_{n'}(z) = f(z + x_{n'})$ form the fundamental sequence with respect to the uniform convergence on T_K for every $K \subset \Omega$.*

Remark 1. *All these definitions and theorems carry over practically unchanged to the case of mappings $F : T_K \to \mathbb{C}^N$ or $F : T_\Omega \to \mathbb{C}^N$. Since component-wise convergence is equivalent to the convergence of mappings, we get that the vector function $F(z) = (f_1(z), \dots, f_N(z))$ is almost periodic if and only if its components are almost periodic. Therefore for any $\varepsilon > 0$ the set $E_{\varepsilon,K}$ of common almost periods of functions $f_1, \dots, f_N$ is also relatively dense. In particular, this implies that a sum or a product of any finite number of almost periodic functions is also an almost periodic function.*

In the rest of the article, only the cases of functions and sets on $\mathbb{R}^d$ or on $T_\Omega \subset C^d$ will be considered.

Next we give the basic definition of our article.

Definition 3. *We shall say that functions f, g on $\mathbb{R}^d$ converge weakly at infinity, if*

$$\lim_{x\to\infty, x\in G} |f(x) - g(x)| = 0,$$

where $G \subset \mathbb{R}^d$ is a set with the property

$$G \supset \bigcup_{k=1}^{\infty} B_{\mathbb{R}}(x_k, R_k) \quad \text{for some sequence of balls} \quad B_{\mathbb{R}}(x_k, R_k), \quad R_k \to \infty. \tag{1}$$

Definition 4. *Functions f, g on T_Ω converge weakly at infinity if for each fixed $y^0 \in \Omega$ the functions $f(x+iy^0)$, $g(x+iy^0)$ of the variable $x \in \mathbb{R}^d$ weakly converge at infinity.*

Theorem 3. *If almost periodic functions f, g on $\mathbb{R}^d$ or T_Ω converge weakly at infinity, then they coincide identically.*

Proof. Let f, g be almost periodic functions on $\mathbb{R}^d$. Fix $x^0 \in \mathbb{R}^d$ and $\varepsilon > 0$. Let E_ε be the set of ε-almost periods of the almost periodic function $h = f - g$. Taking into account (1) and relative density of E_ε, we get that for large n there is a point $\tau_n \in E_\varepsilon \cap B_{\mathbb{R}}(x_n - x^0, R_n)$. Hence $x_0 + \tau_n \in B_{\mathbb{R}}(x_n, R_n)$ and $|h(x_0 + \tau_n)| < \varepsilon$. Also, $|h(x^0 + \tau_n) - h(x^0)| < \varepsilon$, therefore, $|h(x^0)| < 2\varepsilon$. The choice of ε and x^0 was arbitrary, therefore $h(x) \equiv 0$. In the case of functions on T_Ω we take $z^0 = x^0 + iy^0 \in T_\Omega$ and a compact set $K \subset \Omega$ such that $y^0 \in K$, then replace E_ε by $E_{\varepsilon,K}$ and x^0 by z^0. Theorem is proved. □

2 Almost periodic distributions, measures, multisets

Let $D(\mathbb{R}^d)$ be the space of test functions on $\mathbb{R}^d$, i.e., C^∞-functions with compact supports, equipped with the topology of uniform convergence of derivatives of all orders of functions from $D(\mathbb{R}^d)$, provided that all their supports are subsets of some fixed compact from R^d, let $D'(\mathbb{R}^d)$ be the space of distributions on $\mathbb{R}^d$, that is, the set of continuous linear functionals on $D(\mathbb{R}^d)$. The distribution space $D'(T_\Omega)$ is similarly defined as continuous linear functionals on the space $D(T_\Omega)$, consisting of C^∞-functions with compact support in T_Ω.

