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COMPOSITION OF SLICE ENTIRE FUNCTIONS AND BOUNDED L-INDEX IN DIRECTION

We study the following question: "Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of bounded l-index, $\Phi : \mathbb{C}^n \to \mathbb{C}$ be a slice entire function, $n \geq 2$, $l : \mathbb{C} \to \mathbb{R}_+$ be a continuous function. What is a positive continuous function $L : \mathbb{C}^n \to \mathbb{R}_+$ and a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that the composite function $f(\Phi(z))$ has bounded L-index in the direction $\mathbf{b}$?". In the present paper, early known results on boundedness of L-index in direction for the composition of entire functions $f(\Phi(z))$ are generalized to the case where $\Phi : \mathbb{C}^n \to \mathbb{C}$ is a slice entire function, i.e. it is an entire function on a complex line $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ for any $z^0 \in \mathbb{C}^n$ and for a given direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. These slice entire functions are not joint holomorphic in the general case. For example, it allows consideration of functions which are holomorphic in variable z_1 and continuous in variable z_2.

Key words and phrases: slice entire function, entire function, bounded L-index in direction, composite function, bounded l-index.

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1 Introduction

The present paper is devoted to compositions of slice entire functions and theory of functions having bounded L-index in direction. Let us introduce some notation from [1, 2]. Let $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_+^* = [0, +\infty)$, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direciton, $L : \mathbb{C}^n \to \mathbb{R}_+$ be a continuous function, $F : \mathbb{C}^n \to \mathbb{C}$ an entire function. The slice functions on a line $\{z^0 + t\mathbf{b}\colon t \in \mathbb{C}\}$ for a fixed $z^0 \in \mathbb{C}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ and $l_{z^0}(t) = L(z^0 + t\mathbf{b})$. Besides, we denote by $\langle a, c\rangle = \sum_{j=1}^n a_j\overline{c_j}$ the Hermitian scalar product in $\mathbb{C}^n$, where a, $c \in \mathbb{C}^n$.

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}^n$ be the class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ for each $z^0 \in \mathbb{C}^n$ and let $\mathcal{H}_{\mathbf{b}}^n$ be the class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}^n$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e. for

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every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}^{p} F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is an entire function of complex variable $t \in \mathbb{C}$ for given $z \in \mathbb{C}^n$. It is easy to check that for any $p \in \mathbb{N}$ the derivative $\partial_{\mathbf{b}}^{p} F$ is also joint continuous.

In this research, we will often call this derivative as directional derivative because if F is entire function in $\mathbb{C}^n$ then the derivatives of the function $g_z(t)$ matches with directional derivatives of the function F. The assumption of joint continuity together with the holomorphity assumption in one direction do not imply holomorphy in whole n-dimensional complex space.

A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}^{n}$ is said [1] to be of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{C}^n$ the inequality

$$\frac{|\partial_{\mathbf{b}}^{m} F(z)|}{m! L^m(z)} \le \max_{0 \le k \le m_0} \frac{|\partial_{\mathbf{b}}^{k} F(z)|}{k! L^k(z)}, \tag{1}$$

is true. The least integer m_0, obeying (1), is called the L-index in the direction $\mathbf{b}$ of the function F and is denoted by $N_{\mathbf{b}}(F, L)$. If such m_0 does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$, and the function F is called of unbounded L-index in the direction $\mathbf{b}$ in this case. For $n = 1$, $\mathbf{b} = 1$, $L(z) = l(z)$, $z \in \mathbb{C}$ inequality (1) defines a function of bounded l-index with the l-index $N(F, l) \equiv N_1(F, l)$ [10], and if in addition $l(z) \equiv 1$, then we obtain a definition of index boundedness with index $N(F) \equiv N_1(F, 1)$ [11, 12]. Other approach to use a concept of bounded index in the investigations of functions of several complex variables was considered in papers by F. Nuray and R. Patterson [15, 14, 13]. They use all possible partial derivatives in the definition.