Definition 5. *A distribution $f \in D'(\mathbb{R}^d)$ (or $f \in D'(T_\Omega)$) is called almost periodic, if for any test-function φ the function $(f, \varphi(\cdot - t))$ is almost periodic in the variable $t \in \mathbb{R}^d$.*

Definition 6. *A distribution $f \in D'(T_\Omega)$ is called almost periodic, if for any test-function $\varphi \in D(T_\Omega)$ the function $(f, \varphi(\cdot - z))$ is almost periodic in the variable $z \in T_\omega$. Here ω is the open subset of Ω such that for all $z \in T_\omega$ the condition $\zeta - z \in \operatorname{supp}\varphi$ implies $\zeta \in T_\Omega$.*

A particular case of distributions are complex-valued measures. Such measures will be denoted by μ, and the measure, which is the variation of μ, by $|\mu|$. A measure μ on $\mathbb{R}^d$ is called translation bounded if

$$\sup_{x\in\mathbb{R}^d} |\mu|(B_{\mathbb{R}}(x,1)) < \infty.$$

Similarly, a measure on T_Ω is called translation bounded if for any compact $K \subset \Omega$

$$\sup_{x\in\mathbb{R}^d} |\mu|(B_{\mathbb{R}}(x,1) \times K) \le C,$$

where C is a constant depending on K. Note that every nonnegative almost periodic measure is translation bounded. To prove this we should take a nonnegative test function $\varphi(z) \in D(T_\Omega)$ such that $\varphi(z) = 1$ on $B_{\mathbb{R}}(0,1) \times K$, where K is a compact subset of Ω (for the case $T_{\mathbb{R}^d}$ we should take nonnegative $\varphi \in D(R^d)$, $\varphi(x) = 1$ on $B_{\mathbb{R}}(0,1)$). The function

$$\int \varphi(z-t)\mu(dz) \tag{2}$$

is almost periodic, hence it is bounded in $t \in \mathbb{R}^d$. On the other hand, for all $t \in \mathbb{R}^d$

$$\mu(B_{\mathbb{R}}(t,1)\times K) \le \int \varphi(z-t)\mu(dz).$$

If a measure $\mu \in D'(T_\Omega)$ is translation bounded, then we can use any continuous function with compact support as test functions in Definition 5. This follows from the fact that any such a function can be uniformly approximated by C^∞-functions supported on a fixed compact set. On the other hand, there are signed almost periodic measures for which (2) are not almost periodic for an appropriate continuous compactly supported φ ([4]). Note that if (2) is bounded for all continuous φ with compact support, then the complex measure μ is translation bounded ([9]).

Let $D = \{a, p\}, p \in \mathbb{N}$, be a discrete multiset in T_Ω or in $\mathbb{R}^d$. It can be identified with a sequence $\{a_n\}$ without condensation points in T_Ω (or in $\mathbb{R}^d$) such that each point from T_Ω or in $\mathbb{R}^d$ can occur in this sequence at most a finite number of times. In the case of $T_\Omega \subset \mathbb{C}$ a discrete multiset is also called a divisor (see [6]).

Definition 7. *([6]) A discrete multiset $D \subset \mathbb{R}^d$ is called almost periodic if for all $\varepsilon > 0$ there is a relatively dense set $E_\varepsilon \subset \mathbb{R}^d$ such that a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ corresponds to any $\tau \in E_\varepsilon$ with the property*

$$\sup_{n\in\mathbb{N}} |a_n - \tau - a_{\sigma(n)}| < \varepsilon.$$

A discrete multiset $D \subset T_\Omega$ is called almost periodic if for all $\varepsilon > 0$ and compact set $K \subset \Omega$ there is a relatively dense set $E_{\varepsilon,K} \subset \mathbb{R}^d$ such that a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ corresponds to any $\tau \in E_{\varepsilon,K}$ with the property

$$\sup |a_n - \tau - a_{\sigma(n)}| < \varepsilon,$$

where supremum is taken over all $n \in \mathbb{N}$ such that either a_n, or $a_{\sigma(n)}$ belongs to T_K.

We also need a notion of *bounded density*. For a discrete multiset $D \subset \mathbb{R}^d$, $D = \{a_n\}$, this means that

$$\sup_{x\in\mathbb{R}^d} \#\{n :\ a_n \in B_{\mathbb{R}}(x,1)\} < \infty.$$

Also, $D \subset T_\Omega$ is of bounded density if for every compact $K \subset \Omega$

$$N(K) := \sup_{x\in\mathbb{R}^d} \#\{n :\ a_n \in B_{\mathbb{R}}(x,1)\times K\} < \infty. \tag{3}$$

It is easy to check that each almost periodic multiset is of bounded density. For $T_\Omega \subset \mathbb{C}^d$ the proof can be found in [6]. For convenience, we present it here. The proof for $D \subset \mathbb{R}^d$ differs only in the corresponding simplifications.

Set $\eta = \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$ (in the case $\Omega = \mathbb{R}^d$ set $\eta = \frac{1}{2}$). Take $R < \infty$ such that every ball $B_{\mathbb{R}}(x,R)$ intersects with $E_{\eta,K}$. Fix $\tau \in B_{\mathbb{R}}(x,R) \cap E_{\eta,K}$ and take the bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that for each $a_n \in T_K$

$$|a_n - \tau - a_{\sigma(n)}| < \eta.$$

For $a_n \in B_{\mathbb{R}}(x,1)\times K$ we get

$$|\mathrm{Re}\ a_{\sigma(n)}| \le |\mathrm{Re}\ a_{\sigma(n)} - \mathrm{Re}\ a_n + \tau| + |\mathrm{Re}\ a_n - x| + |x - \tau| < \eta + 1 + R,$$

$$\mathrm{Im}\ a_{\sigma(n)} = \mathrm{Im}\ a_n + (\mathrm{Im}\ a_{\sigma(n)} - \mathrm{Im}\ a_n).$$

Since $\mathrm{Im}\ a_n \in K$ and $|\mathrm{Im}\ a_{\sigma(n)} - \mathrm{Im}\ a_n| < \eta$, we get $\mathrm{Im}\ a_{\sigma(n)} \in K_1$, where $K_1 = \{y : \operatorname{dist}(y, K) \le \eta\}$. Thus,

$$\#\{n : a_n \in B_{\mathbb{R}}(x,1) \times K\} \le \#\{n : a_\sigma(n) \in B_{\mathbb{R}}(0, 1+\eta+R) \times K_1\},$$

and we obtain (3).

Note that the measure

$$\mu_D = \sum_n \delta_{a_n},$$

corresponds to each discrete multiset $D = \{a_n\}$, where δ_{a_n} is the unit mass at the point a_n.

Theorem 4. *A discrete multiset D is almost periodic iff the measure μ_D is almost periodic.*

For $D \subset \mathbb{C}$ this theorem was proved in [6], and for $D \subset \mathbb{R}^d$ in [3]. Here we give a new, much simpler proof for $D \subset T_\Omega$. The proof for $D \subset \mathbb{R}^d$ differs only in the corresponding simplifications.

Proof. Let a discrete multiset D be almost periodic and $K \subset \Omega$ be a compact set. Take a function $\varphi \in C^\infty(T_\Omega)$ such that $\operatorname{supp}\varphi \subset B_{\mathbb{R}}(0,1/2) \times K$. Let $\varepsilon > 0$ be arbitrary and $\delta < (1/2)\operatorname{dist}(K, \partial\Omega)$ such that for $|z - z'| < \delta$

$$|\varphi(z) - \varphi(z')| < \frac{\varepsilon}{N(K)},$$

where $N(K)$ is defined in (3). Pick $\tau \in E_{\delta,K}(D)$ and the corresponding bijection σ. We have

$$\int \varphi(z-\tau)\mu_D(dz) - \int \varphi(z)\mu_D(dz) = \sum_n \varphi(a_n - \tau) - \sum_n \varphi(a_n) =$$

$$= \sum_n [\varphi(a_n - \tau) - \varphi(a_{\sigma(n)})].$$

The number of terms in the letter sum does not exceed $2N(K)$, moreover, $|a_n - \tau - a_{\sigma(n)}| < \delta$, hence the difference between integrals does not exceed ε. Therefore the points of the set $E_{\delta,K}(D)$ are ε-almost periods of the function $(\mu_D(\varphi(\cdot - t))$. This reasoning is valid for every φ with compact support, therefore the measure μ_D is almost periodic.