A detailed review of papers on compositions of functions and boundedness of index is presented in [3]. There was considered the following question: *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of bounded l-index, $\Phi : \mathbb{C}^n \to \mathbb{C}$ be an entire function, $l : \mathbb{C} \to \mathbb{R}_+$ be a continuous function. What are a positive continuous function L and a direction $\mathbf{b} \in \mathbb{C}^n$ such that the composite function $f(\Phi(z))$ has bounded L-index in the direction $\mathbf{b}$?* There was presented an answer to the question in [3]. Similar questions were considered for analytic functions in the unit ball in [5], for entire functions of bounded $\mathbf{L}$-index in joint variables in [4], for entire and analytic functions of bounded $l - M$-index in [6], for analytic functions in $\mathbb{C} \times \mathbb{D}$ [7].

Here we will consider the question of whether $\Phi : \mathbb{C}^n \to \mathbb{C}$ is a slice entire function, i.e. it belongs to the class $\widetilde{H}_{\mathbf{b}}^{n}$.

Note that the positivity and continuity of the function L are mild restrictions. Therefore, we impose additional assumptions on the function L.

For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L\colon \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda(\eta) = \sup_{z \in \mathbb{C}^n} \sup_{t \in \mathbb{C}} \left\{ \frac{L(z + t\mathbf{b})}{L(z)} : |t| \le \frac{\eta}{L(z)} \right\}.$$

By $Q_{\mathbf{b}}^{n}$ we denote the class of functions L such that $\lambda(\eta)$ is finite for any $\eta > 0$. We also use notation $Q = Q_1^1$ for the class of positive continuous function $l(z)$, when $z \in \mathbb{C}$, $\mathbf{b} = 1$, $n = 1$, $L \equiv l$.

There was obtained the following result for entire functions.

Theorem 1 ([3]). *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, *f be an entire function in* $\mathbb{C}$, Φ *be an entire function in* $\mathbb{C}^n$ *such that* $\partial_{\mathbf{b}}\Phi(z) \neq 0$ *and*

$$\left|\partial_{\mathbf{b}}^j \Phi(z)\right| \leq K \left|\partial_{\mathbf{b}}\Phi(z)\right|^j, \quad K \equiv \text{const} > 0,$$

for all $z \in \mathbb{C}^n$ *and every* $j \leq p$, *where p is defined by* (2).

Let $l \in Q$, $l(w) \geq 1$, $w \in \mathbb{C}$ *and* $L \in Q^n_{\mathbf{b}}$, *where* $L(z) = \left|\partial_{\mathbf{b}}\Phi(z)\right| l(\Phi(z))$. *The entire function f has bounded l-index if and only if* $F(z) = f(\Phi(z))$ *has bounded L-index in the direction* $\mathbf{b}$.

To prove the main theorem we need auxiliary propositions.

Theorem 2 ([2]). *Let* $L \in Q^n_{\mathbf{b}}$. *A function* $F \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$ *is of bounded L-index in the direction* $\mathbf{b}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for every* $z \in \mathbb{C}^n$ *one has*

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : \ 0 \leq k \leq p\right\}. \tag{2}$$

For $n = 1$ Theorem 2 is Sheremeta's result [16]. W. K. Hayman [9] proved Theorem 2 for entire functions of bounded index. Analogs of the Hayman Theorem are also known for other classes of holomorphic functions of bounded index [8, 16, 17].

Our main result is following

Theorem 3. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, *f be an entire function in* $\mathbb{C}$, $\Phi \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$ *such that* $\partial_{\mathbf{b}}\Phi(z) \neq 0$ *and*

$$\left|\partial_{\mathbf{b}}^j \Phi(z)\right| \leq K \left|\partial_{\mathbf{b}}\Phi(z)\right|^j, \quad K \equiv \text{const} > 0, \tag{3}$$

for all $z \in \mathbb{C}^n$ *and every* $j \leq p$, *where p is defined by* (2).

Let $l \in Q$, $l(w) \geq 1$, $w \in \mathbb{C}$ *and* $L \in Q^n_{\mathbf{b}}$, *where* $L(z) = \left|\partial_{\mathbf{b}}\Phi(z)\right| l(\Phi(z))$. *The entire function f has bounded l-index if and only if the slice entire function* $F(z) = f(\Phi(z))$ *has bounded L-index in the direction* $\mathbf{b}$.

Note that the assumptions for every $j \in \{1, \ldots, p\}$ $\partial_{\mathbf{b}}\Phi(z) \neq 0$ and $|\partial_{\mathbf{b}}^j\Phi(z)| \leq K|\partial_{\mathbf{b}}\Phi(z)|^j$, in Theorem 3 are generated by method of proof. In fact, we can remove them and prove a more general proposition with some greater function L.