On the other hand, let μ_D be the almost periodic measure on T_Ω, which corresponds to a discrete multiset $D = \{a_n\}$. Fix a compact set $K \subset \Omega$ and $\varepsilon < \frac{1}{4}\min\{1, \operatorname{dist}(K, \Omega)\}$. Put

$$\widetilde{K} = \{y \in \Omega : \operatorname{dist}(y, K) \le \varepsilon\}.$$

Choosing a sufficiently large K, we can assume that either $D \subset K$, or $D \setminus T_{\tilde{K}} \ne \varnothing$. Since μ_D is almost periodic we get that it is translation bounded, hence for some $N < \infty$

$$\mu_D(B_{\mathbb{R}}(x,1) \times \widetilde{K}) \le N, \quad \forall x \in \mathbb{R}^d,$$

therefore,

$$\#\{n : a_n \in B_{\mathbb{R}}(x,1) \times \widetilde{K}\} \le N,\ \forall x \in \mathbb{R}^d. \tag{4}$$

Set $\delta = \varepsilon/(4N+1)$. Let A be any connected component of the set $\bigcup_n B_{\mathbb{C}}(a_n, 2\delta)$ such that $A \cap T_K \neq \varnothing$. There exists $a_{n'} \in A$ such that $B_{\mathbb{C}}(a_{n'}, 2\delta) \cap T_K \neq \varnothing$. If $A \cap \partial B_{\mathbb{C}}(a_{n'}, \varepsilon) \neq \varnothing$, then the connected set $A \cap B_{\mathbb{C}}(a_{n'}, \varepsilon)$ contains at least $\varepsilon/(4\delta) > N$ points of D, which contradicts (4). Hence,

$$A \subset B_{\mathbb{C}}(a_{n'}, \varepsilon) \subset B_{\mathbb{R}}(\operatorname{Re} a_{n'}, 1) \times \widetilde{K}$$

and, by (4), $\#\{n : a_n \in A\} \le N$.

By $\varphi(z)$ denote any C^∞-function on $\mathbb{C}^d$ such that

$$0 \le \varphi(z) \le 1, \quad \varphi(0) = 1, \quad \operatorname{supp} \varphi \subset B_{\mathbb{C}}(0,1), \tag{5}$$

Let $\alpha = \int \varphi(z)\omega(dz)$, where ω is the Lebesgue measure on $\mathbb{C}^d$. Put

$$\Psi(z) := \int \varphi\left(\frac{z-w}{\delta}\right) \mu_D(dw) = \sum_n \varphi\left(\frac{z-a_n}{\delta}\right).$$

Since

$$\operatorname{dist}(\widetilde{K}, \partial\Omega) \ge \operatorname{dist}(K, \partial\Omega) - \varepsilon \ge \varepsilon,$$

we see that $\Psi(z)$ is defined and almost periodic on $T_{\widetilde{K}}$. Let τ be ρ-almost period of $\Psi(z)$ with $\rho < \min\{1; 2^{-2d}\alpha/(N\omega_{2d})\}$, where $\omega_{2d} = \omega(B_{\mathbb{C}}(0,1))$. We have

$$|\Psi(z+\tau) - \Psi(z)| < \rho, \quad \forall z \in T_{\widetilde{K}}. \tag{6}$$

On the other hand,

$$\Psi(z) = 0 \quad \text{for} \quad z \notin \cup_n B_{\mathbb{C}}(a_n, \delta) \quad \text{and} \quad \Psi(z+\tau) = \Psi(a_n) \ge 1 \quad \text{for } z = a_n - \tau.$$

Therefore the set $A \setminus \cup_n B_{\mathbb{C}}(a_n, \delta)$ does not contain any point $a_n - \tau$. If A' is another connected component of the set $\cup_n B_{\mathbb{C}}(a_n, 2\delta)$, then for the same reason $A' \setminus \cup_n B_{\mathbb{C}}(a_n, \delta)$ does not contain any point $a_n - \tau$ as well. Thus the set A contains all balls $B_{\mathbb{C}}(a_n, \delta)$, for which $a_n \in A$ and all balls $B_{\mathbb{C}}(a_n - \tau, \delta)$, for which $a_n - \tau \in A$, and do not intersect balls $B_{\mathbb{C}}(a_n, \delta)$ with $a_n \notin A$ and balls $B_{\mathbb{C}}(a_n - \tau, \delta)$ with $a_n - \tau \notin A$. We get

$$\alpha\delta^{2d} \#\{n : a_n \in A\} = \sum_{n: a_n \in A} \int \varphi\left(\frac{z-a_n}{\delta}\right) \omega(dz) = \int_A \Psi(z)\omega(dz),$$

$$\alpha\delta^{2d} \#\{n : a_n - \tau \in A\} = \sum_{n: a_n - \tau \in A} \int \varphi\left(\frac{z+\tau-a_n}{\delta}\right) \omega(dz) = \int_A \Psi(z+\tau)\omega(dz),$$

Note that

$$\int_A \omega(dz) \le \sum_{a_n \in A} \int_{B_{\mathbb{C}}(a_n, 2\delta)} \omega(dz) = N\omega_{2d}(2\delta)^{2d}.$$

By (6),

$$|\#\{n : a_n \in A\} - \#\{n : a_n - \tau \in A\}| \le \frac{\int_A |\Psi(z) - \Psi(z+\tau)|\omega(dz)}{\delta^{2d}\alpha} < \frac{\rho N \omega_{2d} 2^{2d}}{\alpha} < 1.$$

Therefore,

$$\#\{n : a_n \in A\} = \#\{n : a_n - \tau \in A\},$$

which allows to construct a bijection σ between the sets $\{n : a_n - \tau \in A\}$ and $\{n : a_n \in A\}$.

This construction works for every connected component of the set $\cup_n B_{\mathbb{C}}(a_n, 2\delta)$, hence there exists a bijection σ of a part S_1 of $\mathbb{N}$ to the part S_2 of $\mathbb{N}$. It follows from the inequality $\operatorname{diam} A \le 2\varepsilon$ that

$$|a_n - \tau - a_{\sigma(n)}| < 2\varepsilon. \tag{7}$$

If $D \subset T_K$, we have $S_1 = S_2 = \mathbb{N}$, and theorem is proved. If $D \setminus T_{\widetilde{K}} \neq \varnothing$, we have only

$$\{n : a_n \in T_K\} \subset S_1 \cup S_2 \subset \{n : a_n \in T_{\widetilde{K}}\}.$$

For $a \in D \setminus T_{\tilde{K}}$ put

$$\eta < \frac{1}{2} \min\{\operatorname{dist}(\mathsf{Im}\ a, \widetilde{K}), \operatorname{dist}\{\mathsf{Im}\ a, \partial\Omega\}\}$$

and consider the function

$$\Psi(z) = \int \varphi\left(\frac{w - z}{\eta}\right) \mu_D(dw) = \sum_{n: a_n \in D} \varphi\left(\frac{a_n - z}{\eta}\right).$$

In view of the choice of η, this function is well-defined and almost periodic on T_ω with $\omega = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) > \eta\}$. Furthermore, $\Psi(a) \ge \varphi(0) = 1$, hence $\Psi(a + t)$ is strictly positive for some large enough $t \in \mathbb{R}^d$. Therefore the set $\{n : a_n \in D \setminus \widetilde{K}\}$ is unbounded and countable, as well as the sets $\mathbb{N} \setminus S_1$ and $\mathbb{N} \setminus S_2$. For points a_n with $n \notin S_1$ condition (7) need not be required, therefore the bijection $\sigma : S_1 \to S_2$ can can be extended to a bijection $\mathbb{N} \to \mathbb{N}$. The theorem is proved. □

3 Uniqueness theorems for almost periodic distributions, measures, multisets

Definition 8. *We shall say that distributions $f, g \in D'(\mathbb{R}^d)$ converge weakly at infinity, if for any $\varphi \in D(\mathbb{R}^d)$ the functions $(f, \varphi(\cdot - t))$ and $(g, \varphi(\cdot - t))$ of the variable $t \in \mathbb{R}^d$ converge weakly at infinity.*

Also, we shall say that distributions $f, g \in D'(T_\Omega)$ converge weakly at infinity, if for any $\varphi \in D(T_\Omega)$ the functions $(f, \varphi(\cdot - z))$ and $(g, \varphi(\cdot - z))$ of the variable $z \in T_\omega$ converge weakly at infinity ($\omega \subset \Omega$ is defined in Definition 6).