Theorem 4. *Let* $l \in Q$, $f : \mathbb{C} \to \mathbb{C}$ *be an entire function of bounded l-index,* $\Phi \in \widetilde{\mathcal{H}}^n_{\mathbf{b}}$, $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. *Suppose that* $l(w) \geq 1$, $w \in \mathbb{C}$, *and* $L \in Q^n_{\mathbf{b}}$ *with*

$$L(z) = \max\{1, |\partial_{\mathbf{b}}\Phi(z)|\}\, l(\Phi(z)) \tag{4}$$

and for all $z \in \mathbb{C}^n$ *and* $k \in \{1, 2, \ldots, N(f, l) + 1\}$ *one has*

$$|\partial_{\mathbf{b}}^k \Phi(z)| \leq K(l(\Phi(z)))^{1/(N(f,l)+1)} |\partial_{\mathbf{b}}\Phi(z)|^k. \tag{5}$$

where $K \geq 1$ *is a constant. Then the slice entire function* $F(z) = f(\Phi(z))$ *has bounded L-index in the direction* $\mathbf{b}$.

2 PROOF OF MAIN THEOREM

Proof of Theorem 3. Below we repeat the considerations from [3]. But now we consider the more general case where $\Phi(z)$ is an entire slice function. At first, we prove that

$$\partial_{\mathbf{b}}^k F(z) = f^{(k)}(\Phi(z)) \left(\partial_{\mathbf{b}}\Phi(z)\right)^k + \sum_{j=1}^{k-1} f^{(j)}(\Phi(z)) Q_{j,k}(z), \tag{6}$$

$$\text{where } Q_{j,k}(z) = \sum_{\substack{n_1+2n_2+\ldots+kn_k=k \\ 0\le n_1\le j-1}} c_{j,k,n_1,\ldots,n_k} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2} \ldots \left(\partial_{\mathbf{b}}^k\Phi(z)\right)^{n_k},$$

and $c_{j,k,n_1,\ldots,n_k}$ are some non-negative integer coefficients. We also deduce that

$$f^{(k)}(\Phi(z)) = \frac{\partial_{\mathbf{b}}^k F(z)}{\left(\partial_{\mathbf{b}}\Phi(z)\right)^k} + \frac{1}{\left(\partial_{\mathbf{b}}\Phi(z)\right)^{2k}} \sum_{j=1}^{k-1} \partial_{\mathbf{b}}^j F(z) \left(\partial_{\mathbf{b}}\Phi(z)\right)^j Q^*_{j,k}(z), \tag{7}$$

$$\text{where } Q^*_{j,k}(z) = \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^k\Phi(z)\right)^{m_k},$$

and $b_{j,k,m_1,\ldots,m_k}$ are some integer coefficients.

The validity of (6) and (7) will be checked by the method of mathematical induction. Obviously, for $k=1$ equalities (6) and (7) hold. Assume that they are true for $k=s$. Let us prove them for $k=s+1$. Evaluate directional derivative in (6)

$$\begin{aligned}
\partial_{\mathbf{b}}^{s+1} F(z) &= f^{(s+1)}(\Phi(z)) \left(\partial_{\mathbf{b}}\Phi(z)\right)^{s+1} + s f^{(s)}(\Phi(z)) \left(\partial_{\mathbf{b}}\Phi(z)\right)^{s-1} \partial_{\mathbf{b}}^2\Phi(z) + \\
&+ \sum_{j=1}^{s-1} \left(f^{(j+1)}(\Phi(z)) \partial_{\mathbf{b}}\Phi(z) Q_{j,s}(z) + f^{(j)}(\Phi(z)) \partial_{\mathbf{b}} Q_{j,s}(z)\right) = \\
&= f^{(s+1)}(\Phi(z)) \left(\partial_{\mathbf{b}}\Phi(z)\right)^{s+1} + f^{(s)}(\Phi(z)) \left(s \left(\partial_{\mathbf{b}}\Phi(z)\right)^{s-1} \partial_{\mathbf{b}}^2\Phi(z) + \partial_{\mathbf{b}}\Phi(z) Q_{s-1,s}(z)\right) + \\
&+ \sum_{j=2}^{s-1} f^{(j)}(\Phi(z)) \left(\partial_{\mathbf{b}}\Phi(z) Q_{j-1,s}(z) + \partial_{\mathbf{b}} Q_{j,s}(z)\right) + f'(\Phi(z)) \partial_{\mathbf{b}} Q_{1,s}(z).
\end{aligned}$$