It follows from Theorem 3

Theorem 5. *If two almost periodic distributions or measures $f, g \in D'(\mathbb{R}^d)$ converge weakly at infinity, then $f \equiv g$. The similar assertion is valid for $f, g \in D'(T_\Omega)$.*

Definition 9. *We shall say that two discrete multisets $F = \{a_n\}, H = \{b_n\} \subset \mathbb{R}^d$ converge weakly at infinity, if there is a set $G \subset \mathbb{R}^d$ satisfying (1) such that under an appropriate numbering*

$$\lim_{n \to \infty, n \in N(G)} a_n - b_n = 0,$$

where $\mathbb{N}(G) = \{n \in \mathbb{N} : a_n \text{ or } b_n \in G\}$.

Definition 10. *We shall say that two discrete multisets $F = \{a_n\}, H = \{b_n\} \subset T_\Omega$ converge weakly at infinity, if for every $K \subset \Omega$ there is a set $G = G(K) \subset \mathbb{R}^d$ satisfying (1) such that under an appropriate numbering*

$$\lim_{n\to\infty, n\in\mathbb{N}(G,K)} a_n - b_n = 0,$$

где $\mathbb{N}(G,K) = \{n \in \mathbb{N} : a_n \in G \times K \ \text{ or } \ b_n \in G \times K\}$.

Theorem 6. *If two discrete multisets F, H converge weakly at infinity, then they are identical.*

Proof. It follows from theorems 4 and 5 that we have to check the weak convergence of measures μ_F and μ_H at infinity. The latter means that for any $\varphi \in D(\mathbb{R}^d)$ (or $\varphi \in D(T_\Omega)$) the almost periodic functions of the variable $t \in \mathbb{R}^d$

$$\Psi_F(t) = (\mu_F, \varphi(\cdot - t)) = \sum_n \varphi(a_n - t)$$

and

$$\Psi_H(t) = (\mu_H, \varphi(\cdot - t)) = \sum_n \varphi(b_n - t)$$

converge weakly at infinity. To be specific consider the case $F = \{a_n\}, H = \{b_n\} \subset T_\Omega$. The is similar for $F, H \subset \mathbb{R}^d$.

Suppose that $\operatorname{supp}\varphi \subset B_{\mathbb{R}}(0,1) \times K$ for compact $K \subset \Omega$. Take $\varepsilon > 0$ and then $\delta > 0$ such that $|\varphi(z) - \varphi(z')| < \varepsilon/(N(K))$ for $|z - z'| < \delta$, where $N(K)$ is the constant from (3). Let a set $G \subset \mathbb{R}^d$ satisfy (1) with balls $B_{\mathbb{R}}(x_k, R_k)$, $k \in \mathbb{N}$. It is easy to see that having reduced by 3 times the radii of these balls and changing the location of their centers, we can assume that $\operatorname{dist}(B(x_k, R_k), 0) \to \infty$. For sufficiently large k and for a_n, $b_n \in B_{\mathbb{R}}(x_k, R_k) \times K$ we have $|a_n - b_n| < \delta$. Also assume that $R_k > 2$.

Let $t \in B_{\mathbb{R}}(x_k, R_k/2)$ and $a_n - t \in \operatorname{supp}\varphi$. Then $a_n \in B_{\mathbb{R}}(x_k, R_k) \times K$, and the same is valid for $b_n - t$. Therefore if $a_n - t \in \operatorname{supp}\varphi$ or $b_n - t \in \operatorname{supp}\varphi$, we get $|a_n - b_n| < \delta$ and

$$|\Psi_F(t) - \Psi_H(t)| \le \sum_n |\varphi(a_n - t) - \varphi(b_n - t)| < \frac{\varepsilon}{N(K)} \cdot N(K) = \varepsilon.$$

Hence the almost periodic functions $\Psi_F(t)$ и $\Psi_H(t)$ converge weakly at infinity. □

4 Uniqueness theorems for delta-subharmonic and meromorphic functions

It follows immediately from the definition that any partial derivative of an almost periodic distribution from $D'(\mathbb{R}^d)$ or $D'(T_\Omega)$ is also an almost periodic distribution. Since any subharmonic function on any region from $\mathbb{R}^d$ is locally integrable, it can be considered as a distribution. Thus, if u is a subharmonic almost periodic function on $D'(\mathbb{R}^d)$ or $D'(T_\Omega)$, then its Riesz measure Δu is also an almost periodic distribution, and the same is true for the difference of subharmonic functions, the so-called delta-subharmonic functions.

It follows from Theorem 5

Theorem 7. *If two delta-subharmonic functions u, v on $\mathbb{R}^d$ or T_Ω have weakly converging at infinity Riesz measures Δu and Δv, then $u = v + h$ with a harmonic function h.*

The last part of the proof uses the fact that the condition $\Delta h = 0$ in the sense of distributions implies that h is an ordinary harmonic function.

Definition 11. *(see [10], [5]) A meromorphic function $f(z)$ on the strip $S_{a,b} = \{z \in \mathbb{C} : \mathrm{Re}\, z \in \mathbb{R}, a < \mathrm{Im}\, z < b\}$, $-\infty \le a < b \le +\infty$, is called almost periodic, if in any smaller strip $S_{\alpha,\beta}$, $a < \alpha < \beta < b$, the function $\rho_S(f(z+t), f(z))$, where ρ_S is the spherical distance, is almost periodic in the variable $t \in \mathbb{R}$.*

In [5] the following properties of meromorphic almost periodic functions are proved:

- The distance between any pole and any zero of meromorphic almost periodic functions is bounded from below by a strictly positive constant depending on the strip in which this pole and zero lie,
- Every meromorphic almost periodic function on $S_{a,b}$ is a ratio of two holomorphic almost periodic functions in $S_{a,b}$; the converse assertion is only valid if distances between poles and zeros of this ratio are uniformly bounded from below by a strictly positive constant in any smaller strip. In particular, every holomorphic almost periodic function in a strip is simultaneously a meromorphic almost periodic function.

It was proved in [9] that for any holomorphic function f on $S_{a,b}$ the function $\log |f|$ is an almost periodic distribution, hence the measure μ_Z corresponding to the multiset of zeros Z_f is almost periodic. Also, if f is an almost periodic meromorphic function, then the measures μ_Z and μ_P corresponding to the multiset of zeros Z_f and the multiset of poles P_f of f are also almost periodic. Therefore, Theorem 7 implies

Theorem 8. *If multisets of poles P_f and P_g of meromorphic almost periodic functions f, g in a strip $S_{a,b}$ converge weakly at infinity and the same is true for multisets of zeros Z_f and Z_g, then $P_f = P_g$, $Z_f = Z_g$, hence, f/g is a holomorphic almost periodic function on $S_{a,b}$ without zeros.*

If f, g are holomorphic almost periodic functions on $S_{a,b}$ and multisets of zeros Z_f, Z_g converge weakly at infinity, then $Z_f = Z_g$, and we obtain Theorem 6 from [7].

Note that the linear-fractional mapping of a meromorphic almost periodic function f is a meromorphic almost periodic function as well. Then instead of zeros and poles one can consider A_1 -points and A_2 -points, $A_1 \ne A_2$, that is zeros of functions $f - A_1$ and $f - A_2$. Also, for $T_\Omega = \mathbb{C}$ we obtain the following theorem:

Theorem 9. *Let f, g be meromorphic almost periodic functions on $\mathbb{C}$ and let A_j-points of f converge weakly at infinity to A_j-points of g for three pairwise distinct values A_1, A_2, A_3. Then either $f = g$, or f and g have the forms*

$$f = T\left(\frac{1 - h_1}{h_2 - h_1}\right), \qquad g = T\left(\frac{h_2 - h_1 h_2}{h_2 - h_1}\right), \tag{8}$$

where h_1, h_2 are distinct entire functions without zeros, and T is a linear-fractional mapping that moves the triple point $0, 1, \infty$ to the triple point A_1, A_2, A_3.

At the final stage of the proof we use the following theorem from [8]: If two meromorphic functions on $\mathbb{C}$ have the same multisets of A -points for three distinct values of A_1, A_2, A_3, then these functions either coincide, or have form (8).

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Теореми єдиності розглядаються для різних типів майже періодичних об'єктів: функцій, мір, розподілів, мультимножин, мероморфних функцій.