Since

$$\begin{aligned}
& s \left(\partial_{\mathbf{b}}\Phi(z)\right)^{s-1} \partial_{\mathbf{b}}^2\Phi(z) + \\
&+ \sum_{\substack{n_1+2n_2+\ldots+sn_s=s \\ 0\le n_1\le s-2}} c_{s-1,s,n_1,\ldots,n_s} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1+1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right) \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s} = \\
&= \sum_{\substack{m_1+2m_2+\ldots+sm_s=s+1 \\ 0\le m_1\le s-1}} \tilde{c}_{s,s+1,m_1,\ldots,m_s} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{m_s} = Q_{s,s+1}(z),
\end{aligned}$$

$$\begin{aligned}
\partial_{\mathbf{b}} Q_{1,s}(z) &= \sum_{2n_2+\ldots+sn_s=s} c_{1,s,0,n_2,\ldots,n_s} \Big(n_2 \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2-1} \left(\partial_{\mathbf{b}}^3\Phi(z)\right)^{n_3+1} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s} + \\
&+ \ldots + n_s \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2} \left(\partial_{\mathbf{b}}^3\Phi(z)\right)^{n_3} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s-1} \partial_{\mathbf{b}}^{s+1}\Phi(z)\Big) = \\
&= \sum_{2m_2+\ldots+(s+1)m_{s+1}=s+1} \tilde{c}_{1,s+1,0,m_2,\ldots,m_{s+1}} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{m_s} \left(\partial_{\mathbf{b}}^{s+1}\Phi(z)\right)^{m_{s+1}} =
\end{aligned}$$

$$= Q_{1,s+1}(z), \text{ and}$$
$$\partial_{\mathbf{b}}\Phi(z)Q_{j-1,s}(z) + \partial_{\mathbf{b}}Q_{j,s}(z) =$$
$$= \sum_{\substack{n_1+2n_2+\ldots+sn_s=s\\ 0\le n_1\le j-2}} c_{j-1,s,n_1,\ldots,n_s} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1+1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s} +$$
$$+ \sum_{\substack{n_1+2n_2+\ldots+kn_s=s\\ 0\le n_1\le j-1}} c_{j,s,n_1,n_2,\ldots,n_s} \Big(n_1 \left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1-1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2+1} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s} +$$
$$+ \ldots + n_s \left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1} \left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s-1} \partial_{\mathbf{b}}^{s+1}\Phi(z)\Big)$$
$$= \sum_{\substack{m_1+2m_2+\ldots+(s+1)m_{s+1}=s+1\\ 0\le m_1\le j-1}} \tilde{c}_{j,s+1,m_1,\ldots,m_{s+1}} \left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1} \ldots \left(\partial_{\mathbf{b}}^s\Phi(z)\right)^{n_s} \left(\partial_{\mathbf{b}}^{s+1}\Phi(z)\right)^{n_{s+1}} = Q_{j,s+1}(z),$$

we obtain (6) with $s+1$ instead of k.

By the mathematical induction as (6) it can be proved equality (7).

Let f be an entire function of bounded l-index. By Theorem 2 inequality (2) is valid for $n=1$, $F=f$, $L=l$, $\mathbf{b}=1$. Taking into account (3) and (6), for $k=p+1$ we have

$$\frac{1}{L^{p+1}(z)} \left|\partial_{\mathbf{b}}^{p+1}F(z)\right| \le \frac{|f^{(p+1)}(\Phi(z))|}{L^{p+1}(z)} \left|\partial_{\mathbf{b}}\Phi(z)\right|^{p+1} + \sum_{j=1}^{p} \frac{|f^{(j)}(\Phi(z))||Q_{j,p+1}(z)|}{L^{p+1}(z)} \le$$
$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\}\left(C + \sum_{j=1}^{p}\frac{|Q_{j,p+1}(z)|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\right) \le$$
$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\}\times$$
$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}} \frac{\left|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2}\ldots\left(\partial_{\mathbf{b}}^{p+1}\Phi(z)\right)^{n_{p+1}}\right|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\right) \le$$
$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}:\ 0\le k\le p\right\}\left(C + \sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}}K^{p+1}}{l^{p+1-j}(\Phi(z))}\right) \le$$
$$\le C_1 \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}: 0\le k\le p\right\}.$$

Using equality (7), we can estimate the quotient $\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}$:

$$\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} \le \frac{|\partial_{\mathbf{b}}^k F(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k} + \sum_{j=1}^{k-1}\frac{\left|\frac{\partial^j F(z)}{\partial \mathbf{b}^j}\right||Q^*_{j,k}(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2k-j}} \le$$
$$\le \max\left\{\frac{1}{L^j(z)}\left|\partial_{\mathbf{b}}^j F(z)\right| : 1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q^*_{j,k}(z)|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\right) \le$$
$$\le \max\left\{\frac{1}{L^j(z)}\left|\partial_{\mathbf{b}}^j F(z)\right| : 1\le j\le k\right\}\times$$

$$\times\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2}\ldots\left(\partial_{\mathbf{b}}^k\Phi(z)\right)^{m_k}|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\right)\le$$

$$\le\max\left\{\frac{1}{L^j(z)}\left|\partial_{\mathbf{b}}^jF(z)\right|:1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\frac{|b_{j,k,m_1,\ldots,m_k}|K^k}{l^{k-j}(\Phi(z))}\right)$$

$$\le C_2\max\left\{\frac{1}{L^j(z)}\left|\partial_{\mathbf{b}}^jF(z)\right|:1\le j\le k\right\}.$$

Hence, it follows that

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|\le C_1C_2\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^kF(z)\right|:\ 0\le k\le p\right\}. \tag{8}$$

Therefore, by Theorem 2 inequality (8) means that the function F has bounded L-index in the direction $\mathbf{b}$.

Conversely, suppose that F is a function of bounded L-index in the direction $\mathbf{b}$. Then it satisfies (2). In view of (3) and (7), we obtain

$$\frac{|f^{(p+1)}(\Phi(z))|}{l^{p+1}(\Phi(z))}\le\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{l^{p+1}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}+\sum_{j=1}^{p}\frac{|\partial_{\mathbf{b}}^jF(z)||Q^*_{j,p+1}(z)|}{l^{p+1}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2p+2-j}}\le$$

$$\le\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^kF(z)\right|:0\le k\le p\right\}\left(C+\sum_{j=1}^{p}\frac{|Q^*_{j,p+1}(z)|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(p+1-j)}}\right)\le$$

$$\le\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^kF(z)\right|:0\le k\le p\right\}\times$$

$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}|b_{j,p+1,m_1,\ldots,m_{p+1}}|\frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{m_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{m_2}\ldots\left(\partial_{\mathbf{b}}^{p+1}\Phi(z)\right)^{m_{p+1}}|}{l^{p+1-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(p+1-j)}}\right)\le$$

$$\le\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^kF(z)\right|:0\le k\le p\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}\frac{|b_{j,p+1,m_1,\ldots,m_{p+1}}|K^{2p+2-2j}}{l^{p+1-j}(\Phi(z))}\right)\le$$

$$\le C_3\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^kF(z)\right|:0\le k\le p\right\}.$$

Accordingly to equality (6) we estimate

$$\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^kF(z)\right|\le\frac{|f^{(k)}(\Phi(z))||\varphi'(z)|^k}{L^k(z)}+\sum_{j=1}^{k-1}\frac{|f^{(j)}(\Phi(z))||Q_{j,k}(z)|}{L^k(z)}\le$$

$$\le\max\left\{\frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))}:1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}(z)|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k}\right)\le$$

$$\le C_4\max\left\{\frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))}:1\le j\le k\right\}.$$

It implies that

$$\frac{|f^{(p+1)}(\Phi(z))|}{l^{p+1}(\Phi(z))} \le C_3C_4 \max\left\{\frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))} : 0 \le j \le p\right\}.$$

Thus, by Theorem 2 ($n = 1$, $F = f$, $L = l$, $\mathbf{b} = 1$) the function f has bounded l-index. □

Proof of Theorem 4. Let f be an entire function of bounded l-index. Denote

$$L_0(z) = l(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|.$$

Taking into account (6) and (4), for $k = p + 1$ we have

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \frac{|f^{(p+1)}(\Phi(z))|}{L_0^{p+1}(z)} |\partial_{\mathbf{b}}\Phi(z)|^{p+1} + \sum_{j=1}^{p} \frac{|f^{(j)}(\Phi(z))||Q_{j,p+1}(z)|}{L_0^{p+1}(z)} \le$$

$$\le \frac{|f^{(p+1)}(\Phi(z))|\,|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}{l^{p+1}(\Phi(z))\,|\partial_{\mathbf{b}}\Phi(z)|^{p+1}} + \sum_{j=1}^{p} \frac{|f^{(j)}(\Phi(z))|}{l^j(\Phi(z))} \cdot \frac{|Q_{j,p+1}(z)|l^j(\Phi(z))}{|\partial_{\mathbf{b}}\Phi(z)|^{p+1}\,l^{p+1}(\Phi(z))}. \quad (9)$$

By Theorem 2 inequality (2) is valid for $n = 1$, $F = f$, $L = l$, $\mathbf{b} = 1$ and $p = N(f, l)$.

$$(\forall \tau \in \mathbb{C}):\quad \frac{|f^{(p+1)}(\tau)|}{l^{p+1}(\tau)} \le C \max\left\{\frac{|f^{(k)}(\tau)|}{l^k(\tau)} :\ 0 \le k \le p\right\}.$$

Applying to (9) these inequalities with $\tau = \Phi(z)$, we obtain

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} :\ 0 \le k \le p\right\}\left(C + \sum_{j=1}^{p} \frac{|Q_{j,p+1}(z)|l^{j-p-1}(\Phi(z))}{|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\right) \le$$

$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} :\ 0 \le k \le p\right\}\Bigg(C +$$

$$+ \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}} \frac{|\left(\partial_{\mathbf{b}}\Phi(z)\right)^{n_1}\left(\partial_{\mathbf{b}}^2\Phi(z)\right)^{n_2}\ldots\left(\partial_{\mathbf{b}}^{p+1}\Phi(z)\right)^{n_{p+1}}|}{l^{p+1-j}(\Phi(z))\,|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\Bigg). \quad (10)$$

In view of condition (5) inequality (10) yields

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} : 0 \le k \le p\right\} \times$$

$$\times\left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}}K^{p+1}l(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}{l^{p+1-j}(\Phi(z))\,|\partial_{\mathbf{b}}\Phi(z)|^{p+1}}\right) \le$$

$$\le \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} : 0 \le k \le p\right\}\left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}}K^{p+1}}{l^{p-j}(\Phi(z))}\right). \quad (11)$$

We will use that $l(\Phi(z)) \ge 1$. Then from (11) it follows that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le C_1 \max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} : 0 \le k \le p\right\}, \quad (12)$$

where

$$C_1 = C + K^{p+1}\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}}.$$

Applying equality (7), we can estimate the quotient $\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}$

$$\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))} \le \frac{|\partial_{\mathbf{b}}^k F(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^k} + \sum_{j=1}^{k-1}\frac{|\partial_{\mathbf{b}}^j F(z)||Q^*_{j,k}(z)|}{l^k(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2k-j}} \le$$

$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^j\Phi(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}: 1\le j\le k\right\}\Big(1+\sum_{j=1}^{k-1}\frac{|Q^*_{j,k}(z)|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Big) \le$$

$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^j\Phi(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}: 1\le j\le k\right\}\Big(1+$$

$$+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\frac{|(\partial_{\mathbf{b}}\Phi(z))^{m_1}(\partial_{\mathbf{b}}^2\Phi(z))^{m_2}\ldots(\partial_{\mathbf{b}}^k\Phi(z))^{m_k}|}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Big). \qquad (13)$$

Inequalities (5) and $l(w)\ge 1$ imply that $|\partial_{\mathbf{b}}^s\Phi(z)|\le Kl^{s/2}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^s$, because $s/2\ge 1/(N(f,l)+1)$ for $s\in\{1,2,\ldots,N(f,l)+1\}$. Applying this inequality to (13), we deduce

$$\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}\le\max\left\{\frac{|\partial_{\mathbf{b}}^j F(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}: 1\le j\le k\right\}\Big(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\times$$

$$\times|b_{j,k,m_1,\ldots,m_k}|K^{m_1+m_2+\ldots+m_k}\frac{(l(\Phi(z)))^{(m_1+2m_2+\ldots+km_k)/2}|\partial_{\mathbf{b}}\Phi(z)|^{m_1+2m_2+\ldots+km_k}}{l^{k-j}(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^{2(k-j)}}\Big)\le$$

$$\le C_2\max\left\{\frac{|\partial_{\mathbf{b}}^j\Phi(z)|}{l^j(\Phi(z))|\partial_{\mathbf{b}}\Phi(z)|^j}: 1\le j\le k\right\},$$

where

$$C = 1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|K^{m_1+m_2+\ldots+m_k}.$$

Then from inequality (12) we get

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\le C_1\max\left\{\frac{|f^{(k)}(\Phi(z))|}{l^k(\Phi(z))}: 0\le k\le p\right\}\le C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^j F(z)|}{L_0^j(z)}: 0\le j\le p\right\}, \quad (14)$$

$p=N(f,l)$. The last inequality is proved for all z such that $\partial_{\mathbf{b}}\Phi(z)\ne 0$.

Remind that $L(z)=l(\Phi(z))\max\{1,|\partial_{\mathbf{b}}\Phi(z)|\}$. Rewrite inequality (14) in the following form:

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\cdot\frac{L^{p+1}(z)}{L_0^{p+1}(z)}\le C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}\frac{L^k(z)}{L_0^k(z)}:\ 0\le k\le p\right\}.$$

Then

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\le C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^k F(z)\right|}{L^k(z)}\frac{L^k(z)}{L_0^k(z)}:\ 0\le k\le p\right\}\le$$

$$\leq C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^kF(z)\right|}{L^k(z)}\colon\ 0\leq k\leq p\right\}\max\left\{\frac{L^k(z)}{L_0^k(z)}\colon\ 0\leq k\leq p\right\}=$$

$$=C_1C_2\frac{(L_0(z)/L(z))^{p+1}}{\min\limits_{0\leq k\leq p}(L_0(z)/L(z))^k}\max\left\{\frac{\left|\partial_{\mathbf{b}}^kF(z)\right|}{L^k(z)}\colon\ 0\leq k\leq p\right\}. \quad (15)$$

Let $t_0=t(z)=L_0(z)/L(z)$ and $k_0\leq p$ ($k_0\in\mathbb{Z}_+$) be such that $(t_0)^{k_0}=\min_{0\leq k\leq p}t_0^k$. One should observe that $t_0\in(0,1]$ and $p+1-k_0\geq 1$. Hence,

$$\frac{t_0^{p+1}}{t_0^{k_0}}=t_0^{p+1-k_0}\leq t_0\leq 1.$$

Therefore,

$$\frac{(L_0(z)/L(z))^{p+1}}{\min_{0\leq k\leq p}(L_0(z)/L(z))^k}=t_0^{p+1-k_0}\leq t_0\leq 1.$$

Thus, from inequality (15) we get

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\leq C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^kF(z)\right|}{L^k(z)}\colon\ 0\leq k\leq p\right\} \quad (16)$$

for all z such that $\partial_{\mathbf{b}}\Phi(z)\neq 0$.

If $\partial_{\mathbf{b}}\Phi(z)=0$ then for any $k\in\{1,\ldots,N(f,\ell)+1\}$ inequality (5) implies $\partial_{\mathbf{b}}^k\Phi(z)=0$. In view of (6) it means that for each $k\in\{1,\ldots,N(f,l)+1\}$ $\quad\partial_{\mathbf{b}}^kF(z)=0$. Thus, for the points z such that $\partial_{\mathbf{b}}\Phi(z)=0$ inequality (16) is also satisfied.

Therefore, by Theorem 2 this inequality means that the function F has bounded L-index in the direction $\mathbf{b}$. $\square$

References

[1] Bandura A., Skaskiv O., *Slice Holomorphic Functions in Several Variables with Bounded L-Index in Direction*, Axioms, 2019, 8 (3), Article ID 88. doi: 10.3390/axioms8030088

[2] Bandura A.I., Skaskiv O.B., *Some criteria of boundedness of the L-index in direction for slice holomorphic functions of several complex variables*. J. Math. Sci. 2020, **244** (1), 1-21. doi: 10.1007/s10958-019-04600-7

[3] Bandura A. *Composition of entire functions and bounded L-index in direction*. Mat. Stud. 2017, **47** (2), 179–184. doi: 10.15330/ms.47.2.179-184

[4] Bandura A. I., Skaskiv O. B. *Boundedness of L-index for the composition of entire functions of several variables*, Ukr. Math. J. 2019, **70** (10), 1538–1549. doi: 10.1007/s11253-019-01589-9

[5] Bandura A.I. *Composition, product and sum of analytic functions of bounded L-index in direction in the unit ball*, Mat. Stud. 2018, **50** (2), 115–134. doi: 10.15330/ms.50.2.115-134

[6] Bandura A.I., Sheremeta M.M., *Bounded l-index and $l-M$-index and compositions of analytic functions*. Mat. Stud. 2017, **48** (2), 180-188. doi: 10.15330/ms.48.2.180-188

[7] Bandura A. I., Skaskiv O. B., Tsvigun V. L., *The functions of Bounded L-Index in the Collection of Variables Analytic in* $\mathbb{D}\times\mathbb{C}$. J. Math. Sci., 2020, **246** (2), 256–263. doi: 10.1007/s10958-020-04735-y

[8] Bandura A., Petrechko N., Skaskiv O., *Maximum modulus in a bidisc of analytic functions of bounded* **L***-index and an analogue of Hayman's theorem.* Mat. Bohemica., 2018, **143** (4), 339–354. doi: 10.21136/MB.2017.0110-16

[9] Hayman W.K., *Differential inequalities and local valency.* Pacific J. Math., 1973, **44** (1), 117–137.

[10] Kuzyk A.D., Sheremeta M.N., *Entire functions of bounded l-distribution of values.* Math. Notes 1986, **39** (1), 3–8. doi:10.1007/BF01647624

[11] Lepson B., *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index.* Proc. Sympos. Pure Math. 1968, **11**, 298–307.

[12] Macdonnell J. J., *Some convergence theorems for Dirichlet-type series whose coefficients are entire functions of bounded index.* Doctoral dissertation, Catholic University of America, Washington, USA, 1957

[13] Nuray F., Patterson R.F., *Multivalence of bivariate functions of bounded index.* Le Matematiche. 2015, **70**, 225–233. doi:10.4418/2015.70.2.14.

[14] Nuray F., Patterson R.F., *Entire bivariate functions of exponential type.* Bull. Math. Sci. 2015, **5**, 171–177. doi:10.1007/s13373-015-0066-x.

[15] Nuray F. *Bounded index and four dimensional summability methods.* Novi Sad J. Math. 2019, **49**, 73–85. doi:10.30755/NSJOM.08285

[16] Sheremeta M.N., *Entire functions and Dirichlet series of bounded l-index.* Russian Math. (Iz. VUZ) 1992, **36** (9), 76–82.

[17] Sheremeta M., Analytic functions of bounded index. VNTL Publishers, Lviv, 1999.

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Бандура А.І., Скасків О.Б. *Композиція цілих на зрізках функцій та обмежений L-індекс за напрямком* // Буковинський матем. журнал — 2021. — Т.9, №1. — С. 29–38.

Розглядається таке питання: "нехай $f : \mathbb{C} \to \mathbb{C}$ — ціла функція обмеженого l-індексу, $\Phi : \mathbb{C}^n \to \mathbb{C}$ — ціла на зрізках функція, $n \geq 2$, $l : \mathbb{C} \to \mathbb{R}_+$ — неперервна функція. Для якої додатної неперервної функції $L : \mathbb{C}^n \to \mathbb{R}_+$ та для якого напрямку $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ складена функція $f(\Phi(z))$ має обмежений L-індекс за напрямком $\mathbf{b}$?". У поданій статті раніше відомі результати про обмеженість L-індекс за напрямком для композиції цілих функцій $f(\Phi(z))$ узагальнені на випадок, коли $\Phi : \mathbb{C}^n \to \mathbb{C}$ — ціла на зрізках функція, себто ця функція ціла на кожній комплексній прямій $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ для будь-якого $z^0 \in \mathbb{C}^n$ та заданого напрямку $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. Такі цілі на зрізках функцій не голоморфні за сукупністю змінних у загально випадку. Наприклад, такий підхід дозволяє розгляд функцій, голоморфних за змінною z_1 і неперервних за змінноюю z_2.